Charge relaxation in inhomogeneous media

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The example of a 2-D two-phase system and a comb structure is used to show that charge relaxation in inhomogeneous media has a non-Maxwellian character. Under quantum Hall effect conditions, the charge relaxes in a complicated oscillatory manner only due to the inhomogeneity of the system. The effect of the frequency on the quantum Hall effect is studied. The frequency results in destruction of the σ_{xy} plateau (of constant magnitude).

1. INTRODUCTION

As is well known, the relaxation of excess charge density in a homogeneous isotropic conducting medium has an exponential character:

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0)\exp(-t/t_\sigma). \tag{1}$$

here $t_\sigma = 1/4\pi\sigma$ is the Maxwellian relaxation time, and σ is the conductivity of the medium. It has been shown recently that in a 3-D anisotropic case and in systems of reduced dimensionality, the charge relaxes in a quantitatively different manner.[1,2]

The purpose of the present work is to study charge relaxation in inhomogeneous media. Using the example of a 2-D two-phase system and a comb structure, we will show that the inhomogeneity of the medium also appreciably changes the nature of charge relaxation. Inclusion of inhomogeneity is fundamentally necessary for the spreading of charge under quantum Hall effect (QHE) conditions, when we have $\sigma_{xx} = 0$, $\sigma_{xy} = \text{const}$. Another cause of charge spreading under QHE conditions—because of retardation—was discussed in Ref. 3.

A general approach to the study of the properties of a 2-D two-phase system was developed in Refs. 4 and 5. We will apply this method to the study of charge relaxation in a 2-D medium at the flow threshold (at equal concentrations of the phases), and as a result, a generalization of the law (1) to the two-phase case will be obtained. We will show that the relaxation time in an homogeneous medium is determined by the conductivity of the poorly conducting phase. In a metal–dielectric mixture, the charge relaxes according to a power law:

$$\rho(t) \sim \rho_0/(\sigma t)^{1/2}. \tag{2}$$

Charge spreading under QHE conditions has a more complex oscillatory character. In addition to charge relaxation, this approach makes it possible to study the influence of the frequency on the destruction of the σ_{xy} plateau (of constant magnitude) σ_{xy}.

A comb-structure model was proposed in Ref. 6 for studying anomalous diffusion in highly inhomogeneous media. The properties of a random walk on a comb structure were studied in detail in Ref. 7. The present work examines charge spreading on this structure. This spreading also turns out to be non-Maxwellian.

2. CHARGE RELAXATION IN A 2-D TWO-PHASE MEDIUM

Let us consider a 2-D inhomogeneous medium obtained by random mixing of two phases with conductivities σ_1 and σ_2 ($\sigma_1 > \sigma_2$). This model was studied in detail in Refs. 4 and 5, where the effective characteristics of a highly inhomogeneous medium at the flow threshold were determined. Exact solutions were obtained as a result of the invariance of the 2-D equations of electrostatics

$$\operatorname{div}\mathbf{j} = 0, \operatorname{curl}\mathbf{e} = 0 \tag{3}$$

and of Ohm's law

$$\mathbf{j} = \sigma\mathbf{e} \tag{4}$$

relative to the linear transformations

$$\mathbf{j} = a\mathbf{j}' + b[\mathbf{n}\mathbf{e}'], \quad \mathbf{e} = c\mathbf{e}' + d[\mathbf{n}\mathbf{j}'], \tag{5}$$

where $\mathbf{j}$ is the current density, $\mathbf{e}$ is the electric field, σ is the conductivity, and $\mathbf{n}$ is the unit normal to the plane.

In addition to the equations

$$\partial\rho/\partial t + \operatorname{div}\mathbf{j} = 0, \operatorname{curl}\mathbf{e} = 0 \tag{6}$$

and Ohm's law, charge spreading in a conducting medium is also described by the Poisson equation

$$\operatorname{div}\mathbf{e} = 4\pi\rho. \tag{7}$$

Similar equations also hold for the averaged quantities $\mathbf{E} = \langle\mathbf{e}\rangle$, $\mathbf{J} = \langle\mathbf{j}\rangle$. We carry out a Fourier transformation with respect to time and, using the Poisson equation (7), transform Eqs. (6) to the form (3)

$$\operatorname{div}[(i\omega + 4\pi\sigma)\mathbf{e}] = 0, \operatorname{curl}\mathbf{e} = 0. \tag{8}$$

We repeat the familiar arguments of Ref. 4. We choose the coefficients in the transformations (5) in the form

$$a = c = 0, \quad b = d^{-1} = [(i\omega + 4\pi\sigma_1)(i\omega + 4\pi\sigma_2)]^{1/2}.$$

Then for a fixed value of the frequency ω, the primed system will differ from the initial one in the fact that the phase exchange places:

$$(i\omega + 4\pi\sigma)' = (i\omega + 4\pi\sigma_1)(i\omega + 4\pi\sigma_2)/(i\omega + 4\pi\sigma).$$

Macroscopically, these systems are equivalent. Hence, at the flow threshold, the effective conductivity is

$$\sigma_{\text{eff}}(\omega) = [(i\omega + 4\pi\sigma_1)(i\omega + 4\pi\sigma_2)]^{1/2}. \tag{9}$$

Using Eq. (9) for the effective conductivity at a fixed frequency and the Poisson equation for average quantities, we obtain an equation for the averaged concentration

$$\operatorname{div}[\sigma_{\text{eff}}(\omega)\mathbf{E}] = 4\pi[(i\omega + 4\pi\sigma_1)(i\omega + 4\pi\sigma_2)]^{1/2}\langle\rho\rangle = 0. \tag{10}$$

Correspondingly, we have an expression for the Green's

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function averaged over a random arrangement of the phases at the flow threshold

$$G(\omega)=1/[(i\omega+4\pi\sigma_1)(i\omega+4\pi\sigma_2)]^{1/2}. \qquad (11)$$

From Eq. (11) it is easy to find the expression for the Green's function in the t representation:

$$G(t)=I_0(2\pi(\sigma_1-\sigma_2)t)\exp(-2\pi(\sigma_1+\sigma_2)t), \qquad (12)$$

where $I_0(x)$ is a modified Bessel function. Using the known asymptotic expression for $I_0(x)$, we obtain for long times

$$G(t)\sim\exp(-4\pi\sigma_2 t)/[(\sigma_1-\sigma_2)t]^{1/2}. \qquad (13)$$

The result (13) has a clear meaning. In a randomly inhomogeneous two-phase medium, relaxation at late times is limited by inclusions of the poorly conducting phase: $t_\sigma = 1/4\pi\sigma_2$ (in the well-conducting phase, the charge spreads quickly). The power-law behavior in the metal-dielectric mixture is also easy to understand. At the flow threshold, the correlation radius is infinite, and therefore, metallic phase inclusions of all sizes are possible. The absence of characteristic scales from the problem leads to the power-law relaxation (2).

Let us note that the frequency dependence of the effective conductivity was also obtained in Ref. 8; charge relaxation was not studied in that work.

3. CHARGE SPREADING AND DESTRUCTION OF THE σ_{xy} PLATEAU AT A FIXED FREQUENCY UNDER QUANTUM HALL EFFECT CONDITIONS

Let us consider the charge relaxation in a two-phase medium placed in a magnetic field **H**. This relaxation is described by the same equations (6) and (7), except that instead of Eq. (4), it is necessary to use the generalized Ohm's law:

$$\mathbf{j}=\sigma_{xx}\mathbf{e}+\sigma_{xy}[\mathbf{ne}], \qquad (14)$$

where $\sigma_{xx} = \sigma/(1+\beta^2)$, $\sigma_{xy} = \sigma\beta/(1+\beta^2)$, $\beta = e\mathbf{H}\tau/mc$ is the Hall factor. When Eqs. (7) and (14) are taken into account, Eqs. (8) assume the form

$$\mathrm{div}\,((i\omega+4\pi\sigma_{xx})\mathbf{e}+4\pi\sigma_{xy}[\mathbf{ne}])=0, \ \mathrm{curl}\,\mathbf{e}=0. \qquad (15)$$

Repeating the reasoning given above, we find an expression for the averaged Green's function in a magnetic field:

$$G(\omega)=1\Big/\left[(i\omega+4\pi\sigma_{xx}^{(1)})(i\omega+4\pi\sigma_{xx}^{(2)})\times\left(1+\left(\frac{\sigma_{xy}^{(1)}-\sigma_{xy}^{(2)}}{2i\omega+\sigma_{xx}^{(1)}+\sigma_{xx}^{(2)}}\right)^2\right)\right]^{1/2}. \qquad (16)$$

Let us consider the limit of the ideal quantum Hall effect $\sigma_{xx} = 0$, $\sigma_{xy} = \mathrm{const}$. In this case we have

$$G(\omega)=1/[{}^1/_4(\sigma_{xy}^{(1)}-\sigma_{xy}^{(2)})^2-\omega^2]^{1/2}. \qquad (17)$$

Correspondingly, in the t representation we obtain

$$G(t)=J_0({}^1/_2|\sigma_{xy}^{(1)}-\sigma_{xy}^{(2)}|t), \qquad (18)$$

where J_0 is a Bessel function. Thus, under quantum Hall effect conditions, charge relaxation has an oscillatory character and is possible only in an inhomogeneous medium.

Let us clarify the result obtained. As follows from the continuity equation (6) and Ohm's law (14), the charge in a Hall system spreads along the phase boundaries owing to surface currents. Oscillations of charge density are obtained as a result of the superposition of surface currents moving at different velocities in different phases. At the flow threshold, the conditions of the two phases are equivalent, and therefore, the change in charge density is described only by the conductivities of the phases.

We will study the influence of frequency on σ_{xy}^e. (It was shown earlier that the effective characteristics of a medium consisting of a metallic and Hall mixture are constant and equal to the values in the Hall phase as long as flow in the Hall phase takes place.[9]) For this purpose, we shall use the general Dykhne relation between the quantities σ_{xx}^e and σ_{xy}^e at any concentrations:[5]

$$(\sigma_{xy}^e)^2+\sigma_{xy}^e(a/d-c/b)-ac/bd+(\sigma_{xx}^e)^2=0. \qquad (19)$$

Here $a = -c = 1$, $b = -\sigma_{xy}^{(1)}$, $d = -\sigma_{xy}^{(2)}$. This relation was obtained as the result of the symmetry of the 2-D system relative to the direction of the magnetic field. In the QHE limit at a fixed frequency we also have

$$\sigma_{xx}^e(\omega)=[{}^1/_4(\sigma_{xy}^{(1)}-\sigma_{xy}^{(2)})^2-\omega^2]^{1/2}. \qquad (20)$$

Then from Eq. (19) we obtain the following dependence of σ_{xy}^e on the frequency ω:

$$\sigma_{xy}^e(\omega)={}^1/_2(\sigma_{xy}^{(1)}-\sigma_{xy}^{(2)})+[{}^1/_2[(\sigma_{xy}^{(1)})^2+(\sigma_{xy}^{(2)})^2]+\omega^2]^{1/2} \qquad (21)$$

4. CHARGE RELAXATION ON A COMB STRUCTURE

The comb structure (see Fig. 1) is arranged so that the charge transfer on this structure in the x direction is possible only along its axis (for $y=0$), i.e., $\sigma_{xx} = \sigma_1\delta(y)$ ($j_x = \sigma_{xx}e_x$). The conductivity along the edges is conventional: $\sigma_{yy} = \sigma_2$. Thus, the conductivity of the comb structure is described by the tensor (see also Ref. 7)

$$\sigma=\begin{pmatrix}\sigma_1\delta(y) & 0\\ 0 & \sigma_2\end{pmatrix}. \qquad (22)$$

From the continuity equation with the conductivity tensor (22) and the Poisson equation (7), we obtain an equation for the electric potential φ:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\varphi}{\partial t}+4\pi\left(\sigma_1\delta(y)\frac{\partial^2}{\partial x^2}+\sigma_2\frac{\partial^2}{\partial y^2}\right)\varphi=0. \qquad (23)$$

We perform a Laplace transformation with respect to time and a Fourier transformation with respect to the x coordinate. The Green's function in the mixed (p, k, y) representation is described by the equation

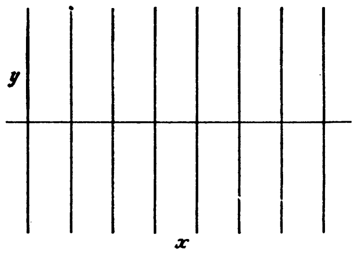

FIG. 1. Comb structure: ribs going off to infinity are fastened to the conducting axis ($y=0$).

$$\left[\left(-k^2+\frac{\partial^2}{\partial y^2}\right)p+4\pi\left(-\sigma_1 k^2\delta(y)+\sigma_2\frac{\partial^2}{\partial y^2}\right)\right]\Phi(k,p,y)=\delta(y). \tag{24}$$

We seek the solution of Eq. (24) in the form

$$\Phi(k,p,y)=f(k,p)\exp(-\lambda|y|). \tag{25}$$

Substituting Eq. (25) into Eq. (24), we find the parameter λ:

$$\lambda=|k|[p/(p+4\pi\sigma_2)]^{1/2}. \tag{26}$$

Integrating over the circle $y=0$ (discontinuity of the derivative), we obtain an expression for the function $f(k,p)$

$$f(k,p)=1/[2\lambda(p+4\pi\sigma_2)+4\pi\sigma_1 k^2]. \tag{27}$$

Knowing the expression (25), one can readily set up the corresponding Green's function of the equation for the concentration:

$$G(k,p,y)=\frac{2\lambda\delta(y)+k^2-\lambda^2}{2\lambda(p+4\pi\sigma_2)+4\pi\sigma_1 k^2}\exp(-\lambda|y|). \tag{28}$$

As an example, we find the expression for the Green's function averaged over y:

$$\bar{G}(k,p)=\int_{-\infty}^{+\infty}G(k,p,y)\,dy=\frac{k^2}{\lambda[\lambda(p+4\pi\sigma_2)+2\pi\sigma_1 k^2]}. \tag{29}$$

In the (x,t) representation, the latter is

$$\bar{G}(x,t)=\exp(-2\pi\sigma_2 t)\left\{\frac{vt}{x^2+v^2t^2}+\sigma_2\int_0^t\frac{v\tau}{x^2+v^2\tau^2}\right.$$
$$\left.\times\left[I_0\left(\frac{\sigma_2}{2}[t^2-\tau^2]^{1/2}\right)+I_1\left(\frac{\sigma_2}{2}[t^2-\tau^2]^{1/2}\right)\frac{t}{[t^2-\tau^2]^{1/2}}\right]\frac{d\tau}{\pi}\right\}. \tag{30}$$

Here we have written $v=2\pi\sigma_1$, and $I_0(x)$ and $I_1(x)$ are modified Bessel functions. For $\sigma_2=0$, the Green's function has the simple form

$$\bar{G}(x,t)=\frac{1}{\pi}\frac{vt}{x^2+v^2t^2}. \tag{31}$$

A similar expression is obtained when the charges are located in a plane, and the field has all three spatial components.[2]

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[1] M. I. D'yakonov and A. S. Furman, Zh. Eksp. Teor. Fiz. **92**, 1012 (1987) [Sov. Phys. JETP **65**, 574 (1987)].
[2] A. O. Govorov and A. V. Chaplik, Poverkhnost' **12**, 5 (1987).
[3] A. O. Govorov and A. V. Chaplik, Zh. Eksp. Teor. Fiz. **95**, 1976 (1989) [Sov. Phys. JETP **68**, 1143 (1989)].
[4] A. M. Dykne, Zh. Eksp. Teor. Fiz. **59**, 110 (1970) [Sov. Phys. JETP **32**, 63 (1971)].
[5] A. M. Dykhne, Zh. Eksp. Teor. Fiz. **59**, 641 (1970) [Sov. Phys. JETP **32**, 348 (1971)].
[6] G. H. Weiss and S. Havlin, Physica A **174**, 474 (1986).
[7] V. E. Arkhincheev and E. M. Baskin, *Authors' Abstracts of 14th Conference on Semiconductor Theory* (1989), p. 102.
[8] A. M. Dykhne, *Proceedings of International Kiev Conference on Plasma Theory* (1984), p. 1323.
[9] V. E. Arkincheyev and E. G. Batyev, Solid State Commun. (1989) (in press).

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