Properties of the photon polarization operator in an electric field: effective charge of the electron in an external field

G. K. Artimovich

P. N. Lebedev Physics Institute, USSR Academy of Sciences
(Submitted 19 September 1989; revision submitted 18 December 1989)
Zh. Eksp. Teor. Fiz. **97**, 1393–1406 (May 1990)

A compact expression is proposed for the polarization operator in a constant electromagnetic field. We derive the asymptotic behavior of the polarization operator in weak and strong electric fields, and appropriate applicability conditions are found for the crossed-field approximation. We show that in an electric field, the imaginary part of the polarization operator never vanishes. In the subthreshold domain it is exponentially small if the field is weak, while above threshold it oscillates as a result of interference among the individual components of the electron wave function. We go on to examine the effective charge of the electron determined by the polarization operator. A modest generalization of the effective-charge concept enables one to extend to the crossed-field case the analogy between the field dependence and the virtual photon momentum-transfer dependence for effective charges.

INTRODUCTION

The properties of the photon polarization operator in a constant magnetic field have been studied in detail,[1-5] as have its properties in a constant crossed field.[6-8] Thus far, however, the electric-field case has received scant attention; the latter therefore constitutes the main thrust of the present paper. We consider all three cases—magnetic, electric, and crossed fields—in only one section, where we deal with the effective charge of the electron.

The paper is organized as follows. Section 1 is concerned with general questions bearing upon the tensor structure of the photon polarization operator in an arbitrary constant field. We introduce a compact expression for the polarization operator, an expansion in terms of four tensors in standard form. The coefficients of that expansion are functions of four invariant parameters derived from the electromagnetic field tensor $F_{\mu\nu}$ and the photon momentum k_μ. In Section 2, we calculate the corrections that distinguish the polarization operator in a weak electric field from the polarization operator in a crossed field; those corrections establish the limits of applicability of the crossed-field approximation. The asymptotic behavior of the polarization operator at a material surface is also studied in Sec. 2 for a strong electric field. In Sec. 3, we study the effect of an electric field on pair creation by a virtual photon. We derive the dependence of the imaginary part of one of the polarization-operator expansion components on the momentum transfer of the virtual photon. Finally, Sec. 4 is concerned with the fundamentally important question of the effective charge of an electron immersed in an external field.

In this paper we employ a system of units with $\hbar = c = 1$. Electromagnetic quantities are measured in Heaviside units, so that the fine structure constant is $\alpha = e/4\pi \approx 1/137$. Four-dimensional quantities may be written in Pauli notation,

$$a_\mu=(a_1, a_2, a_3, ia_0),\ a^2=a_\perp{}^2+a_\parallel{}^2,$$

where $a_\perp^2 = a_1^2 + a_2^2$ and $a_\parallel^2 = a_3^2 - a_0^2$. We make use of the tensor invariants $F_{\mu\nu}$ and $F^*_{\mu\nu} = i\varepsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}/2$;

$$\varepsilon=[(\mathcal{F}^2+\mathcal{G}^2)^{1/2}-\mathcal{F}]^{1/2},$$
$$\eta=\mathcal{G}/\varepsilon=\pm[(\mathcal{F}^2+\mathcal{G}^2)^{1/2}+\mathcal{F}]^{1/2},$$

where

$$\mathcal{F}=\frac{1}{4}F_{\mu\nu}F_{\mu\nu}=\frac{1}{2}(H^2-E^2),$$
$$\mathcal{G}=\frac{1}{4}F_{\mu\nu}F_{\mu\nu}{}^*=\mathbf{EH}.$$

We also make use of the invariant dimensionless parameters

$$\lambda=-k^2/m^2,\ \varkappa=|e|m^{-3}[(F_{\mu\nu}k_\nu)^2]^{1/2}.$$

All invariant quantities in the present paper are functions of the four invariants ε, η, $\varkappa$, and λ. Noninvariant expressions are always rendered in a reference system in which **E** and **H** are parallel, and point along axis 3. The parameters ε and η in that system have a simple physical meaning: $\varepsilon = |\mathbf{E}|$, $\eta = \pm|\mathbf{H}|$, and the expression for $\varkappa$ takes the form

$$\varkappa=|e|m^{-3}[\eta^2k_\perp{}^2+\varepsilon^2(k_\perp{}^2-k^2)]^{1/2}.$$

1. GENERAL EXPRESSION FOR THE PHOTON POLARIZATION OPERATOR

In an arbitrary constant field, the photon polarization operator is a function of the operators

$$-i\partial_\mu,\ -iF_{\mu\nu}\partial_\nu,\ -iF_{\mu\nu}{}^*\partial_\nu,\ -iF_{\mu\nu}F_{\nu\lambda}\partial_\lambda, \tag{1}$$

which commute with the momentum operator $-i\partial_\mu$. The polarization operator is therefore diagonal in the momentum representation:

$$\Pi_{\mu\nu}(k, k', F)=(2\pi)^4\delta(k-k')\Pi_{\mu\nu}(k, F). \tag{2}$$

The polarization vector of a photon with momentum k_μ can be expanded in terms of four independent vectors

$$k_\mu,\ L_\mu=F_{\mu\nu}k_\nu,\ L_\mu{}^*=F_{\mu\nu}{}^*k_\nu,$$
$$G_\mu=\frac{k^2}{L^2}F_{\mu\nu}F_{\nu\lambda}k_\lambda+k_\mu. \tag{3}$$

 0038-5646/90/050787-08$03.00

The symmetry, transversality, and C and P invariance of the operator $\Pi_{\mu\nu}(k,F)$ imply that it can be expanded in terms of four tensors (see (Refs. 7, 8),

$$\frac{L_\mu L_\nu}{L^2},\quad \frac{L_\mu^* L_\nu^*}{L^{*2}},\quad \frac{L_\mu L_\nu^*+L_\nu L_\mu^*}{LL^*},\quad \frac{G_\mu G_\nu}{G^2}, \tag{4}$$

where

$$L=(L_\mu L_\mu)^{1/2}=(\eta^2 k_\perp^2-\varepsilon^2 k_\parallel^2)^{1/2},$$
$$L^*=(L_\mu^* L_\mu^*)^{1/2}=(\varepsilon^2 k_\perp^2-\eta^2 k_\parallel^2)^{1/2}, \tag{5}$$
$$G^2=G_\mu G_\mu=\frac{(\varepsilon^2+\eta^2)^2}{L^4}k_\perp^2 k_\parallel^2 k^2.$$

The vectors L_μ, L_μ^*, and G_μ are orthogonal to k_μ, and the vector G_μ is also orthogonal to L_μ and L_μ^*. On the other hand, the latter two vectors are only orthogonal if at least one of the invariants ε, η vanishes, i.e., for purely magnetic, purely electric, or crossed fields.

In general, if instead of L_μ^* we use the vector

$$\tilde{L}_\mu=L_\mu^*+CL_\mu, \tag{6}$$

where

$$C=-\frac{L_\mu L_\mu^*}{L^2}=-\frac{\varepsilon\eta k^2}{L^2}, \tag{7}$$

we obtain a set of mutually orthogonal vector k_μ, L_μ, $\tilde{L}_\mu$, and G_μ. We therefore have

$$\frac{k_\mu k_\nu}{k^2}+\frac{L_\mu L_\nu}{L^2}+\frac{\tilde{L}_\mu \tilde{L}_\nu}{\tilde{L}^2}+\frac{G_\mu G_\nu}{G^2}=\delta_{\mu\nu}, \tag{8}$$

where

$$\tilde{L}=(\tilde{L}_\mu \tilde{L}_\mu)^{1/2}=\left(-\frac{L^2G^2}{k^2}\right)^{1/2}. \tag{9}$$

Making use of Eqs. (6), (8), and (9), we can express the tensor $G_\mu G_\nu/G^2$ in terms of the tensors

$$\frac{L_\mu L_\nu}{L^2},\quad \frac{L_\mu^* L_\nu^*}{L^{*2}},\quad \delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2},\quad \frac{L_\mu L_\nu^*+L_\nu L_\mu^*}{LL^*}. \tag{10}$$

Thus, in addition to (4), one could also take these four tensors as a basis for the expansion of $\Pi_{\mu\nu}(k,F)$.

The general expression for the polarization operator in the one-loop approximation has been derived by Batalin and Shabad.[9,10] In a previous paper,[11] the present author described in detail the procedure for expanding $\Pi_{\mu\nu}(k,F)$ in terms of the tensors (10); here we cite only the final result:

$$\Pi_{\mu\nu}(k,F)=\pi_1\frac{L_\mu L_\nu}{L^2}+\pi_2\frac{L_\mu^* L_\nu^*}{L^{*2}}+\pi_3\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+\pi_4\frac{L_\mu L_\nu^*+L_\nu L_\mu^*}{LL^*}, \tag{11.1}$$

where

$$\pi_1=\frac{2\alpha m^2}{\pi}\left(\frac{\varkappa m^2}{e}\right)^2\frac{1}{(\varepsilon^2+\eta^2)^2}\int\!\!\int_0^\infty ds\,ds'\,B_0[\eta^2B_1-\varepsilon^2B_2+(\varepsilon^2-\eta^2)B_3-2\varepsilon^2\eta^2B_4]\times\exp(-im^2\theta), \tag{11.2}$$

$$\pi_2=\frac{2\alpha m^2}{\pi}\left(\frac{\varkappa m^2}{e}\right)^2\frac{1}{(\varepsilon^2+\eta^2)^2}\left[1+\lambda(\eta^2-\varepsilon^2)\left(\frac{e}{\varkappa m^2}\right)^2\right]\times\int\!\!\int_0^\infty ds\,ds'\,B_0[\varepsilon^2B_1-\eta^2B_2+(\eta^2-\varepsilon^2)B_3+2\varepsilon^2\eta^2B_4]\exp(-im^2\theta), \tag{11.3}$$

$$\pi_3=-\frac{2\alpha m^2\lambda}{\pi}\int\!\!\int_0^\infty ds\,ds'\left[B_0B_3\exp(-im^2\theta)-\frac{ss'}{(s+s')^4}\exp(-im^2\theta_0)\right]+\pi_0, \tag{11.4}$$

$$\pi_4=\frac{2\alpha m^2}{\pi}\left(\frac{\varkappa m^2}{e}\right)^2\frac{\varepsilon\eta}{(\varepsilon^2+\eta^2)^2}\left[1+\lambda(\eta^2-\varepsilon^2)\left(\frac{e}{\varkappa m^2}\right)^2\right]^{1/2}\times\int\!\!\int_0^\infty ds\,ds'\,B_0[B_1+B_2-2B_3+(\eta^2-\varepsilon^2)B_4]\exp(-im^2\theta), \tag{11.5}$$

$$\theta=s+s'+\left(\frac{\varkappa m^2}{e}\right)^2\frac{A_1-A_2}{\varepsilon^2+\eta^2}-\lambda\frac{\varepsilon^2A_1+\eta^2A_2}{\varepsilon^2+\eta^2},$$
$$\theta_0=s+s'-\lambda\frac{ss'}{s+s'}, \tag{11.6}$$

$$A_1=\frac{\sin e\eta s\cdot\sin e\eta s'}{e\eta\sin[e\eta(s+s')]},\qquad A_2=\frac{\operatorname{sh} e\varepsilon s\cdot\operatorname{sh} e\varepsilon s'}{e\varepsilon\operatorname{sh}[e\varepsilon(s+s')]}, \tag{11.7}$$

$$B_0=\frac{e^2\varepsilon\eta}{\sin[e\eta(s+s')]\cdot\operatorname{sh}[e\varepsilon(s+s')]},$$
$$B_1=\frac{\sin e\eta s\cdot\sin e\eta s'\cdot\operatorname{ch}[e\varepsilon(s+s')]}{\sin^2[e\eta(s+s')]},$$
$$B_2=\frac{\cos[e\eta(s+s')]\cdot\operatorname{sh} e\varepsilon s\cdot\operatorname{sh} e\varepsilon s'}{\operatorname{sh}^2[e\varepsilon(s+s')]}, \tag{11.8}$$
$$B_3=\frac{\sin 2e\eta s\cdot\operatorname{sh} 2e\varepsilon s'}{4\sin[e\eta(s+s')]\cdot\operatorname{sh}[e\varepsilon(s+s')]}$$
$$B_4=\frac{\sin^2 e\eta s\cdot\operatorname{sh}^2 e\varepsilon s'}{\varepsilon\eta\sin[e\eta(s+s')]\cdot\operatorname{sh}[e\varepsilon(s+s')]}.$$

The function π_0 in (11.4) defines the photon polarization operator in vacuum:

$$\Pi_{\mu\nu}(k)=\pi_0\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right).$$

It is not hard to see that the functions A_i $(i=1,2)$ and B_i $(i=0,\ldots,4)$ are scalars. It is also immediately clear from Eqs. (11.1)–(11.8) that the functions π_i $(i=1,2,3)$ are scalars, while π_4 is a pseudoscalar. The latter property is related to the fact that $(L_\mu L_\nu^*+L_\nu L_\mu^*)/LL^*$ is a pseudotensor, while $\Pi_{\mu\nu}(k,F)$ must be a true tensor. The function π_4 will vanish if either ε of η is zero, a result we can demonstrate explicitly by expanding π_4 in powers of ε and η, retaining only the first nonvanishing term, and replacing s and s' with

$$v=\frac{(s+s')^2}{ss'},\qquad x=m^2\varkappa^{2/3}\frac{(ss')^{2/3}}{(s+s')^{1/3}}. \tag{12}$$

We then obtain

$$\pi_4=-\frac{2\alpha e^2\varepsilon\eta}{9\pi m^2}\int\limits_4^\infty dv\frac{7v+2}{v^2[v(v-4)]^{1/2}}\left(\frac{v}{\varkappa}\right)^{2/3}f'''(z)+\ldots, \tag{13}$$

where

$$z=\left(\frac{v}{\varkappa}\right)^{2/3}\left(1-\frac{\lambda}{v}\right), \tag{14}$$

and

$$f(z)=i\int\limits_0^\infty dx\exp\left[-i\left(xz+\frac{x^3}{3}\right)\right]$$

is related to the Airy function.

To conclude this section, we point out that in the special case of a magnetic field, our result is identical to the result previously obtained by Tsai (see Ref. 2), and for crossed fields, it is the same as a result first obtained by Narozhnyi.[6]

2. ASYMPTOTIC BEHAVIOR OF THE POLARIZATION OPERATOR AT A MATERIAL SURFACE IN WEAK AND STRONG ELECTRIC FIELDS

In a purely electric field ($\eta=0$, $\varepsilon\neq0$), the function π_4 goes to zero, and the vectors L_μ and L_μ^* become orthogonal.

Let us first consider a weak electric field (we shall identify the criteria for weakness below). Assuming $\eta=0$ in Eqs. (11.2)–(11.8), expanding the integrands in powers of ε, and transforming to the variables (12), we obtain

$$\pi_1=\frac{4\alpha m^2}{3\pi}\int\limits_4^\infty\frac{dv}{v[v(v-4)]^{1/2}}\left(\frac{\varkappa}{v}\right)^{2/3}\left\{(v-1)f'(z)+\frac{2}{15}\frac{\beta^2}{\varkappa}\left(\frac{v}{\varkappa}\right)^{1/3}\left[(v^2+4v-2)f'''(z)-\frac{1}{6}(v^2+v-2)f^{(6)}(z)\right]+\ldots\right\}, \tag{15.1}$$

$$\pi_2=\frac{4\alpha m^2}{3\pi}\int\limits_4^\infty\frac{dv}{v[v(v-4)]^{1/2}}\left(\frac{\varkappa}{v}\right)^{2/3}\times\left\{\left[1-\lambda\left(\frac{\beta}{\varkappa}\right)^2\right](v+2)f'(z)+\frac{2}{15}\frac{\beta^2}{\varkappa}\left(\frac{v}{\varkappa}\right)^{1/3}\left[(v^2-v+3)f'''(z)-\frac{1}{6}(v^2+4v+4)f^{(6)}(z)\right]+\ldots\right\}, \tag{15.2}$$

$$\pi_3=-\frac{4\alpha m^2\lambda}{\pi}\int\limits_4^\infty\frac{dv}{v^2[v(v-4)]^{1/2}}\left\{f_1(z)-\frac{2}{3}\frac{\beta^2}{\varkappa}\left(\frac{v}{\varkappa}\right)^{1/3}\left[f'(z)-\frac{1}{30}(v+2)f^{(4)}(z)\right]+\ldots\right\}+\pi_0, \tag{15.3}$$

where

$$f_1(z)=\int\limits_0^\infty\frac{dx}{x}e^{-ixz}\left[\exp\left(-\frac{ix^3}{3}\right)-1\right]$$

is a function which was introduced and investigated in Ref. 7. Instead of ε in Eqs. (15.1)–(15.3), we make use of the dimensionless parameter

$$\beta=\frac{|e\varepsilon|}{m^2}. \tag{16}$$

If we put $\beta=0$ in (15.1)–(15.3), we obtain the result quoted above for the polarization operator in a crossed field (see Refs. 6–8). Terms proportional to β^2 in (15.1)–(15.3) are corrections to the cross fields, and an ellipsis represents terms that are small compared to β^2 as $\beta\to0$.

At a material surface we have $\lambda=0$, the quantity π_3 vanishes, and $\pi_{1,2}$ take the form

$$\pi_{1,2}=\frac{2\alpha m^2}{3\pi}\int\limits_4^\infty\frac{dv}{v[v(v-4)]^{1/2}z}\left\{(2v+1\mp3)f'(z)+\frac{2}{15}\frac{\beta^2}{\varkappa}z^{1/2}\left[(2v^2+3v+1\pm5(v-1))f'''(z)-\frac{1}{6}(2v^2+5v+2\mp3(v+2))f^{(6)}(z)\right]+\ldots\right\}, \tag{17}$$

with $z=(v/\varkappa)^{2/3}$; the upper sign in (17) corresponds to π_1, and the lower to π_2.

For $\varkappa<1$, the most important contribution to the integral in (17) comes from $v\sim1$, so that we have $z\gg1$. Taking advantage of the well known asymptotic behavior of $f(z)$ as $z\to\infty$ (see Refs. 7, 8), we obtain

$$\mathrm{Re}\,\pi_{1,2}=-\frac{\alpha m^2\varkappa^2}{90\pi}\left[(11\mp3+\ldots)+\frac{2}{7}\beta^2(37\pm11+\ldots)\right]+\ldots,$$

$$\mathrm{Im}\,\pi_{1,2}=-\frac{(3\mp1)}{16}\left(\frac{3}{2}\right)^{1/2}\alpha m^2\varkappa e^{-8/3\varkappa}\left[(1+\ldots)+\frac{32}{15}\frac{\beta^2}{\varkappa^3}(1+\ldots)\right]+\ldots, \tag{18}$$

where the ellipses in parentheses represent terms small compared to unity as $\varkappa\to0$. The approximation (17) therefore holds for the real part of the functions $\pi_{1,2}$ when we have

$$\beta\ll1, \tag{19}$$

and for the imaginary part when we have

$$\beta\ll\varkappa^{3/2}. \tag{20}$$

Interestingly enough, the right-hand side of (19) makes no reference to $\varkappa$; in other words, in calculating $\mathrm{Re}\,\pi_{1,2}$, we can assume that we are dealing with crossed fields regardless of the relationship between β and $\varkappa$, as long as the field is much weaker than the characteristic quantum electrodynamic value

$$F_0=\frac{m^2c^3}{e\hbar}\approx4.4\cdot10^{13}\ \mathrm{Oe}, \tag{21}$$

which is far stronger than fields currently attainable in the laboratory. Normally, in calculations such as these, one can assume crossed fields as long as the parameter β is small compared to $\varkappa$.

The imaginary part of the polarization operator yields the probability of pair creation by a polarized photon. Specifically, the total relative probability for pair creation by a photon with polarization e_μ is (see Refs. 7, 8)

$$W(e)=-\frac{1}{k_0}\sum_{i=1,2}(ee_i)^2\,\mathrm{Im}\,\pi_i, \qquad (22)$$

where

$$e_{1\mu}=\frac{L_\mu}{L}, \quad e_{2\mu}=\frac{L_\mu^*}{L^*}.$$

Substituting the expressions in (18) for $\mathrm{Im}\,\pi_{1,2}$ into (22) and averaging over the photon polarization (note that we have $(ee_i)^2=1/2$), we obtain for the probability of pair creation by an unpolarized photon

$$W=\left(\frac{3}{2}\right)^{1/2}\frac{\alpha m^2}{8k_0}\varkappa e^{-8/3\varkappa}\left(1+\frac{32}{15}\frac{\beta^2}{\varkappa^3}+\ldots\right). \qquad (23)$$

For $\varkappa\gg1$, we may use the Taylor expansion of the function $f(z)$,

$$f(z)=\sum_{n=0}^{\infty}\frac{3^{(n-2)/3}}{n!}\exp\left[-\frac{i\pi(2n-1)}{3}\right]\Gamma\left(\frac{n+1}{3}\right)z^n. \qquad (24)$$

As a result, we obtain

$$\mathrm{Re}\,\pi_{1,2}=\frac{2\alpha m^2\varkappa^{2/3}}{3\pi}\left[\frac{\pi^{1/2}3^{5/3}\Gamma^2(2/3)}{2^{4/3}\cdot7\Gamma(1/6)}(5\mp1+\ldots)\right.$$
$$\left.+\frac{1}{5}\frac{\beta^2}{\varkappa^{2/3}}(1+\ldots)\right]+\ldots,$$
$$\mathrm{Im}\,\pi_{1,2}=-\frac{2\alpha m^2\varkappa^{2/3}}{3\pi}\left[\frac{\pi^{1/2}3^{13/6}\Gamma^2(2/3)}{2^{4/3}\cdot7\Gamma(1/6)}(5\mp1+\ldots)\right.$$
$$\left.+\frac{2^{4/3}\Gamma(1/3)}{3^{7/6}\cdot5}\frac{\beta^2}{\varkappa^{4/3}}(1+\ldots)\right]+\ldots, \qquad (25)$$

$$W=\frac{2\alpha m^2\varkappa^{2/3}}{3\pi k_0}\left[\frac{\pi^{1/2}5\cdot3^{13/6}\Gamma^2(2/3)}{2^{4/3}\cdot7\Gamma(1/6)}+\frac{2^{4/3}\Gamma(1/3)}{3^{7/6}\cdot5}\frac{\beta^2}{\varkappa^{4/3}}+\ldots\right]. \qquad (26)$$

In the present case, the prerequisite for the approximation (17) to hold for the real part of $\pi_{1,2}$ is

$$\beta\ll\varkappa^{1/3}, \qquad (27)$$

and for the imaginary part, it is

$$\beta\ll\varkappa^{2/3}. \qquad (28)$$

Note that Narozhnyi[12] calculated the probability of pair creation by a photon in a weak electric field in a different way. The conditions (20) and (28) for the imaginary part of $\pi_{1,2}$ are the same as the corresponding conditions obtained in Ref. 12, but Narozhnyi obtained twice as large a value for the coefficient of $\beta^2/\varkappa^3$ in Eq. (23). The reason for the discrepancy is not yet known. We also point out that a general expression for the probability of pair creation by a photon in an arbitrarily strong electric field was first obtained by Nikishov.[13]

Let us now consider the case of a strong electric field. Assuming $\eta=0$ and $\lambda=0$ in Eqs. (11.2), (11.3), and (11.6)–(11.8), and transforming to variables

$$x=|e\varepsilon|s,\ y=|e\varepsilon|(s+s'), \qquad (29)$$

we obtain

$$\pi_{1,2}=\frac{2\alpha m^2}{\pi}\left(\frac{\varkappa}{\beta}\right)^2 I_{1,2}, \qquad (30)$$

where

$$I_1=\int_0^\infty\frac{dy}{y\,\mathrm{sh}^2y}\int_0^y dx\,\mathrm{sh}(y-x)\left[\frac{x\,\mathrm{ch}(y-x)}{y}-\frac{\mathrm{sh}\,x}{\mathrm{sh}\,y}\right]e^{-i\varphi/\beta}, \qquad (31)$$

$$I_2=\int_0^\infty\frac{dy}{y^2\,\mathrm{sh}\,y}\int_0^y dx\,x\left[\left(1-\frac{x}{y}\right)\mathrm{ch}\,y-\frac{\mathrm{sh}\,2(y-x)}{2\,\mathrm{sh}\,y}\right]e^{-i\varphi/\beta}, \qquad (32)$$

$$\varphi=y+\left(\frac{\varkappa}{\beta}\right)^2\left[x-\frac{x^2}{y}-\frac{\mathrm{sh}\,x\cdot\mathrm{sh}(y-x)}{\mathrm{sh}\,y}\right]. \qquad (33)$$

We assume $\varkappa\lesssim\beta$. The main contribution to I_1 comes at $x\sim y\sim1$, so we can put $\beta=\infty$. The main contribution to I_2 comes at $x_{\mathrm{eff}}\sim y_{\mathrm{eff}}\sim\beta\gg1$, which facilitates a significant simplification of the integrand. The net result is then

$$\pi_1=\frac{\alpha m^2a^2}{3\pi}(1+\ldots), \qquad (34)$$

$$\pi_2=-\frac{2i\alpha\beta m^2}{\pi}\left[1-\frac{1}{a(1+a^2/4)^{1/2}}\ln\frac{(1+a^2/4)^{1/2}+a/2}{(1+a^2/4)^{1/2}-a/2}+\ldots\right]. \qquad (35)$$

In the latter two equations we have made use of the scale-invariant parameter

$$a=\varkappa/\beta. \qquad (36)$$

As noted above, (34) and (35) yield the asymptotic behavior for $\beta\gg1$ and $a\lesssim1$. If $a\ll1$, the expression for π_2 simplifies considerably:

$$\pi_2=-\frac{i\alpha m^2\varkappa^2}{3\pi\beta}+\ldots \qquad (37)$$

An interesting result can be obtained by assuming $a\ll1$ from the very outset. We can then put $\varkappa=0$ in (33) and integrate I_1 and I_2 over x.

Integrating by parts, we can reduce the remaining integrals over y to integrals of meromorphic functions. There is a theorem on the series expansion of meromorphic functions in elementary fractions[14] which makes it possible to calculate the imaginary parts of the integrals I_1 and I_2 exactly. The results are

$$\mathrm{Im}\,\pi_1=-\frac{\alpha m^2a^2}{2\beta}\sum_{n=1}^{\infty}\left(\frac{1}{\beta}+\frac{1}{\pi n}+\ldots\right)e^{-\pi n/\beta}, \qquad (38)$$

$$\mathrm{Im}\,\pi_2=-\alpha m^2a^2\sum_{n=1}^{\infty}\left(\frac{1}{3}+\frac{1}{2\pi n\beta}-\frac{1}{\pi^2n^2}+\ldots\right)e^{-\pi n/\beta}, \qquad (39)$$

where the ellipses denote terms that vanish as $\varkappa \to 0$. Equations (38) and (39) apply to both strong and weak fields. As $\beta \to \infty$, Eq. (39) goes into (37), while Im π_1 vanishes in accordance with (34).

Substituting (38) and (39) into (22) and averaging over photon polarization, we obtain for the pair-creation probability

$$W = \frac{\alpha m^2 a^2}{2k_0} \sum_{n=1}^{\infty} \left(\frac{1}{2\beta^2} + \frac{1}{\pi n \beta} + \frac{1}{3} - \frac{1}{\pi^2 n^2} + \dots \right) e^{-\pi n/\beta}. \tag{40}$$

This series resembles the expansion obtained by Schwinger[15] for the imaginary part of the Lagrangian of a constant electric field in the one-loop approximation:

$$\text{Im}\, \mathscr{L}^{(1)} = \frac{\beta^2 m^4}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\pi n/\beta}. \tag{41}$$

Both series contain powers of the quasiclassical exponential $e^{-\pi/\beta}$. Nikishov[16] has shown that up to a constant factor, the quantity $\beta^2 m^4 e^{-\pi/\beta}$ is precisely the probability of pair creation by an electric field per unit four-volume. Based on that result, Ritus[17] has carried out a detailed analysis of the expansion (41); in particular, he proved that the nth term of the series (41) corresponds to the coherent production of n pairs within the very same pair-creation four-volume. Consequently, by analogy with Ref. 17, we may interpret the nth term of the series (40) to be the probability of the coherent production of n pairs by a real photon in an external field.

3. EFFECT OF AN ELECTRIC FIELD ON THE THRESHOLD BEHAVIOR OF THE POLARIZATION OPERATOR

In vacuum, the functions π_1, π_2, and π_4 are equal to zero, and the function π_3 reduces to π_0. The imaginary part of the latter is given by the well known formula

$$\text{Im}\, \pi_0 = -\frac{\alpha m^2}{3} \theta(\lambda - 4)(2+\lambda)(1-4/\lambda)^{1/2}, \tag{42}$$

where $\theta(x)$ is the Heaviside step function. Near threshold ($\lambda = 4$), the behavior of the function Im π_0 is nontrivial. Corrections to Im π_0 that appear when an electric field is turned on are therefore of substantial interest. To calculate these corrections, we make use of Eq. (15.3). Instead of v, we use the variable z defined by Eq. (14); then

$$\pi_3 = -\frac{12\alpha m^2 \lambda \varkappa^{2/3}}{\pi} \left(J_0 + \frac{2}{3} \frac{\beta^2}{\varkappa^{4/3}} J_1 + \dots \right) + \pi_0, \tag{43.1}$$

where

$$J_0 = \int_{(b/2)^{2/3}}^{\infty} \frac{dz}{v^{7/4}(2v+\lambda)(v-4)^{1/2}} f_1(z), \tag{43.2}$$

$$J_1 = -\int_{(b/2)^{2/3}}^{\infty} \frac{dz}{v^{5/4}(2v+\lambda)(v-4)^{1/2}} \left[f'(z) - \frac{v+2}{30} f^{(4)}(z) \right]. \tag{43.3}$$

The latter two equations contain the parameter

$$b = (4-\lambda)^{3/2}/\varkappa. \tag{44}$$

For the sake of definiteness, we assume that arg $b = -\pi$ when $\lambda > 4$. This is consistent with the correct sign for Im π_0 in Eq. (42). The parameter $\varkappa$ may be expressed in terms of β and λ as

$$\varkappa = \beta(\lambda + k_\perp^2/m^2)^{1/2}. \tag{45}$$

For simplicity, we take $\lambda \geqslant 0$ from here on. Then $\varkappa$ will be real and nonnegative, so that the integration contour for J_0 and J_1 will lie on the real axis.

Consider first the case $|b| \gg 1$. If we have arg $b = 0$, then $z \gg 1$ holds over the entire domain of integration for J_0 and J_1. If, on the other hand, we have arg $b = -\pi$, the integration path can be rotated by $-\pi/2$, yielding $|z| \gg 1$. For $|b| \gg 1$, then, one can take advantage of the known asymptotic functions $f_1(z)$ and $f(z)$ as $|z| \to \infty$ (see Refs. 7, 8).

Assume now that the opposite is true, and $|b| \ll 1$ holds. In that event we may put $\lambda = 4$ in the expression for π_3. Furthermore, we shall assume $\varkappa \ll 1$. In the integrals for J_0 and J_1, we may then put $v = 4$, since we have $z_{\text{eff}} - 1$ there. The net result for either case is

$$\text{Im}\, \pi_3 = \text{Im}\, \pi_0 - \frac{\alpha m^2}{4} \left(\frac{3}{8+\lambda} \right)^{1/2} \lambda \varkappa^{1/3} \left[\Phi_0(b) + \frac{\beta^2}{\varkappa^{4/3}} \Phi_1(b) + \dots \right], \tag{46.1}$$

where

$$\Phi_0(b) = \begin{cases} \dfrac{1}{b^{2/3}} e^{-b/3} + \dots, & |b| \gg 1, \quad \arg b = 0, \\ -\dfrac{2}{|b|^{2/3}} \cos \dfrac{|b|}{3} + \dots, & |b| \gg 1, \quad \arg b = -\pi, \\ \dfrac{2^{4/3}\pi^{1/2}}{3^{7/6}\Gamma(1/6)} + \dots, & |b| \ll 1, \end{cases} \tag{46.2}$$

$$\Phi_1(b) = \begin{cases} \dfrac{1}{15} b e^{-b/3} + \dots, & |b| \gg 1, \quad \arg b = 0, \\ \dfrac{2}{15} |b| \sin \dfrac{|b|}{3} + \dots, & |b| \gg 1, \quad \arg b = -\pi, \\ \dfrac{14}{5 \cdot 3^{2/3}} + \dots, & |b| \ll 1. \end{cases} \tag{46.3}$$

For (46.1)–(46.3) to hold, it is necessary that the term proportional to β_2 be small compared to the leading term. Far from threshold, therefore, the parameter b must satisfy

$$1 \ll |b| \ll \varkappa^{4/3} \beta^{-4/3}. \tag{47}$$

The two-sided inequality (47) will hold for any $\lambda \neq 4$ if the field is weak enough and $k_\perp^2$ is large enough. Notice that the field correction to Im π_3 oscillates rapidly for $\lambda > 4$, but that the sign of the function Im π_3 remains unchanged as λ varies, since in the present approximation the second term in (46.1) is much smaller than the first. Thus, Im π_3 is always negative, as it must be by unitarity. Sufficient conditions for the validity of (46.1)–(46.3) near threshold are

$$|b| \ll 1, \quad \varkappa \ll 1. \tag{48}$$

Note that (48) contains no condition $\beta^2/\varkappa^{4/3} \ll 1$, since for $\lambda = 4$ we have $\varkappa \geqslant 2\beta$, so that in a weak field that condition is satisfied automatically.

Equations (46.1)–(46.3) provide a clear-cut demonstration of the effect of an electric field on pair production by a virtual photon. Clearly, when a field is turned on, the pair-production threshold is "smeared out." In the subthreshold regime, the magnitude of Im π_3—i.e., the probability of pair creation by a photon—is exponentially small if the field is weak, behavior that is typical of processes that are forbidden in the absence of a field.[8]

Above threshold, low-level oscillations contribute to the vacuum function Im π_0, the explanation being that when a pair is created the electron wave function leaving the creation region, where the process is unperturbed by the field, is nevertheless distorted by the field; thereupon, components of the wave function whose wave vectors point in different directions will interfere with one another. A detailed study of such interference effects appears in the last section of Ref. 19. The electric field exerts a particularly strong influence near threshold: there, the magnitude of Im π_3 is proportional to the cube root of the field strength.

4. EFFECTIVE CHARGE OF THE ELECTRON IN AN EXTERNAL FIELD

The photon Green's function in a field satisfies the Dyson equation

$$[k^2+\Pi(k,F)]G(k,F)=-i. \tag{49}$$

The most convenient method of solving this equation involves expanding $\pi_{\mu\nu}(k,F)$ in tensors constructed from the mutually orthogonal vector k_μ, L_μ, $\tilde{L}_\mu$, and G_μ:

$$\Pi_{\mu\nu}(k,F)=\Pi_1\frac{L_\mu L_\nu}{L^2}+\Pi_2\frac{\tilde{L}_\mu\tilde{L}_\nu}{\tilde{L}^2}+\Pi_3\frac{G_\mu G_\nu}{G^2}+\Pi_4\frac{L_\mu\tilde{L}_\nu+L_\nu\tilde{L}_\mu}{L\tilde{L}}, \tag{50.1}$$

where

$$\Pi_1=\pi_1+\frac{C^2L^2}{L^{\cdot 2}}\pi_2+\pi_3-\frac{2CL}{L^{\cdot}}\pi_4,$$
$$\Pi_2=\frac{\tilde{L}^2}{L^{\cdot 2}}\pi_2+\pi_3,\qquad \Pi_3=\pi_3,$$
$$\Pi_4=\frac{\tilde{L}}{L^{\cdot}}\left(-\frac{CL}{L^{\cdot}}\pi_2+\pi_4\right). \tag{50.2}$$

The transverse part of the Green's function shares the same tensor structure as the polarization operator:

$$G_{\mu\nu}(k,F)=g_1\frac{L_\mu L_\nu}{L^2}+g_2\frac{\tilde{L}_\mu\tilde{L}_\nu}{\tilde{L}^2}+g_3\frac{G_\mu G_\nu}{G^2}+g_4\frac{L_\mu\tilde{L}_\nu+L_\nu\tilde{L}_\mu}{L\tilde{L}}+g_l\frac{k_\mu k_\nu}{k^2}. \tag{51}$$

Substituting the expansions (50.1) and (51) into Eq. (49) and equating coefficients of like tensors on the two sides of the equation, we obtain

$$g_{1,2}=-i\frac{k^2+\Pi_{2,1}}{(k^2+\Pi_1)(k^2+\Pi_2)-\Pi_4^2},\qquad g_3=-\frac{i}{k^2+\Pi_3},$$
$$g_4=i\frac{\Pi_4}{(k^2+\Pi_1)(k^2+\Pi_2)-\Pi_4^2},\qquad g_l=-\frac{i}{k^2}. \tag{52}$$

The photon Green's function in a constant electromagnetic field was first derived by Batalin and Shabad.[10] The new representation derived here, however, is more compact suited to an investigation of the effective charge of the electron in an external field, upon which we now embark.

The asymptotic form of the transverse part of the Green's function as $k^2 \to 0$ is

$$G_{\mu\nu}^{tr}(k,F)=-\frac{i}{k^2(1+\pi_3'|_{k^2=0})}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+\ldots, \tag{53}$$

where we have written $\pi_3' = \partial\pi_3/\partial k^2$. We introduce the notation

$$\tilde{e}^2=e^2/(1+\pi_3'|_{k^2=0}). \tag{54}$$

Then as $k^2 \to 0$, the asymptotic form of the expression $e^2G_{\mu\nu}^{tr}(k,F)$ formally acquires the vacuum form describing a new "charge" $\tilde{e}$. The Green's function $\tilde{G}_{\mu\nu}(k,F)$ and polarization operator $\tilde{\Pi}_{\mu\nu}(k,F)$, which are obtained by replacing e^2 with $\tilde{e}^2$, satisfy Eq. (49) and have the same tensor structure as $G_{\mu\nu}(k,F)$ and $\Pi_{\mu\nu}(k,F)$, so the equations (52) still apply. Substituting those relations into the identity

$$e^2G_{\mu\nu}^{tr}(k,F)=\tilde{e}^2\tilde{G}_{\mu\nu}^{tr}(k,F), \tag{55}$$

we easily obtain the relationship between Π_i and $\tilde{\Pi}_i$ (we mark the "new" quantities with a tilde to distinguish them from the "old"):

$$\tilde{\Pi}_i=\frac{\tilde{e}^2}{e^2}(k^2+\Pi_i)-k^2\qquad (i=1,2,3),$$
$$\tilde{\Pi}_4=\frac{\tilde{e}^2}{e^2}\Pi_4. \tag{56}$$

The identity (55), viewed as an equation in the unknown quantities $\tilde{\Pi}_i$, therefore has a unique solution. The nontrivial fact is related to the renormalizability of quantum electrodynamics in the presence of an external field, and makes it possible to use the quantity $\tilde{e}$ as a new electrodynamic coupling constant.

Let us clarify the physical meaning of this quantity $\tilde{e}$. Consider the interaction between two well-separated electrons situated in an external field. The field potential due to an electron at rest at the point $\mathbf{x} = 0$ is given by

$$A_0(\mathbf{x})=ie\int\frac{d^3k}{(2\pi)^3}G_{00}(k,F)|_{k_0=0}e^{i\mathbf{kx}}. \tag{57}$$

The large separation between the electrons corresponds to the exchange of virtual photons with $\mathbf{k}^2 \to 0$. Making use the asymptotic behavior of the Green's function for $k_0 = 0$ and $k_0 \to 0$, we obtain an expression for the Fourier component of the interaction energy of the two electrons:

$$eA_0(\mathbf{k})=e^2/\mathbf{k}^2(1+\pi_3'|_{k^2=0,\,k_0=0}). \tag{58}$$

If we employ the notation

$$e_{\text{eff}}^2=e^2/(1+\pi_3'|_{k^2=0,k_0=0}), \tag{59}$$

Eq. (58) becomes identical to the Coulomb expression. it is therefore quite natural to call the quantity $e_{\rm eff}$ the effective charge of the electron in the field. Equating (54) and (59), we see that the effective charge is the same as the quantity $\tilde{e}$ taken at the point $k_0 = 0$. Thus, $\tilde{e}$ is the generalization of the concept of the effective charge of the electron in a field, so we refer to it from here on as the generalized charge.

We now proceed to calculate $\tilde{e}^3$ in strong electric, magnetic, and crossed fields. For the electric field, we obtain from (11.4)

$$\pi_3'|_{\lambda=0}=\frac{2\alpha}{\pi}\int\limits_0^\infty\frac{dy}{y^2}\int\limits_0^y dx\,x\left[\frac{\operatorname{sh}2(y-x)}{2\operatorname{sh}^2 y}e^{-i\varphi/\beta}\right.$$
$$\left.+\left(\frac{x}{y^2}-\frac{1}{y}\right)e^{-iv/\beta}\right]. \qquad (60)$$

In this equation we have used the variables defined in (29); β and φ are given by (16) and (33), respectively. The asymptotic behavior of $\pi'_{3|\lambda=0}$ for $\beta\gg 1$ and $\varkappa/\beta\lesssim 1$ may be calculated with no particular difficulty. The result for $\tilde{e}^2$ is then

$$\tilde{e}^2=e^2\bigg/\left[1-\frac{\alpha}{3\pi}\ln(-i\beta)+\ldots\right]. \qquad (61)$$

For the magnetic field case, we again use the variables (29) but with ε replaced by η. The analytic properties of the integrand enable us to rotate the integration contour with respect to x and y by $-\pi/2$. The resulting expression for $\pi'_3|_{\nu=0}$ differs from (60) in the replacement $\beta\to i\beta$, where in the present case

$$\beta=\frac{|e\eta|}{m^2}. \qquad (62)$$

For $\beta\gg 1$ and $\varkappa/\beta\lesssim 1$, the expression for $\tilde{e}^2$ thus takes the form

$$\tilde{e}^2=e^2\bigg/\left(1-\frac{\alpha}{3\pi}\ln\beta+\ldots\right). \qquad (63)$$

In a crossed field, we obtain the expression for $\pi'_3|_{\lambda=0}$ from Eq. (15.3) by putting $\beta=0$:

$$\pi_3'|_{\lambda=0}=\frac{4\alpha}{\pi}\int\limits_4^\infty\frac{dv}{v^2[v(v-4)]^{1/2}}f_1(z),\qquad z=(v/\varkappa)^{2/3}. \qquad (64)$$

For $\varkappa\gg 1$ we have $z\ll 1$, so we may take advantage of the known asymptotic behavior of $f_1(z)$ as $z\to 0$ (see Refs. 7, 8). The end result is

$$\tilde{e}^2=e^2\bigg/\left[1-\frac{\alpha}{3\pi}\ln(-i\varkappa)^{2/3}+\ldots\right]. \qquad (65)$$

The asymptotic form of $\tilde{e}^2$ for the electric and magnetic fields is independent of the photon momentum k_μ; it is therefore the same as the asymptotic form of the square of the effective charge. In the crossed-field case, on the other hand, for $k^2=0$ and $k_0=0$ the invariant $\varkappa$ vanishes, so the polarization operator does not depend on the field. Therefore, for crossed fields, the effective charge—in contrast to the generalized charge—is the same as the physical charge.

Equations (61), (63), and (65) demonstrate that the dependence of the generalized charge of the electron on the field (or more precisely, on $e\varepsilon$, $e\eta$, or $m^2\varkappa^{2/3}$ for electric, magnetic, and crossed fields, respectively) is similar to the dependence of the renormalization-invariant charge $\tilde{e}^2$—which is given by the exact photon Green's function in vacuum—on k^2 for $k^2\gg m^2$:

$$\tilde{e}^2=e^2\bigg/\left(1-\frac{\alpha}{3\pi}\ln\frac{k^2}{m^2}+\ldots\right). \qquad (66)$$

This remarkable property was previously known to apply to the effective charge determined by the exact Lagrangian of a strong field.[20,21] The asymptotic behavior of the effective charge derived in Refs. 20 and 21 is the same as given by (61) and (63). We thus have yet another confirmation of the fact that the leading terms of such asymptotic expansions do not depend on how the effective charge is determined. The result (65) for crossed fields supplements the picture sketched out in Refs. 20 and 21; it could not have been derived there, since the Lagrangian of crossed fields is identically zero.

It is quite interesting to compare the present results with the expression for the physical charge in terms of the bare charge of the electron e_0 and the square of the cutoff momentum Λ^2 (see Ref. 18):

$$e^2=e_0^2\bigg/\left(1+\frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{m^2}+\ldots\right), \qquad (67)$$

where $\alpha_0=e_0^2/4\pi$. Notice that the sign of the logarithmic term is the opposite of that in (67) for the generalized charge. This suggests that when an electron is located in a strong field, the charge screening due to the interaction of the electron with virtual photons having $|k^2|\lesssim e\varepsilon$, $e\eta$, or $m^2\varkappa^{2/3}$ in an electric, magnetic, or crossed fields, respectively, will disappear.

This stripping of an electron in a strong electric field has also be exhibited by a calculation of the mass shift.[22] Specifically, an electron accelerated by a strong electric field loses that part of its radiative mass produced by its interaction with virtual photons that have $|k^2|\lesssim e\varepsilon$. It remains unclear why there is no similar effect for the mass shift in strong magnetic or crossed fields.

The author is indebted to V. I. Ritus for his attention and support, and to A. I. Nikishov and A. E. Shabad for discussions and valuable comments.

[1]D. H. Constantinescu, Nucl. Phys. **B36**, 121 (1972).
[2]Wu-Yang Tsai, Phys. Rev. **D10**, 2699 (1974).
[3]R. A. Cover and G. Kalman, phys. Rev. Lett. **33**, 1113 (1974).
[4]A. E. Shabad, Ann. Phys. **90**, 166 (1975).
[5]L. F. Urrutia, Phys. Rev. **D17**, 1977 (1978).
[6]N. B. Narozhnyi, Zh. Eksp. Teor. Fiz. **55**, 714 (1968) [Sov. Phys. JETP **28**, 371 (1969)].
[7]V. I. Ritus, Ann. Phys. **69**, 555 (1972).
[8]V. I. Ritus, Trudy Fiz. Inst. Akad. Nauk SSSR **111** (1979).
[9]I. A. Batalin and A. E. Shabad, FIAN Preprint No. 10 (1971).
[10]I. A. Batalin and A. E. Shabad, Zh. Eksp. Teor. Fiz. **60**, 894 (1971) [Sov. Phys. JETP **33**, 483 (1971)].
[11]G. K. Artimovich, FIAN Preprint No. 158 (1987).
[12]N. B. Narozhnyi, Zh. Eksp. Teor. Fiz. **54**, 676 (1968) [Sov. Phys. JETP **27**, 360 (1968)].
[13]A. I. Nikishov, Zh. Eksp. Teor. Fiz. **59**, 1262 (1970) [Sov. Phys. JETP **59**, 690 (1971)].
[14]Yu. V. Sidirov, M. V. Fedoryuk, and M. I. Shabunin, *Lectures on the Theory of Functions of a Complex Variable* [in Russian], Nauka, Moscow (1982).
[15]J. Schwinger, Phys. Rev. **75**, 651 (1949).
[16]A. I. Nikishov, Zh. Eksp. Teor. Fiz. **57**, 1210 (1969) [Sov. Phys. JETP **30**, 360 (1970)].
[17]V. I. Ritus, Dokl. Akad. Nauk SSSR **275**, 611 (1984) [Sov. Phys. Dokl. **29**, 227 (1984)].
[18]V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electro-*

dynamics (2nd Ed), Pergamon, Oxford (1982).
[19]V. I. Ritus and A. I. Nikishov, Trudy Fiz. Inst. Akad. Nauk SSSR **168** (1986).
[20]V. I. Ritus, Zh. Eksp. Teor. Fiz. **69**, 1517 (1975) [Sov. Phys. JETP **42**, 774 (1975)].
[21]V. I. Ritus, Zh. Eksp. Teor. Fiz. **73**, 807 (1977) [Sov. Phys. JETP **46**, 423 (1977)].
[22]G. K. Artimovich and V. I. Ritus, Zh. Eksp. Teor. Fiz. **90**, 816 (1986) [Sov. Phys. JETP **63**, 476 (1986)].

Translated by Marc Damashek