Finite-size effects in conformal theories and asymptotic form of correlation functions in one-dimensional quantum systems

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(Submitted 9 January 1990)
Zh. Eksp. Teor. Fiz. **97**, 1453–1466 (May 1990)

Various vacuum expectation values in one-dimensional quantum systems are studied using finite-size effects. The scaling dimensions of the principal operators are found in a continuum model of a one-dimensional Bose gas with a general form of interaction. In the model of an *XXZ* Heisenberg antiferromagnet the asymptotic forms of the correlators of certain nonlocal operators (disorder operators) are found. It is shown that finite-size effects can be used to find the spectrum of the scaling dimensions in fermion systems.

1. INTRODUCTION

An efficient method, involving the use of conformal invariance at a critical point, has recently appeared for the study of one-dimensional quantum systems.[1–6] In one-dimensional systems a phase transition occurs at zero temperature, and although the conformal symmetry here is exact only at large distances it is fully adequate for the determination of quantities that do not depend on the structure of the interaction on small scales. As a rule, it is these quantities that are of real interest in the theory of one-dimensional systems. Primarily, we have in mind correlation functions, i.e., vacuum expectation values of products of operators at different spatial points.

The principal properties of correlation functions of one-dimensional systems are associated with their power-law decrease at large distances (at zero temperature) and with the continuous dependence of the power exponents on the interaction constants.[7–9] These exponents are sometimes called critical indices, and their determination is one of the problems of the theory.

As is well known, in the two-dimensional case conformal symmetry imposes extremely stringent restrictions on the spectrum of the scaling dimensions of the operators of the theory.[10] It is this circumstance that gives the possibility of finding the critical indices of the correlation functions in one-dimensional quantum systems (i.e., in two-dimensional models of quantum field theory with one space and one time dimension). The central charge and the dimensions of the operators of the effective conformal theory that arises in the long-wavelength limit can be determined using the so-called finite-size effects.[1,2,11]

We recall the basic formulas. Let the conformal theory be specified on an infinite strip of width L in the spatial direction. Then to each conformal primary operator $\phi_{\Delta,\bar\Delta}$ there corresponds an infinite set ("tower") of eigenstates $|\phi_{k,\bar k}\rangle$ of the Hamiltonian, the energies

$$E_L{}^\varphi(k,\bar k)=E_L^{\text{vac}}+\frac{2\pi v}{L}(h+k+\bar k) \qquad (1a)$$

and momenta

$$P_L{}^\varphi(k,\bar k)=P_\infty{}^\varphi+\frac{2\pi}{L}(s+k-\bar k) \qquad (1b)$$

(with $\langle \text{vac}|\phi_{\Delta,\bar\Delta}|\text{vac}\rangle\neq 0$). Here, Δ and $\bar\Delta$ are the conformal dimensions of the operator ϕ, $h=\Delta+\bar\Delta$ and $s=\Delta-\bar\Delta$ are the scaling dimension and spin, respectively, $k,\bar k\geqslant 0$ are integers, E_L^{vac} is the ground-state energy, and p_∞^ϕ is the momentum of the lowest state $|\phi\rangle$ from this tower when the system has infinite length. Finally, the parameter v in (1a) takes into account the possible difference in the units of measurement of the spatial and temporal quantities; in other words, v is simply the velocity of acoustic excitations in the system (the group velocity on the Fermi surface). Thus, to determine the spectrum of dimensions of the primary operators it is sufficient to find the energies $E_L^\phi(0,0)\equiv E_L^\phi$ of the lowest excitation of each tower to order L^{-1}:

$$E_L{}^\varphi-E_L^{\text{vac}}=2\pi v h L^{-1}. \qquad (2)$$

We note that states with $k,\bar k\neq 0$ from the tower corresponding to the primary operator ϕ correspond to the so-called "descendent" operators of ϕ (Ref. 10). They have a well-defined scaling (but not conformal) dimension $h+k+\bar k$ and, as we shall see below, also give a contribution to the correlation functions.

As is well known,[1,2] the central charge of the Virasoro algebra that arises is related to the volume correction $\sim L^{-1}$ to the ground-state energy of the system. For example, in the case of periodic boundary conditions,

$$E_L^{\text{vac}}=\varepsilon_0 L-\frac{\pi c v}{6L}. \qquad (3a)$$

Here ε_0 is the ground-state energy density in infinite volume. The first term in (3) depends on the way in which the theory is regularized, and the second term is universal.

Often it is more convenient to use another version of this formula. By virtue of the conformal invariance, the determination of the corrections to the energy in powers of L^{-1} at zero temperature ($T=0$) is equivalent to the determination of the temperature corrections to the free energy for $L=\infty$. Then (3a) can be rewritten in the form

$$f(T)=\varepsilon_0-\frac{\pi c T^2}{6v}, \qquad (3b)$$

where $f(T)$ is the free-energy density at temperature T.

The long-wavelength asymptotic form of the pair (equal-time) correlator of the fields ϕ is

$$\langle\phi(x)\phi(0)\rangle\propto x^{-2h}\cos(P_\infty{}^\varphi x). \qquad (4)$$

The oscillatory factor $\cos(P_\infty^\phi x)$ arises because of the presence of the gap P_∞^ϕ in the spectrum of the momentum operator. Henceforth in this paper, for simplicity, we consider

 0038-5646/90/050820-08$03.00

only equal-time correlators, the asymptotic form of which is determined by the scaling dimension h of the field ϕ. Unequal-time correlators depend also on the spin s of the operator ϕ.

In Refs. 3–6 Eqs. (1)–(3) were used to determine critical indices in one-dimensional exactly solvable models. In this case the energies of the lowest excited states can be found exactly by means of the Bethe method.[12] An already extensive literature has been devoted to these questions (see, e.g., Refs. 1–6 and 13–20). However, in all these papers the possibilities of the new method were not exploited in full measure, and certain important aspects have not been reflected in the literature. First, the method described turns out to be applicable to models with interaction of general form, making it possible to obtain results of practically the same degree of completeness as in exactly solvable models. Second, the results obtained previously are valid only for Bose systems, and in systems of Fermi particles the critical indices differ, generally speaking, from boson critical indices. Finally, it is found that, by using finite-size effects, it is possible to find also the vacuum expectation values of certain nonlocal operators. It is to the study of these questions that the present paper is devoted.

We shall consider two basic types of one-dimensional system: a continuum model of a spinless Fermi or Bose gas with interaction of general form, and a lattice model of a Heisenberg XXZ antiferromagnet—a model which, for brevity, we shall sometimes call simply a spin chain. The second-quantized Hamiltonian of the first model has the form (in units in which the particle mass is equal to 1/2)

$$\hat{H}=\int_0^L dx\,\partial_x\psi^*(x)\,\partial_x\psi(x)+{}^1/_2 g \times \iint_0^L dx\,dy\,\psi^*(x)\psi^*(y)V(x-y)\psi(x)\psi(y), \tag{5a}$$

where L is the length of the system, $V(x)$ is a certain pair-interaction (pair-repulsion) potential of quite general form, and $g>0$ is the coupling constant. The operators ψ^*, ψ satisfy standard equal-time (anti) commutation relations. Sometimes, in order to indicate the type of statistics explicitly, we shall write ψ_B or ψ_F. The number N of particles in the system is conserved; in the thermodynamic limit $N\to\infty$, $L\to\infty$ the quantity $\rho=N/L$ is the equilibrium density. We note that for $V(x)=\delta(x)$ there exists an exact solution of this model.[12] In the language of first quantization the Hamiltonian (5) has the form

$$\hat{H}=\sum_{i=1}^N\frac{\partial^2}{\partial x_i{}^2}+g\sum_{i<j}^N V(x_i-x_j). \tag{5b}$$

We shall also use the first-quantization representation occassionally for reasons of convenience.

The Hamiltonian of the spin chain has the form

$$\hat{H}={}^1/_2\sum_{x=1}^L(\sigma_x^1\sigma_{x+1}^1+\sigma_x^2\sigma_{x+1}^2+\cos\gamma\,\sigma_x^3\sigma_{x+1}^3),\quad 0\leqslant\gamma<\pi. \tag{6}$$

Here, L is the number of lattice sites, σ^j are the Pauli matrices, and γ is the anisotropy parameter. For $\gamma=0$ we have isotropic (XXX) Heisenberg antiferromagnet. The XXZ antiferromagnet (6) admits an exact solution: All the eigenstates of the Hamiltonian can be constructed in explicit form by the Bethe method.[12] The simplest eigenstate, in which all the spins point in the same direction (i.e., the total spin is equal to $L/2$), is the "bare" (nonphysical) vacuum. The physical vacuum of the antiferromagnet has the minimum possible total spin (0 or $\frac{1}{2}$, depending on the parity of the number L of sites) and is obtained by filling the bare vacuum. Here, the number N of reversed spins plays the role of the number of particles in the model (5).

The models described have many physical properties in common, and are solved by similar methods. In particular, in both cases the low-energy spectrum characterized by the absence of a gap and by the presence of only one sound velocity v. The calculations of the central charge from formula (3) give $c=1$ (Refs. 1–3), i.e., both systems fall in the universality class of the Gaussian model.[21] This corresponds to the fact that the lowest excitations are phonons describable by the free theory.

In systems of particles with internal degrees of freedom there are several branches of gapless excitations, with, generally speaking, different sound velocities. The analysis of this more complicated case lies outside the scope of the present paper.

We shall say a few words about the long-wavelength approximation that we are considering. The characteristic distances x_c over which, in a Bose or Fermi gas, individual particles can be "felt" is of the order of $L/N=1/\rho$. Over large distances $x\gg x_c$ a certain effective theory, describing phonon excitations, arises. In this case all information about short distances is contained in the constant v. The long-wavelength excitations, corresponding to a linear dispersion law (small energies ε), lead to effective asymptotic scale and conformal invariance. At the same time, this implies that deviations in Eqs. (1)–(3) from the exact conformal theory can appear only for small values of L, i.e., series in inverse powers of L in these formulas can be used effectively.

We not briefly describe the content of the article. In Sec. 2 we calculate the energies of the lowest excitations in finite volume in the Bose-gas model (5) and obtain the spectrum of scaling dimensions. In Sec. 3 it is shown how, on the basis of these data, the asymptotic series for the correlation functions is obtained. Very important examples (the pair correlator of the densities and the one-particle density matrix) are analyzed in detail. It is found that the theory of finite-size effects has a natural extension that makes it possible to find the vacuum expectation values of certain nonlocal operators; Sec. 4 is devoted to these questions. The account is given principally for the example of the spin chain (6), for which certain previously unknown correlators, of interest in their own right, are found. Finally, in Sec. 5 the case of fermion statistics is considered and the critical indices in the Fermi-gas model (5) are found.

A brief account of some of the results of this paper was published in Ref. 22.

2. SCALING DIMENSIONS IN THE BOSE-GAS MODEL

In this section we find the scaling dimensions of the operators in a Bose system described by the Hamiltonian (5). As is clear from the Introduction, for this it is necessary to classify those low-lying excitations which become gapless as $L \to \infty$. We shall assume that periodic boundary conditions are imposed, i.e., the particles are on a circle of length L.

The excitation spectrum of the system (5) is depicted schematically in Fig. 1. Universal characteristics of the spectrum are the absence of an energy gap at small momenta and the vanishing of the energy at momenta that are multiples of $2\pi\rho$. These properties hold for a wide class of potentials $V(x)$, both long-range and short-range. We must consider this spectrum for large but finite values of L. In this case it becomes quasi-discrete, i.e., consists of individual closely spaced points (Fig. 2). In particular, an energy gap of order L^{-1} appears. We are interested in those states which possess zero energy in the limit $L \to \infty$. From these it is necessary to choose one state with the minimum energy for each branch of excitation. These will be the states $|\phi\rangle$ corresponding to the primary operators. Simplifying slightly, one can say that to each gapless branch in the excitation spectrum corresponds its own primary operator. Points in the spectrum that are neighbors of $|\phi\rangle$ are states from the corresponding tower with $k > 0$ or $\bar{k} > 0$ in (1a).

The energies E_L^ϕ of the states $|\phi\rangle$ can be found by direct calculation if a method of constructing the physical vacuum is explicitly known, as it is, e.g., in the model with δ-function interaction or in other exactly solvable models. This was done in Refs. 3–6. In the general case it turns out that the states $|\phi\rangle$ correspond to distinct points of the spectrum, the energies of which can be found from thermodynamic considerations without detailed knowledge of the structure of the vacuum.

We shall demonstrate this using the system (5) as an example. We confine ourselves first to excitations for which the number of particles is conserved. The simplest excitation satisfying the properties listed above is the creation of one phonon with the minimum possible momentum $\pm 2\pi/L$ (the points A_0, $\bar{A}_0$ in Fig. 2). The energy, obviously, is equal to $2\pi v/L$, and there is no gap in the momentum spectrum: $P_\infty^\phi = 0$. Form (1a) and (1b) we immediately obtain $h = \pm s = 1$. We denote the corresponding primary operators by $\phi_\pm^0$. Thus,

$$\langle \phi_+{}^0(x)\phi_+{}^0(0)\rangle = \langle \phi_-{}^0(x)\phi_-{}^0(0)\rangle \propto x^{-2},$$
$$\langle \phi_+{}^0(x)\phi_-{}^0(0)\rangle = \langle \phi_-{}^0(x)\phi_+{}^0(0)\rangle = 0. \tag{7}$$

More interesting is the excitation with momentum equal to $2\pi\rho$ (the point A_1). Here the gap in the momentum spectrum is equal to $2\pi\rho$. Physically, this corresponds to "rotation" of the entire system as a whole with the smallest possible angular momentum (the first level of the rotator). In other words, this state is obtained from the vacuum by going over a uniformly moving reference frame. Because of the periodic boundary conditions, the velocity of this reference frame is quantized. We thereby obtain an entire family of states $|\phi_{0,m}\rangle$ (m is an integer) satisfying the necessary conditions (the points A_2 etc. in Fig. 2). Their momenta are multiples of $2\pi\rho$: $P_{0,m} = 2\pi m\rho$, and the energies are found in an elementary manner by considering the motion of the system as a whole (we recall that in our units the mass of each particle is equal to 1/2):

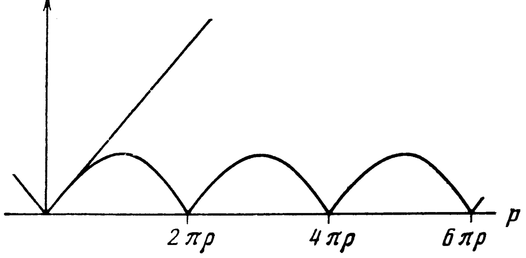

FIG. 1. Excitation spectrum of a one-dimensional spinless Bose gas.

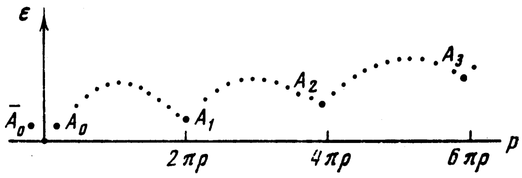

FIG. 2. Spectrum of the same system in a finite volume.

$$\delta E_{0,m} = N(2\pi m/L)^2 = \frac{2\pi v}{L}\frac{2\pi\rho m^2}{v}. \tag{8}$$

From this we calculate the dimensions of the operators $\phi_{0,m}$:

$$h_{0,m} = 2\pi\rho v^{-1} m^2, \quad m \in \mathbf{Z}; \quad s = 0. \tag{9}$$

It should be noted that this way of arguing is not fully consistent, since it does not take into account the quantum nature of the ground state.

The same result can be obtained more rigorously as follows. We consider a change to another inertial reference frame (for reasons of convenience, here we shall use the language of first quantization). The complete N-particle wave function then transforms in accordance with the rule

$$\Psi(\{x_i\}) \to \tilde{\Psi}(\{x_i\}) = \exp\Big(iq\sum_{k=1}^{N} x_k\Big)\Psi(\{x_i\}),$$

and, from the condition that the wave function be unique, we obtain a discrete set of momenta $q = 2\pi m/L$. By acting on $\tilde{\Psi}$ with the Schrödinger operator (5b) we obtain (8).

The states $|\phi_\pm^0\rangle$ and $|\phi_{0,m}\rangle$ are excitations in the sector with a constant number of particles. Besides these there are also excitations that arise when particles are added to the system. As usual, when working with a variable number of particles it is necessary to modify the Hamiltonian: $\hat{H} \to \hat{H} - \mu_0\hat{N}$, where

$$\mu_0 = \lim_{L\to\infty} \frac{\partial E_L^{vac}}{\partial N}\Big|_{\rho=\mathrm{const}}$$

is the chemical potential and $\hat{N}$ is the particle-number operator. In the case of boson statistics the addition of n paticles to the system leads to an energy shift

$$\delta E_{n,0}^B = {}^1/_2 n^2(\partial E_L^{vac}/\partial N^2) = \frac{n^2}{2L}(\partial^2\varepsilon_0/\partial\rho^2) = \frac{2\pi v}{L}\frac{vn^2}{8\pi\rho}. \tag{10}$$

Here we have used the well known thermodynamic relation

$$v^2=2\rho(\partial^2\varepsilon_0/\partial\rho^2).$$

We denote the corresponding primary fields by $\phi_{n,0}$; we find their dimensions from (10);

$$h_{n,0}=\frac{vn^2}{8\pi\rho}. \tag{11}$$

Finally, we can construct all possible combinations of the excitations described, i.e., add n particles and then "rotate" the system as a whole on the mth level. Corresponding to this are operators $\phi_{n,m}$ with dimensions

$$h_{n,m}=\frac{n^2}{R^2}+\frac{m^2R^2}{4},\qquad n, m\in\mathbf{Z}, \tag{12}$$

where we have denoted

$$R^2=8\pi\rho v^{-1}. \tag{13}$$

For the case $V(x)=\delta(x)$ the dimensions $h_{0,1}$ and $h_{1,0}$ were first found in Ref. 3, in which they were expressed in terms specific for integrable systems. It can be shown that the dimensions from Ref. 3 coincide with ours if we express them in terms of the sound velocity. In our opinion, (12)–(13) is a preferable form of writing the result, since these expressions have meaning for arbitrary potentials and all the information about the potential is collected in the single parameter v (or R).

For completeness we give the corresponding results in the case of the antiferromagnet.[3,14] Here too there are primary operators $\phi^0_\pm$ with dimension 1 and operators $\phi_{n,m}$ with dimensions determined by Eq. (12) with

$$R^2=2\pi(\pi-\gamma)^{-1}. \tag{14}$$

In exactly solvable models the central charge can be calculated using Eq. (3b). For $V(x)=\delta(x)$ in (5), and for the XXZ magnet (6), one obtains $c=1$ (Ref. 3). From the results of Ref. 23 it follows that the central charge also has the same value for the potential $V(x)=x^2$ (the Sutherland model). Unfortunately, we do not know how to find $f(T)$ for small T for potentials of general form. Nevertheless, the form of the spectrum of dimensions (12) makes it possible to assume $c=1$ for a wide class of potentials. As already noted, this is entirely natural, since a theory with $c=1$ describes a phonon system in the long-wavelength limit. As is well known, almost all conformal theories with $c=1$ are equivalent to Gaussian models[24] with a spectrum of dimensions of the form (12) and can be characterized by a single continuous parameter, on which the conformal dimensions depend.

More specifically, the Gaussian model is the two-dimensional free theory of a massless scalar field $\varphi(z,\bar z)$ that takes values in a circle of radius R (i.e., it is necessary to identify φ and $\varphi+2\pi R$). The action has the form (here, $z=x+iy$, $\bar z=x-iy$)

$$S_R=\frac{1}{2\pi}\int d^2z\,\partial_z\varphi\,\partial_{\bar z}\varphi \tag{15}$$

and possesses obvious $U(1)$ invariance.

The spectrum of dimensions of the Gaussian model is given[21,24] by Eq. (12); from this the meaning of the introduction of the parameter R becomes clear (in string theory it is called the compactification radius). The operator ϕ^0_+ (ϕ^0_-) can be identified with the (anti) chiral $U(l)$ current $\partial_z\varphi$ ($\partial_{\bar z}\varphi$), and $\phi_{n,m}$ can be identified with the following exponential of the free fields:

$$\phi_{n,m}\to:\exp\{ip\varphi+i\bar p\bar\varphi\}:, \tag{16}$$

where :...: implies suitably defined normal ordering and the conformal dimensions are determined from the relations

$$(\Delta,\bar\Delta)=(p^2/2,\bar p^2/2),\quad h=\Delta+\bar\Delta,\quad s=\Delta-\bar\Delta, \tag{17}$$

$$(p,\bar p)=(nR^{-1}+mR/2,\ nR^{-1}-mR/2),\quad n, m\in\mathbf{Z}. \tag{18}$$

We note that the spectrum (18) is invariant under the "dual" transformation $R\to 2/R$, and $R=2^{1/2}$ is the self-duality point, corresponding to an isotropic antiferromagnet.

3. ASYMPTOTIC FORM OF THE CORRELATION FUNCTIONS

In this section we show how, using the results of Sec. 2, one can obtain asymptotic series for the most important correlation functions in the model (5a)—the pair correlator $H(x=\langle\rho(x)\rho(0)\rangle$ of the densities [here, $\rho(x)=\psi^*(x)\psi(x)$ is the density operator], and the one-particle density matrix $S(x)=\langle\psi^*(x)\psi(0)\rangle$. In the case of the spin chain their analogs are the correlators $H(x)=\langle\sigma^3_x\sigma^3_1\rangle$ and $S(x)=\langle\sigma^+_x\sigma^-_1\rangle$, where $\sigma^\pm=\sigma^1\pm i\sigma^2$.

The operators $\psi(x)$ and $\rho(x)$ themselves do not possess well defined conformal dimensions, but, being local operators, they should be represented in the form of a linear combination of primary operators and their descendants. We shall write this expansion for an arbitrary local operator $\hat O(x)$ (composed, e.g., of ρ, ψ, ψ^*,...) in general form. In order not to encumber the formulas, we now write this expansion somewhat symbolically, without inclusion of the descendents, and discuss the role of the latter separately later. In particular, it will be seen that the "descendants" do not make a contribution to the leading terms of the asymptotic form, so that if we are interested only in the leading terms we make direct use of the expansion in the primary operators. We have

$$\hat O(x)=\sum_\phi C_\phi\exp(iP_\infty{}^\varphi x)\,\phi(x), \tag{19}$$

where the sum over ϕ denotes a sum over the primary operators and the C_ϕ are certain numerical coefficients. The presence of the gap in the spectrum of momenta has been taken into account explicitly in this formula. In the calculation of correlators of the type $\langle\hat O(x)\hat O(y)\rangle$ the products of primary operators ϕ are averaged using the familiar rules of conformal theory.[10,25] In practical calculations it is convenient to make use of the representation of the primary fields in the form of the exponentials (16) and to average them by means of the standard technique of functional integration with the quadratic action (15).

The coefficients C_ϕ in (19) are nonzero only for the operators ϕ that satisfy the following selection rule: $C_\phi\neq 0$ provided that $\langle\text{vac}|\hat O(x)|\hat\phi\rangle\neq 0$ is the thermodynamic limit (here, $\langle\text{vac}|$ is the physical vacuum; for brevity, averaging over the physical vacuum is sometimes denoted simply by angular brackets). For example, using the results of Sec. 2 we obtain in the case of boson statistics

$$\psi_B(x)=\sum_{m=-\infty}^{\infty}c_m\exp(2\pi imx)\,\phi_{1,m}(x), \tag{20a}$$

$$\psi_B^*(x)=\sum_{m=-\infty}^{\infty} c_m^* \exp(-2\pi imx)\,\phi_{-1,-m}(x), \qquad (20b)$$

$$\rho(x)=\phi_+^0(x)+\phi_-^0(x)+\sum_{m=-\infty}^{\infty} c_m' \exp(2\pi imx)\,\phi_{0,m}(x). \qquad (21)$$

In the latter formula $\phi_{0,0}$ denotes the unit operator. The coefficients in (20)–(21) possess, of course, the symmetry

$$c_m=c_{-m},\quad c_m^*=c_{-m}^*,\quad c_m'=c_{-m}'.$$

By means of these expansions it is not difficult to find all the terms of the asymptotic form of the correlators $H(x)$ and $S(x)$, and also their many-point analogs. We note that in the calculation of $H(x)$ terms $\langle\phi_+^0(x)\phi_+^0(0)\rangle = \langle\phi_-^0(x)\phi_-^0(0)\rangle \sim x^{-2}$ arise, the exponent of which corresponds to the canonical dimension of the operator ρ and does not depend on the coupling constant.

We now discuss what is given by taking the descendants into account. Generally speaking, in the expansion (19) we should include all the descendant operators of each primary operator ϕ that satisfies the selection rule given above. According to the general relations of conformal field theory, the replacement of the operator $\phi(x)$ in the correlator $\langle\phi(x)\phi(0)\rangle$ by any particular one of its "descendants" is equivalent to differentiating this correlator with respect to x a certain number of times. Thus, the inclusion of descendants adds to the asymptotic form terms with exponents exceeding the exponents of the leading terms by a natural number. For example, the descendants $\phi_{0,1}$ of the first level make a contribution proportional to

$$x^{-1/2R^2-1}\sin(2\pi\rho x)$$

to $H(x)$. Taking these considerations into account, we write out the resulting asymptotic series ($x \gg \rho^{-1}$):

$$H(x)-\rho^2=\sum_{k=0}^{\infty} A_k x^{-2-k} + \sum_{m=1}^{\infty} x^{-1/2 m^2R^2}\Big[\sum_{k=0}^{\infty} A_{m,k}\cos\Big(2\pi m\rho x+\frac{\pi k}{2}\Big)x^{-k}\Big], \qquad (22)$$

$$S(x)=\sum_{m=0}^{\infty} x^{-2/R^2-1/2 m^2R^2}\Big[\sum_{k=0}^{\infty} B_{m,k}\cos\Big(2\pi m\rho x+\frac{\pi k}{2}\Big)x^{-k}\Big]. \qquad (23)$$

Here, A and B are certain numerical coefficients.[2)]

Expressions (22) and (23) agree with the known exact results. We have in mind the calculation of the density matrix in a system of impenetrable bosons [$V(x)=\delta(x)$, $g=\infty$ in (5)][26,27] and one exact result for the correlator $H(x)$ in the so-called Sutherland model [$V(x)=x^{-2}$ in (5)] with the special coupling-constant value $g=24$ (Ref. 28). In these cases the asymptotic form for the correlators does indeed have the same structure as in (22) and (23).

It is possible also to compare these expressions with the results of Refs. 29 and 30, in which the asymptotic forms of $H(x)$ and $S(x)$ were found by direct calculations with the assumption that the potential $V(x)$ is long-range and the coupling constant g is large. Here, for the parameter R, with all orders of perturbation theory in g^{-1} taken into account, the same expression (13) was obtained. For large g, however, R is a small parameter. This is the reason why only the "primary" terms of the asymptotic form [i.e., the terms with $k=0$ in (22) and (23)] are present in the formulas of Refs. 29 and 30. The other terms cannot be seen against the background of the slowly decaying terms proportional to $x^{-m^2R^2/2}$. The term proportional to x^{-2} in (22) cannot be obtained by the method of Ref. 29 for the same reason. At the same time, in certain models, e.g., in a magnet with $\pi/2<\gamma<\pi$, this term is the leading term.

Here we should like to discuss in a little more detail the meaning of the series (22)–(23) with unknown coefficients. It would appear that a function of highly arbitrary form can be expanded in such a series. Nevertheless, important information is contained in the expansion (22)–(23).

First, in all cases when these expansions are obtained by other methods (see above), the coefficients A and B fall off rapidly and the series converge very well. Moreover, calculations of the correlators by other methods usually reproduce not the full answer but the first terms of the indicated expansions, so that these series appear to be adequate to the problem.

Second, the subseries that correspond to the contribution of the descendants [the sums over k in (22)–(23)], which give powers of x differing by an integar, can be "folded" into a smooth function. At the same time, the sum over the primary operators contains fractional powers of x. For an irrational value of R^2 the number of different noninteger powers is infinite, and this evidently implies the presence of the essential singularity. On the other hand, for a rational value of R^2 in (22)–(23) there is only a finite number of types of branching. This case corresponds to the rational conformal theories,[31] and we thus see a manifestation of the finiteness of the number of fields that are primary with respect to a large chiral algebra.[32]

The natural question arises: What distinguishes the models that are describable in the long-wavelength limit by rational conformal theories? Our hypothesis is that the complete wave function of the ground state in such models is analytic in each variable on a certain finite-sheeted cover of the complex plane (possibly with an excluded point ∞). By making use of the results of Ref. 28 it is easy to check the validity of this hypothesis in the case of the Sutherland model.[3)]

Thus, we have shown that the correlation properties of boson spinless systems are described by the Gaussian model with a suitable "compactification radius" R. For the continuum Bose-gas model R is determined by Eq. (13), while for the Heisenberg antiferromagnet it is determined by Eq. (14).

4. VACUUM EXPECTATION VALUES OF NONLOCAL OPERATORS

In this section we shall study mainly the antiferromagnetic model. We consider the following nonlocal operators in this model:

$$S_{xy}\equiv\prod_{j=x}^{y}\sigma_j^3=\exp\{i\pi q(x,y)\},\qquad (24)$$

where $q(x,y)$ is the operator of the number of reversed spins on the sites from x to y, and

$$T_{x,y}=P_{x,\,x+1}P_{x+1,\,x+2}\ldots P_{y-1,\,y}P_{yx},\qquad (25)$$

where $P_{xy}\equiv\frac{1}{2}(1+\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_y)$ is the permutation operator that interchanges the spins at sites x and y. The operator T_{xy} is the operator of cyclic permutation on the sites from x to y: $x\to x+1, x+1\to x+2, ..., y-1\to y, y\to x$. The determination of the asymptotic forms of the expectation values $\langle\text{vac}|S_{xy}|\text{vac}\rangle$, $\langle\text{vac}|T_{xy}|\text{vac}\rangle$, and $\langle\text{vac}|S_{xy}T_{xy}|\text{vac}\rangle$ in the antiferromagnetic vacuum for $|x-y|\gg 1$ is a component part of the problem of the correlation functions in a system of spin-$\frac{1}{2}$ particles.[35] In addition, these nonlocal expectation values are of interest from the point of view of the quantum method of the inverse problem.[36] By virtue of the translational invariance these expectation values depend only on $|x-y|$.

We shall show that these correlators have a power-law asymptotic form, and that the exponents can be found by a natural generalization of the method described in the Introduction. The idea is as follows. As shown in Ref. 21, in the Gaussian model there is another, nonlocal sector, the spectrum of the primary operators of which is specified as before by Eq. (12), but now with half-integer n and m. The corresponding operators (e.g., $\phi_{0,\frac{1}{2}}$) are the so-called disorder operators, which are expressed in a nonlocal manner in terms of the field φ appearing in the action (15). Application of the operator $\phi_{0,\frac{1}{2}}$ to the ground state of this extended Gaussian model with periodic boundary conditions carries it over into a certain state in the sector with antiperiodic boundary conditions. Therefore, the correlators of an even number of disorder operators have meaning in the original Gaussian model with periodic conditions. From the point of view of the original model, a pair of disorder operators at two points x and y looks like a nonlocal operator acting on the segment from x to y. It is this circumstance that gives the possibility of finding the expectation values of the operators (24) and (25) by the methods of conformal field theory.

We need to consider the Hilbert space that incorporates simultaneously all the states of the spin chain with different numbers of sites and different boundary conditions. We introduce the following operators, acting in this extended space: $a_{x\alpha}^+$ ($\alpha=\pm\frac{1}{2}$), whcih creates a site with spin α between sites of the original chain with the site labels x and $x+1$, and $b_{x\alpha'}$, which annihilates the ith site (with spin β_x) in the case $\beta_x=\alpha$ and gives zero when acting on states for which $\beta_x=-\alpha$. Thus, the operators $a_{x\alpha}^+$ act from the sector with L sites into the sector with $L+1$ sites, while the operators $b_{x\alpha}$, on the contrary, decrease the number of sites by unity. The operator $b_{\pm 1}$ can be identified with the nonlocal operator $\phi_{\pm\frac{1}{2},0}$ in the extended Gaussian model. We introduce also the operator

$$S_x=\prod_{j=x}^{L}\sigma_j^3,\qquad (26)$$

which, as is easily seen, relate the sectors with periodic and with antiperiodic boundary conditions.

In fact, we shall consider the complete wave function $\psi(x_1,..., x_M)$ of the ground state of the magnet, the arguments of which are the coordinates of the reversed spins. For periodic boundary conditions we have $\psi(x_1)=\psi(x_1+L)$, where for brevity we have explicitly indicated only the dependence on x_1. We set $\tilde\psi=S_x^{\psi}$. To find $\tilde\psi(x_1+L)$ we fix the positions of all the reversed spins except the first. Suppose first that $x_1<x$, and that to the right of the site x there are k reversed spins. Then

$$\tilde\Psi(x_1)=S_x\tilde\Psi(x_1)=(-1)^k\Psi(x_1).$$

For $x_1>x$ we have the obvious chain of equalities

$$\tilde\Psi(x_1+L)=S_x\Psi(x_1+L)=(-1)^{k+1}\Psi(x_1+L)=(-1)^{k+1}\Psi(x_1)=-S_x\Psi(x_1)=-\tilde\Psi(x_1),$$

i.e., $\tilde\psi$ belongs to the sector with antiperiodic conditions.

Obviously,

$$T_{xy}=\sum_{s}a_{xs}^{+}b_{ys},\qquad S_{xy}=S_xS_y.\qquad (27)$$

Thus, it is necessary to calculate the correlators $\langle\text{vac}|T_{xy}|\text{vac}\rangle=2\langle a_{x,+1}^+,b_{y,+1}\rangle$ (since $\langle a_{x,+1}^+b_{y,+1}\rangle=\langle a_{x,-1}^+b_{y,-1}\rangle$ and $\langle\text{vac}|S_{xy}|\text{vac}\rangle=\langle\text{vac}|S_xS_y|\text{vac}\rangle$. We denote the vacuum of a chian of L sites with periodic (antiperiodic) boundary conditions by $|L,+\rangle$ ($|L,-\rangle$) (in the thermodynamic limit these states coincide, but now the corrections in L^{-1} are important to us). In the previous notation, $|L,+\rangle=|\text{vac}\rangle$. Working with the extended Hilbert space, we can (and shall) regard the ground states in sectors with another L and with antiperiodic conditions as excitations above the state $|\text{vac}\rangle$. Informally speaking, the operator $a_{x\alpha}^+$ creates an excitation with spin $\frac{1}{2}$ ("half" a magnon). For even L, only an even number of such excitations can exist, while for odd L only an odd number can exist, the ground state being doubly degenerate in the latter case.

We have $\langle L-1,+|b_{x\alpha}|L,+\rangle\neq 0$, $\langle L+1,+|a_{x\alpha}^+|L,+\rangle\neq 0$, and $\langle L,-|S_x|L,+\rangle\neq 0$ in the thermodynamic limit; i.e., the states $|L\pm 1,\pm\rangle$ satisfy the selection rule from Sec. 3. Therefore, to find the asymptotic forms of the expectation values of the operators (27) we can make use of formula (4), finding the dimensions of the operators $a_{x\alpha}^+, b_{x\alpha}$, and S_x from (2). For this it is necessary to calculate the shift of the energy of the states $|L+1,+\rangle$ and $|L,-\rangle$ in comparison with $|L,+\rangle$, and also their momenta. This can be done with the aid of the well-known exact solution (the Bethe ansatz).[12,14] Here it is necessary to modify the Hamiltonian: $\hat H_{XXZ}\to\hat H_{XXZ}-\varepsilon_0$, where ε_0 is the mean energy per site. Without giving these calculations here, we note only that the parameters v in (2) in the case of the magnet is expressed in terms of γ as follows: $v=\pi\gamma^{-1}\sin\gamma$ (Ref. 14).

The results have the form

$$\langle vac|\exp\{i\pi q(x,y)\}|vac\rangle\propto\cos[\pi(x-y)/2]\,|x-y|^{-\lambda},\qquad (28)$$

$$\langle vac|T_{xy}|vac\rangle\propto|x-y|^{-\mu},\qquad (29)$$

$$\langle vac|T_{xy}\exp\{i\pi q(x,y)\}|vac\rangle\propto\cos[\pi(x-y)/2]\,|x-y|^{-\lambda-\mu},\qquad (30)$$

where

$$\mu=R^{-2}/2,\quad \lambda=R^2/8\qquad (31)$$

and R is determined from (14). The complete spectrum of dimensions (with the nonlocal operators taken into account) is given by Eq. (12), with half-integer n and m.

We note that in the isotropic case of the XXX magnet the operator S_x coincides with the spin field of Ref. 24 and has scaling dimension 1/8.

To conclude this section we shall say a few words about nonlocal operators in the Bose-gas model. The analog of the operator S_x is given by the same formula

$$S_{xy}=\exp\,[i\pi q(x,\,y)],$$

where now $q(x,y)$ is the operator of the number of particles on the segment from x to y. For $|x-y|\gg\rho^{-1}$ we find, by means of the technique developed above,

$$\langle vac|S_{xy}|vac\rangle\propto\cos(\pi\rho|x-y|)\,|x-y|^{-R^2/8}. \quad (32)$$

It is convenient to introduce also the analog of the operator S_x:

$$S_x=\exp\,[i\pi q(x,\,L)], \quad (33)$$

which will be needed in the following section.

5. CRITICAL INDICES IN FERMION SYSTEMS

Up to now, when studying the model with the Hamiltonian (5), we had in mind the case of boson statistics. It is known[30] that the critical index of a fermion correlator $S_F(x)=\langle\psi_F^*(x)\,\psi_F(0)\rangle$ differs from that of a boson correlator $S_B(x)=\langle\psi_B^*(x)\psi_B(0)\rangle$. We shall show how $S_F(x)$ can be found in the framework of the conformal approach.

For this we can make use of the results of the preceding section. The operator S_x (33) introduced there implements a Jordan-Wigner transformation from boson to fermion field operators:

$$\psi_F(x)=\psi_B(x)S_x,\qquad \psi_F^*(x)=S_x^*\psi_B^*(x). \quad (34)$$

We thereby quickly find the leading term of the asymptotic form of $S_F(x)$:

$$S_F(x)\propto\cos(\pi\rho x)\,x^{-2/R^2-R^2/8}. \quad (35)$$

It is also easy to write out the entire asymptotic series for $S_F(x)$ and to convince oneself that it agrees with the result of Ref. 30, obtained by other methods.

Less formal arguments are as follows. In fermion systems the momentum of the lowest states (with zero energy in the limit $L\to\infty$), measured in units of $\pi\rho$, can be not only an even but also an odd number. This follows from single considerations based on symmetry properties of the wave function. Let $\psi(x_1,\ldots,x_N)$ be the complete wave function of the system. We act on it with the shift operator $\exp(i\hat{P}a)$, where $a=L/N$ and $\hat{P}$ is the operator of the total momentum. Then, taking into account that ψ should be an eigenfunction of this operator with eigenvalue P, we have

$$\exp(i\hat{P}a)\Psi(x_1,\ldots,x_N)=\Psi(x_1+a,\ldots,x_N+a)=\exp(iPa)\Psi(x_1,\ldots,x_N). \quad (36)$$

On the other hand, cyclic permutation $x_i\to x_{i+1}$ leads to the appearance of a sign factor:

$$\Psi(x_2,x_3,\ldots,x_N,x_1)=(-1)^{N-1}\Psi(x_1,\ldots,x_N).$$

By setting $x_1=x$, $x_2=x+a,\ldots,x_N=x+(N-1)a$, etc., and comparing these formulas, we see that the possible values of the gap in the momentum spectrum depend on whether the number of particles in the system is even or odd:

$$\exp(iPL/N)=(-1)^{N-1}. \quad (37)$$

If we write P in the form $P=2\pi\rho m$, as in Sec. 2, we obtain the condition

$$(-1)^{2m}=(-1)^{N-1}, \quad (38)$$

i.e., depending on the parity of N the value of m should be either an integer or a half-integer. In particular, we arrive at the conclusion that for even N the ground state is doubly degenerate ($P=\pm\pi\rho$). Gapless excitations that conserve the number of particles can change P only by $2\pi m\rho$ (m is an integer). Therefore, the dimensions of the operators $\phi_{0,m}$ do not change and are given by the same formula (9). To avoid confusion, we stress that we are speaking here only of the fact that in both cases the dimensions are expressed in terms of the sound velocity in the same way, but the sound velocity, generally speaking, is different for bosons and fermions.

An important difference between the fermion and boson cases appears when a particle is added to the system. In Fermi systems the thermodynamic formula (10) is true only for even n. The addition of, say, one particle changes the parity of N, and therefore, in accordance with (38), the energy changes by the amount

$$\delta E^F_{1,0}=2\pi vL^{-1}(R^{-2}+R^2/16)=2\pi vL^{-1}h_{1,1/2}, \quad (39)$$

where we have made use of the notation from (12).

In the general case, combining different types of excitations with allowance for the condition (38) we find that the spectrum of dimensions of the Fermi system is determined by the same formula (12) but with a different condition on n and m:

1) If n is even, m is an integer;

2) If n is odd, m is a half-integer (i.e., $m=m'+\frac{1}{2}$, with m' an integer).

In conclusion, the authors would like to express their thanks to V. E. Korepin and A. Yu. Morozov for useful discussions. One of us (A.Z.) is also grateful to A. A. Ovchinnikov for valuable advice and comments.

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2) Strictly speaking, in the expression (22)–(23) one should take into account correction terms that can arise because of irrelevant perturbations of the Hamiltonian.[33] These corrections are numerically small, and, in addition, falloff with distance more rapidly than do the leading terms (they should be comparable with the contributions of the descendants). However, when marginal operators are encountered in the spectrum of dimensions (12), these contributions have a logarithmic form and, in certain cases, give corrections to the leading asymptotic form (as, e.g., in an isotropic antiferromagnet).[34]

3) The authors are grateful to M. A. Ol'shanetskiĭ and A. M. Perelomov for a discussion of this question.

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Translated by P. J. Shepherd