Quantum-mechanical parametric electron resonance in the field of a traveling electromagnetic wave and the Smith–Purcell effect

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The time-dependent problem of the scattering of an electron by a periodic potential existing in a restricted region is solved. The wave functions of the electron are found in the interaction region. A theory of a quantum-mechanical parametric resonance is proposed for the specific case of nonrelativistic free-electron lasers of the diffraction-radiation generator type.

We shall use a semiclassical approximation to determine the resonance states of a charged particle in the interaction region for scattering by a periodic potential existing in a restricted zone (interaction space). The theory proposed essentially deals with the Landau damping (inverse damping) in which the transit time τ (defined below) of a particle crossing the interaction space is an analog of the mean free time.

The problem is of interest in studies of radiation phenomena in free-electron lasers and in the channeling of charged particles in crystals. The difference between these sources of electromagnetic oscillations is in how the periodic potential is created and in its nature.

The quantum-mechanical approach to a study of the states of particles in the interaction region is used mainly in studies of the channeling effect (see, for example, Ref. 1), where, as is known, the states can be separated into above- and below-barrier due to the presence of the interchannel potential barrier. It is also known that a parametric resonance (dechanneling resonance)[2] takes place during channeling, whose theory can be developed using the classical approximation.

We shall consider above- and below-barrier states which appear because of the existence of nonperiodic boundary conditions since the system is not closed. The band of energies which is forbidden in the problem with the Born–von Kármán boundary conditions becomes filled, and this defines the energy of a particle as a continuous function of its momentum. Because of the restricted interaction region, both increasing and decreasing solutions of the Schrödinger equation are taken into account in the interaction region.

We consider a potential which is periodic not only in space, but also in time, and we show that the existence of below-barrier states in combination with the limited residence time in the potential gives rise to a quantum-mechanical parametric resonance.

We consider as a specific application of the theory the determination of the states of an electron in a traveling-wave field, the amplitude of which is small outside the interaction region. This method of imposing a periodic potential is used in nonrelativistic modifications of free-electron lasers (based on the use of the Smith–Purcell effect[3]) of the diffraction-radiation generator type[4] or of the orotron type.[5] These free-electron lasers have open resonators in which one of the mirrors carries a diffraction grating of length L with a period d and the total number of periods is $N = L/d \gg 1$.

The diffraction-radiation field appears because of fluctuations (depending harmonically on time) of the density of an initially unperturbed flux of electrons moving near the surface of a grating along the x axis (this axis is in the plane of the grating and is perpendicular to the grating lines). This field is concentrated between the resonator mirrors and represents a set of fast harmonics which become detached from the grating and slow harmonics which are localized near the grating surface.[4]

The Smith–Purcell effect represents generation of electromagnetic oscillations in the millimeter and submillimeter ranges assuming a constant field and is the result of the interaction of particles with the slow part of the diffraction-radiation field. We shall ignore the edge effects and assume that the interaction region is equal to the grating length.

The diffraction-radiation generators, the orotron, and other modifications of free-electron lasers[6] are devices with a sustained interaction. The condition governing the duration of the interaction $\omega\tau \gg 1$ (where ω is the frequency of the generated radiation, $\tau = L/v_0$, $v_0 = p_0/m$ is the unperturbed velocity of an electron, $p_0 = \hbar k_0$ is the electron momentum governed by the accelerating potential U, and m is the electron mass) is also the condition of adiabaticity of the interaction of a particle with the field and it determines the possibility of existence of metastable states in the interaction region.

A theory of the operation of free-electron lasers of this type is usually based on classical electrodynamics. However, in the classical approach the problem is complicated and it has not been studied much, because it leaves undetermined the nature of the radiation-generating current and the saturation mechanism. Attempts to use a quantum-mechanical approach were reported in Refs. 7 and 8.

We shall show that a one-particle (corresponding to low-density fluxes) semiclassical approximation without allowance for the spin states of an electron can be employed to demonstrate that the lasing mechanism is a quantum-mechanical parametric resonance representing a "horizontal" intraband transition in the presence of a "vertical" interband transition, and that the saturation mechanism is the interband transfer process.

SCHRÖDINGER EQUATION AND ANALYSIS OF ITS SOLUTIONS

1. We consider the interaction of electrons with the field near the surface of a grating (transverse coordinate $z \approx 0$). Then, ignoring the dependence of the field potentials on z and neglecting the small y and z components of the electric field intensity, we shall postulate that the scalar potential is $\Phi(x,t) = 0$ and the longitudinal component of the vector potential $\mathbf{A}(x,t) = \{A(x,t),0,0\}$ is of the form

$$A(x,t) = \sum_{s=-\infty}^{\infty} r_s \exp[i(k_s x - \omega t + \eta_s)], \qquad (1)$$

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where $k_s = \omega/v_{ph}^s = (2\pi/d)s$; the phase velocity of the sth harmonic is $v_{ph}^s \ll c$; c is the velocity of light; $r_s \exp(i\eta_s)$ is the complex amplitude of the sth harmonic of the field which varies adiabatically with the time t and will be assumed to be constant; we take $\pi \geqslant |\eta_s|$ for $s \neq 0$.

It follows from the theory of collisionless plasma[9] that an effective energy exchange between electrons and the field described by Eq. (1) occurs if the resonance condition for synchronization of a particle with one of the harmonics of the field (we consider specifically the first harmonic and omit the index $s = 1$ in future) is satisfied:

$$|\alpha| = \frac{|\Delta v|}{v_0} \ll 1,$$

where $\Delta v = v_0 - v_{ph}$ (we consider here the case $\alpha \geqslant 0$).

Then, the secular one-dimensional Schrödinger equation, written down to within terms of order c^{-1} inclusive, can be simplified by ignoring the interaction with the nonsynchronous part of the field:

$$\frac{1}{2m}\left[\hat{p}^2 + \frac{er}{c}\{\hat{p}, \cos(kx-\omega t+\eta)\}\right]\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}. \tag{2}$$

In Eq. (2) the quantity $\{\hat{p}, \cos(kx - \omega t + \eta)\}$ is the anticommutator of the particle momentum operator and of the real part of the field of the first harmonic; e is the electron charge.

Equation (2) is a quantum-mechanical analog of the classical mechanics equation which describes the parametric resonance[10] that occurs near the doubled frequency $\omega_0 \approx 1/2\omega$.

The period of the potential along the variable x is L ($kL = 2\pi N$), while in terms of the variable t it is τ_{ph}, where $\tau_{ph} = L/v_{ph}$.

We solve Eq. (2) by adopting a coordinate system linked to the wave:.

$$2u = kx - \omega t + \eta, \quad t' = t.$$

Then, separating the motion of the reference system, we seek the wave function of an electron in the form

$$\Psi(u, t') = \varphi(u)\exp[i(\gamma u - \Omega t')], \tag{3}$$

where

$$\gamma = \frac{2k_{ph}}{k} = \frac{4\Omega_{ph}}{\omega}, \quad \hbar\Omega_{ph} = E_{ph} = \frac{\hbar^2}{2m}k_{ph}^2 = \frac{\hbar^2}{2m_*}\gamma^2,$$
$$m_* = \frac{4m}{k^2}, \quad p_{ph} = \hbar k_{ph} = mv_{ph},$$

and $\hbar\Omega$ is the required energy. In the range of frequencies and velocities used in nonrelativistic free-electron lasers[4] the quantity which is quantized is

$$\gamma = \frac{2\pi h}{kL} = \frac{h}{N} \gg 1,$$

where h is an integer.

We use the variables (u, t') in Eq. (2) and substitute the function (3), ignoring the terms that do not contain the quantity γ as a factor. The operator $(\hbar^2/2m)\nabla_x^2$ then transforms to

$$\frac{\hbar^2}{2E_{ph}}\left(\frac{\omega}{2^{1/2}}\nabla_u\right)^2,$$

where $\nabla_u = \partial/\partial u$. It therefore follows that Eq. (2) expressed in terms of the variables (u, t') is formally identical with the Schrödinger equation for a channeled particle[1] in which the role of the mass is played by the energy. We shall use this operator in the more convenient form $\hat{p}_u^2/2m_*$, where $\hat{p}_u = -i\hbar\nabla_u$.

Finally, the Schrödinger equation (2) transforms into a Mathieu equation for the wave function $\varphi(u)$:

$$\frac{d^2\varphi(u)}{du^2} + (\nu^2 - 2R\cos 2u)\varphi(u) = 0, \tag{4}$$

where

$$\nu^2 = \frac{2m_*}{\hbar^2}(E + E_{ph}),$$

and $R = 2e\gamma r/c\hbar k$ is the constant representing the coupling between the electron and the field (which is a classical quantity).

We shall ignore the dissipation of the energy in the investigated system and assume that the Hamiltonian of Eq. (2) is Hermitian, i.e., that $\operatorname{Im}\nu = 0$.

The quantity R differs from zero in a region of length $\Delta x = L = v_0\tau = v_{ph}\tau_{ph}$; in a moving coordinate system the "length" of the interaction region is

$$\Delta u = u_L - u_0 = \frac{1}{2}(kL - \omega\tau) = \frac{1}{2}\alpha\omega\tau_{ph},$$

where $u_L = u(L, t_0 + \tau)$, $u_0 = U(0, t_0)$, $t_0 + \tau > t > t_0$ is the interaction time (for simplicity we shall assume in future that $u_0 = 0$ that $u_L = \Delta u$) and ωt_0 is the phase of the traveling electron.

Solving Eq. (4) in the range $u < 0$, $u > \Delta u$ we obtain the following solutions:

$$\varphi_\nu^{\mathrm{I}}(u) = \exp(i\nu u) + B\exp(-i\nu u), \quad \varphi_\nu^{\mathrm{III}}(u) = D\exp(i\nu u), \tag{5}$$

where $B = \text{const}$ and $D = \text{const}$.

We find ν by assuming that in the wave function of Eqs. (3) and (5) [expressed in terms of the variable (x, t)] we have the following quantity characterizing the incident flux:

$$k_0 = \frac{1}{2}k(\gamma + \nu).$$

Hence the quantity being quantized is

$$\nu = \frac{2\Delta k}{k} = \frac{2\pi g}{kL} = n + q,$$

where $\Delta k = k_0 - k_{ph}$, n is the integral part of the number ν, $1 \geqslant |q| = |f|/N$, and g and f are integers.

We shall show below that the quantity $\frac{1}{2}\hbar kq$ is the reduced electron quasimomentum. Since in this problem the period of the diffraction grating d is an analog of the period of a one-dimensional crystal lattice, it follows that $k = 2\pi/d$ is an analog of a one-dimensional reciprocal lattice "vector." Then the integer f flying within the range $N \geqslant f \geqslant -N$ is equal to the number of the allowed values of q in a band and the density of states in this band is

$$\frac{L}{2\pi} = \frac{Nd}{2\pi}.$$

In the interaction region the solution of the Mathieu equation is[11]

$$\varphi_\nu(u, R)=C_1\varphi_\nu^+(u, R)+C_2\varphi_\nu^-(u, R), \qquad (6)$$

where

$$C_{1,2}=\text{const},$$

$$\varphi_\nu^\pm(u, R)=\exp[\pm(\mu+in)u]F_\nu^\pm(u, R),$$

$$F_\nu^\pm(u, R)=\sum_{l=-\infty}^{\infty} C_l(\nu, R)\exp(\pm 2iul),$$

$\mu'(\nu,R)=\mu(\nu,R)+in$ is a characteristic exponent defined to within an integer n:

$$|\mu(\nu, R)|<1, \quad \mu'(\nu, 0)=\nu.$$

It follows from the theory of the Mathieu equation that, depending on the quantities ν and R, the solutions of this equation form stable [where $\mathrm{Re}\,\mu(\nu,R)=0$] and unstable [$\mathrm{Re}\,\mu(\nu,R)\neq 0$] bands which transform continuously from one to the other. At the band boundaries we have $\mu(\nu,R)=0$. Hence, ν is a function of R:

$$\nu_{t(b)}(R)=n+q_{t(b)}(R),$$

where $q_{t(b)}(0)=0$ and $\nu_{t(b)}(R)$ is the value of the quantity ν at the upper (lower) boundary of the band.

The solutions of the Mathieu equations at the boundaries (edges) of the band are

$$\varphi_n^t(u, R)=\mathrm{Ce}_n(u, R), \quad \varphi_n^b(u, R)=\mathrm{Se}_n(u, R),$$

where $\mathrm{Ce}_n(u, R)$ and $\mathrm{Se}_n(u, R)$ are the periodic Mathieu functions.

It therefore follows that the wave functions (3) and (6) are of the above-barrier ($\mathrm{Re}\,\mu=0$) or below-barrier ($\mathrm{Re}\,\mu\neq 0$) type. The above-barrier functions (5) are the periodic Floquet–Bloch functions, where $\hbar k\mu(\nu,R)$ is the reduced quasimomentum.

Inside an energy band $\nu_t(R)\geqslant\nu\geqslant\nu_b(R)$ the quantity η varies continuously on transition from one band to another and, therefore, the electron energy

$$E=E_n(q)=\frac{\hbar^2}{2m}\left(\frac{k\nu}{2}\right)^2-E_{ph}=\frac{\hbar^2}{2m_e}(\nu^2-\gamma^2)$$

is a continuous function of the electron quasimomentum $1/2\hbar k\nu$.

It follows from the expression for the energy that for $\nu\leqslant\gamma$ ($E_n\leqslant 0$) the electron is in a bound state. The energy of the ground state is E_{ph}. The energy spectrum represents bands of energies of above- and below-barrier states which transform continuously from one to the other. In the wave functions of Eqs. (3) and (6) the quantity $1/2\hbar kq$ is the reduced quasimomentum.

At the boundaries of the nth unstable band the electron energy is governed by the values of the quantities $\nu_{t(b)}$. Then the width of a below-barrier band is

$$\Delta_n(R)=\frac{\hbar^2}{2m_e}(2n+q_t+q_b)(q_t-q_b).$$

It follows from the theory of the Mathieu equation that the width of an unstable band decreases rapidly as its number increases. If R is small, we find that $\Delta_n\sim R^2$ is small even for $n=2$.

The quantity $\mu(\nu, R)$ for a below-barrier band governs the decay constant of the initial state of an electron. We can in fact represent the wave function of Eqs. (3) and (6) by returning to the variables (x, t) and going over from the index ν to the index n:

$$\Psi_n(x, t; q, R)=C_1\Psi_n^+(x, t; q, R)+C_2\Psi_n^-(x, t; q, R),$$

where

$$\Psi_n^\pm(x, t; q, R)=\tilde{\varphi}_n^\pm(x, t; q, R)\exp\left[\pm\frac{\Gamma_n}{2}\left(\frac{x}{v_{ph}}-t\right)+i(k_n^\pm x-\Omega_n^\pm t)\right],$$

$$\tilde{\varphi}_n^\pm(x, t; q, R)=\varphi_n^\pm(u; q, R)\times\exp\left\{\frac{1}{2}[\eta(\pm\mu_n+i\gamma)\mp i\nu(kx-\omega t)]\right\},$$

$$k_n^\pm=\frac{1}{2}k(\gamma\pm\nu), \qquad \hbar\Omega_n^\pm=\frac{\hbar^2}{2m_e}(\gamma\pm\nu)^2,$$

$$\Gamma_n=\Gamma_n(q, R)=\omega(\mu_n+iq), \quad \mu_n=\mu_n(q, R).$$

It follows from the above expression for the wave function that for $\mu_n(q,R)>0$, the interaction of the initial state of an electron decays within the interaction region ($\hbar k_n^+=p_0$). The decay constant is then $\mathrm{Re}\,\Gamma_n=\omega\mu_n$. The lifetime of this state is $T=(\omega\mu_n)^{-1}$ and the width of the level is $\hbar\omega\mu_n$. The condition for the existence of a metastable state is $T\gg\tau$, i.e.,

$$\mu_n^{-1}\gg\omega\tau, \quad \mu_n\ll 1.$$

In addition to decay of the initial state, the wave function of an electron with the momentum $\hbar k_n^-$ increases with time (which decreases with depth in the interaction region). The growth rate has the value $\mathrm{Re}\,\Gamma_n$. The quantity $\mathrm{Im}\,\Gamma_n$ determines the detuning of the transition frequency.

The values of the energy in the field of the traveling wave found above are quantizable and positive, which means that these diffraction-radiation generators and the orotron are lasers utilizing free–free transitions.

2. The explicit form of the dependence of μ_n on ν and R can be found for the case of coupling of an electron to the field using the relationship[11]

$$\mathrm{ch}\,\pi\mu_n=\cos\pi q+\frac{\pi R^2\sin\pi q}{4\nu(\nu^2-1)}, \qquad (7)$$

where the smallness of the quantity μ_n implies the smallness of q.

The second term of the right-hand side of Eq. (7) has two resonant values of ν, namely $\nu=1+q$, $\nu=q$. Expanding the right- and left-hand sides of Eq. (7) in terms of the small quantities μ and q, we find (to lowest order) in the first resonance case

$$\mu_1(q, R)=\frac{1}{2}(R^2-4q^2)^{1/2}.$$

Hence, it follows that for values of $|q|>1/2R$ the quantity μ_1 is purely imaginary, whereas the wave functions are of the above-barrier type.

For q lying within the interval

$$\frac{1}{2}R>q>-\frac{1}{2}R,$$

the solution of the Mathieu equation are unstable and the wave functions are of the below-barrier type. The numbers f lie within the interval

$$\frac{1}{2}NR \geqslant f \geqslant -\frac{1}{2}NR$$

(it follows from the condition of quantization of the quantity ν that RN is an integer) and the density of states in the below-barrier energy band is

$$\frac{RL}{2\pi} = \frac{df}{2\pi}.$$

It follows from the above expression that the minimum value of the quantity defined by it is $R_{\min} \approx N^{-1}$.
At the limits of the instability region we have

$$|q| = \frac{1}{2}R, \quad \nu_{t(b)} = 1 \pm \frac{1}{2}R.$$

The width of a below-barrier energy band is

$$\Delta_1(R) = \frac{\hbar^2}{2m}k^2R = \frac{\hbar^2}{2E_{ph}}\left(\frac{\omega}{2}R^{1/2}\right)^2.$$

The solutions of the Mathieu equation at the edges of a below-barrier band, considered to lowest order in R, are standing waves $\cos u$ and $\sin u$ with the corresponding wave functions described by Eq. (3).

The approximate wave functions $\varphi_1^{\pm}(u;q,R)$ can be found by determining, in accordance with the familiar scheme,[11] the approximate values of the functions $F_\nu^{\pm}(u,R)$. Ignoring the details of the calculations, we find—to the first nonvanishing order in μ_1, q, and R, going over from the parameters q, R to ζ, R,—that

$$F_1^{\pm}(u, \zeta, R) = C_0\{1+\exp[i(\zeta \mp u)]\},$$

where $C_0 = \text{const}$,

$$\zeta = \operatorname{arctg}\frac{\mu_1}{q}, \quad \pi \geqslant \zeta \geqslant 0.$$

Hence, we obtain the dependences of the quantities q and ν on R throughout the full range of existence of a quantum-mechanical parametric resonance:

$$q(\zeta, R) = \frac{1}{2}R\cos\zeta,$$

and the energy distribution $E_1[q(\zeta,R)]$ in a below-barrier band, considered as a function of the parameters of the system. The quantity $\mu_1(q,R)$ regarded as a function of ζ and R is described by

$$\mu(\zeta, R) = \frac{1}{2}R\sin\zeta.$$

The required functions are

$$\varphi_1^{\pm}(u; \zeta, R) = 2C_0\cos\left(u \mp \frac{1}{2}\zeta\right)\exp\left(\pm\mu_1 u + i\frac{1}{2}\zeta\right).$$

The unknown coefficients B, D, and $C_{1,2}$ (including $2C_0$) can be found from the condition of continuity of the approximate wave function of Eq. (6) and its derivative of the points $u_0 = 0$ and $u_L = \Delta u$.

The result is

$$C_1 = -\varkappa^{-1}\exp[-\mu_1(\Delta u + i\zeta)],$$
$$C_2 = \varkappa^{-1}\exp(\mu_1\Delta u),$$
$$B = \varkappa^{-1}\operatorname{sh}(\mu_1\Delta u), \quad D = 2i\varkappa^{-1}\sin\zeta,$$

where

$$\varkappa = \varkappa(\zeta, R) = \operatorname{sh}(\mu_1\Delta u + i\zeta).$$

The approximate wave function described by Eqs. (3) and (6) for the first below-barrier energy band can then be described as follows in terms of the variables (u, t') and (x, t):

$$\Psi_1(u, t'; \zeta, R) = \varkappa^{-1}\{\operatorname{sh}[\mu_1(\Delta u - u) + i\zeta]\exp(iu) + \operatorname{sh}[\mu_1(\Delta u - u)]\exp(-iu)\}\exp[i(\gamma u - \Omega_1 t')], \quad (8)$$
$$\Psi_1(x, t; \zeta, R) = \varkappa^{-1}\{G^+\operatorname{sh}[\mu_1(\Delta u - u) + i\zeta]\exp[i(k_0 x - \Omega_1^+ t)] + G^-\operatorname{sh}[\mu_1(\Delta u - u)]\exp[i(k_1 x - \Omega_1^- t)]\},$$

where

$$G^{\pm} = G^{\pm}(x, t) = \exp\left\{\frac{1}{2}i[\eta(\gamma \pm 1) \mp q(kx - \omega t)]\right\} \qquad k_1 \equiv k_1^-.$$

It follows from the above expressions that the wave function for a below-barrier band increases with time (it decreases with depth in the interaction region) and this defines the investigated below-barrier (tunnel) transition as a quantum-mechanical parametric resonance. The tunnel barrier is the band of energies which are forbidden in the problem with the Born–von Kármán boundary conditions.

An electron in the zone in which a quantum-mechanical parametric resonance exists (which is a below-barrier band) is a two-level system with the energies E_1^+ and E_1^- and with the intraband ("horizontal") transition frequency

$$\Delta\omega_{1,1} = \nu\omega,$$

where the energy difference $\hbar\Delta\omega_{1,1} = 2\Delta p v_{ph}$ is a classical quantity and $\Delta p = p_0 - p_{ph}$. The detuning of the transition frequency is

$$\frac{1}{2}R|\cos\zeta| \omega \ll \omega.$$

The populations of the states are governed by the squares of the moduli of the coefficients in front of the plane wave in the first function (8). For $\nu > 0$, the population inversion is independent of time and is given by

$$\vartheta_{1,1}(\zeta, R) = |\varkappa|^{-2}\sin^2\zeta > 0,$$

where

$$|\varkappa|^{-2} = \operatorname{ch}(2\mu_1\Delta u) + \sin^2\zeta - 1.$$

We can easily show that reversal of the sign of the variable u in Eqs. (5) and (6) (or the assumption $\nu < 0$) leads to a "negative" inversion of the populations, which is equal to the value obtained above.

Using the wave function (8), we can obtain the distribution of the charge density ρ_1 and of the current density j_1 in the interaction region:

$$\rho_1(u; \zeta, R) = e|\varkappa|^{-2}\{\sin^2\zeta - 1 - \cos\zeta\cos 2u + \operatorname{ch}[2\mu_1(\Delta u - u)][1 + \cos\zeta\cos 2u - \sin\zeta\sin 2u\,\operatorname{th}2\mu_1(\Delta u - u)]\}, \quad (9)$$
$$j_1(u; \zeta, R) = v_{ph}\rho_1(u; \zeta, R).$$

In the second resonance case we find from Eq. (7) that

$\mathrm{Re}\,\mu_0 = 0$ and $\mathrm{Im}\,\mu_0 = \pm 2^{-1/2}(R^2 + 2q^2)^{1/2}$ apply near the ground state. At the boundary of an energy band we have $q = 2^{-1/2}iR$. However, as pointed out above, we are ignoring the states with complex values of ν.

In the limiting case of full synchronization of the particle and wave ($\nu = 0$), we find that $\Delta u = 0$. It follows that for $v_0 = v_{ph}$, "channeling" of a particle across the interaction region takes place, i.e., the passage across this region is free of scattering. The wave function of an electron in the ground state $\Psi_0(x,t)$ is the wave function of free motion with a momentum p_{ph}.

The existence of two resonance values of ν implies the possibility of an interband ("vertical") transition from a narrow band of above-barrier states to a narrow subband of resonance states adjoining the ground state.

The wave function in a below-barrier band, regarded as a linear combination of the functions (8) depending on the discrete parameter q, differs little from the wave function at the center of the band (where $\zeta = \pi/2$), since the coefficients of its expansion (due to the smallness of q) can be expanded in powers of q:

$$b(q) = \delta(q) + qb' + q^2 b'' + \ldots,$$

where

$$\delta(q) = \begin{cases} 1, & q=0, \\ 0, & q \neq 0, \end{cases} \qquad b', b'' = \text{const}.$$

We can similarly represent the wave function in the subband near the ground state.

It therefore follows that an electron in the states with the wave functions $\Psi_1(x,t;\pi/2,R)$, $\Psi_0(x,t)$ considered to lowest order in the small quantity q is a two-level system with the energies E_1 and E_{ph}, and a transition frequency given by

$$\Delta\omega_{1,0} = \Omega_1 - \Omega_{ph} = \frac{1}{2}\omega[1 + (2\gamma)^{-1}].$$

It therefore follows that the frequency of a "horizontal" transition differs from the doubled frequency of a "vertical" transition by $(2\gamma)^{-1}$ ("electron frequency shift").

The population of the states with the energy E_1 is

$$\frac{1}{e}\rho_1\left(u; \frac{\pi}{2}, R\right),$$

so that the difference between these populations to lowest order in R is

$$\vartheta_{1,0}(u,R) = \left|\varkappa\left(\frac{\pi}{2}, R\right)\right|^{-2} \mathrm{ch}[R(\Delta u - u)] - (2\pi N)^{-1}.$$

The strength of the spontaneous ("vertical") transition $P_{1,0}$ is readily calculated using the method familiar from quantum electronics[12] and is given by

$$P_{1,0}\left(u; \frac{\pi}{2}, R\right) = \left|\varkappa\left(\frac{\pi}{2}, R\right)\right|^{-2} R\omega\Delta p v_{ph}\,\mathrm{sh}[R(\Delta u - u)].$$

It follows from the above expression that the measured power is a classical quantity.

In the present approximation characterized by $\gamma \gg N$ (representing the free-electron laser approximation) the dependence of the spontaneous radiation power on the variable u (i.e., on the variables x and t) differs little from linearity because the "length" of the interaction region is

$$\Delta u = \pi N \gamma^{-1} \ll \pi.$$

Hence, a quantum-mechanical parametric resonance represents effectively the possibility of a "horizontal" (intraband) transition of frequency $\Delta\omega_{1,0} \approx 2\Delta\omega_{1,1}$ in the presence of a spontaneous "vertical" (interband) transition.

SMITH–PURCELL EFFECT

The Smith–Purcell effect is the result of the interaction of an electron with all the harmonics of the field (1). However, our theory of a quantum-mechanical parametric resonance is also a theory of this effect because in the presence of a resonance the nonsynchronous part of the field is in fact a time-dependent perturbing correction to the Hamiltonian of the Schrödinger equation.

We now consider how the quantities responsible for the existence of a quantum-mechanical parametric resonance vary with the number of the synchronous harmonics, oscillation frequency, and grating parameters.

Synchronize an electron with a sth harmonic of the field ($s \neq 1$) we assume that

$$2u_s = k_s x - \omega t + \eta_s.$$

We can then easily show that

$$R_s = a_1 r_s (ds^{-1})^3,$$
$$\gamma_s = a_2 \omega (ds^{-1})^2,$$

where

$$a_1 = \frac{em}{2\pi^3\hbar^2 c}, \qquad a_2 = \frac{4em}{c\hbar^2}, \qquad r_s > r_{s+1}, \qquad v_{ph}^s > v_{ph}^{s+1}.$$

A reduction in the quantity R_s with increasing harmonic number reduces $\mu_1(\zeta, R_s)$ and the density of states in a below-barrier energy band, which in the final analysis reduces the population inversion $\vartheta_{1,1}(\zeta, R_s)$ [note that the product is $\mu_1(\zeta, R_s)\Delta u_s = \text{const}$].

Using the above expressions for R_s and γ_s, we can now write down the existence condition for metastable states

$$\mu_1\left(\frac{\pi}{2}, R_s\right) = \frac{1}{2} a_1 r_s \omega (ds^{-1})^3 \ll 1.$$

This leads to the familiar (in theory and practical applications of nonrelativistic free-electron lasers) fact that an increase in the oscillation frequency requires a reduction in the grating period and this imposes restrictions on the upper limit of the frequencies which can be generated.

When the interaction is with higher harmonics, the relative desynchronization increases:

$$\alpha_s = \nu_s \gamma_s^{-1}, \quad \nu_s = \nu(\zeta, R_s) = 1 + q(\zeta, R_s).$$

Hence we can readily obtain the value of the accelerating potential difference necessary for the existence of a quantum-mechanical parametric resonance, which is

$$U(s) = \frac{m}{8e\pi^2}\frac{\omega^2 d^2}{s^2}(1 - \nu_s\gamma_s^{-1})^{-2}.$$

This dependence of U on s explains the observation familiar from the experimental studies of the diffraction-radiation and orotron generators that an increase in the accelerating voltage increases the output radiation power until lasing is suppressed.

It follows from the above theory of the quantum-mechanical parametric resonance that an increase in the voltage results in gradual synchronization of an electron from the high harmonics right down to the first, which has the maximum phase velocity and amplitude.

The frequencies of the transitions due to the interaction with the harmonics characterized by $s \neq 1$ are the same as for the interaction with the first harmonic; the only difference is the "electron frequency shift" which increases with the number s.

We can now describe the saturation mechanism as follows: at values of the accelerating voltage higher that needed to synchronize an electron with the first harmonic a nonradiative interband transfer process takes place from a below-barrier band to higher above-barrier bands.

When the accelerating potential U ($s \neq 1$) corresponds to saturation, we have $\nu \approx 1 + {}^{1}/_{2}R$, which should make it possible to determine experimentally the value of the constant representing the coupling of an electron to the field (amplitude of the first harmonic).

In conclusion, we should point out that formulation of the problem and of the above conditions demonstrates that the theory of the Smith–Purcell effect is a model analog of the inverse Landau damping for a low-density plasma.

[1]V. A. Bazylev and N. K. Zhevago, *Radiation Emitted by Fast Particles in Matter and in External Fields* [in Russian], Nauka, Moscow (1987), Chap. 6.

[2]M. A. Kumakhov and R. Wedell, Phys. Status Solidi B **76**, 119 (1976).

[3]S. J. Smith and E. M. Purcell, Phys. Rev. **92**, 1069 (1953).

[4]V. P. Shestopalov, *Diffraction Electronics* [in Russian], Kharkov State University (1976), Chaps. 1 and 7.

[5]F. S. Rusin and G. D. Bogomolov, Author's Certificate No. 195 557 [in Russian], appl. 16 February 1965; publ. 4 May, 1967 [Byull. Izobret. No. 10, 49 (1967)].

[6]A. Gover and P. Sprangle, IEEE J. Quantum Electron **QE-17**, 1196 (1981).

[7]Yu. A. Pobedin and A. A. Yatsenko, Pis'ma Zh. Tekh. Fiz. **7**, 904 (1981) [Sov. Tech. Phys. Lett. **7**, 388 (1981)].

[8]S. G. Glass and H. Mendlowitz, Phys. Rev. **174**, 57 (1968).

[9]E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics*, Pergamon Press, Oxford (1981), Chap. 3.

[10]L. D. Landau and E. M. Lifshitz, *Mechanics, 2nd ed.*, Pergamon Press, Oxford (1969).

[11]N. W. McLachlan, *Theory and Application of Mathieu Functions*, Clarendon Press, Oxford (1947).

[12]R. H. Pantell and H. E. Puthoff, *Fundamentals of Quantum Electronics*, Wiley, New York (1969).

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