Scattering of scalar wave fields by absolutely reflecting rough surfaces

S. Z. Dunin and G. A. Maksimov

Moscow Engineering-Physics Institute
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A theory is proposed for the scattering of a scalar field by statistically rough surfaces. This theory does not use the customary approximation, based on the Rayleigh assumption, in which the field throughout the half-space is sought in the same form as that of the field at infinity. The components of the scattering amplitude and of the mean field which stem from waves scattered by the surface more than once are calculated. This new approach is shown to be valid if the mean slope of the roughness is small in comparison with both unity and the ratio of the wavelength to the mean height of the roughness. Close to the surface, the mean field has a structure more complex than previously believed.

1. INTRODUCTION

The scattering of waves by rough surfaces arises in several problems in optics, acoustics, seismology, radiophysics, and electronics. The basic theoretical and practical task here is to determine the relationship between the characteristics of the scattered field and the properties of the scattering surface. Once this relationship has been established, one can work from the known characteristics of the surface to calculate the scattered field. Inversely, one can work from measurements of the scattered field to find information about the surface structure. Because of the wide range of applications and the importance of this problem, many papers have been published on the scattering of waves by statistically rough surfaces (see Refs. 1–3 and the bibliographies there). However, there is no really complete and systematic theory which meets the needs of experimentalists.

The existing approximate calculation methods yield a description of various limiting cases, but their own range of applicability needs to be examined.[4] The methods used most widely are the small-perturbation method[1,2,5–7] and the tangent-plane method (the Kirchhoff approximation).[1,3–6,8] Certain other approaches are reviewed in Refs. 2, 3, and 9. The range of applicability of the small-perturbation method is limited by the requirement that the Rayleigh parameter be small, $K\sigma\cos\theta \ll 1$, where σ is the characteristic height of the roughness, K is the wave number, and θ is the angle between the wave propagation direction and the normal to the mean plane. Under this condition, the boundary conditions can be transferred to the mean plane. The range of applicability of the Kirchoff approximation is given by the condition[4,10] $DR\cos^3\chi \gg 1$, where R is the radius of curvature of the surface, and χ is the local glancing angle. Under this inequality, the small diffraction corrections to the tangent-plane method can be assumed small. Liska and McCoy[10] have pointed out that diffraction corrections are taken into account in the single-scattering approximation of the waves.

Several recent papers[11–14] have used the Rayleigh assumption,[15,16] assuming a small roughness slope to attempt to develop a description which combines these two approaches. The Rayleigh assumption can be summarized by saying that the asymptotic representation of the field far from the scattering surface is also used to describe the field near the surface. It has been shown[15,17] that the use of the Rayleigh hypothesis in the case of a sinusoidal surface requires an upper limit on the surface slope angle θ_*: $\tan\theta_* < 0.448$. For a randomly rough surface, the applicability of the Rayleigh assumption remains an open question. It is thus exceedingly important to develop a theory which does not lean on the Rayleigh hypothesis and which has both the small-perturbation method and the Kirchoff approximation as limiting cases.

In this paper we propose a systematic theory for the scattering of scalar fields by absolutely reflecting, statistically rough surfaces. This theory does not assume the Rayleigh assumption. The range of applicability of this theory is pointed out. The values of the field or of its normal derivative are not specified at the outset; they are instead regarded as solutions of integral equations which follow from the exact formulation of the problem. A solution of these equations is sought as an expansion in the number of times the waves are scattered by the surface. When only singly scattered waves are considered, one obtains the Kirchoff approximation. In order to describe the results of the small-perturbation method adequately, the doubly and triply scattered waves must also be taken into account.

2. EXPANSION OF THE FIELD IN THE NUMBER OF TIMES THE WAVES ARE SCATTERED BY THE SURFACE

According to Green's theorem,[18] a scalar field in the inner region of a half-space with a rough boundary specified by the equation $z=\eta(\mathbf{r})\,[\mathbf{R}=(\mathbf{r},z)]$ can be expressed in terms of the value of the field and of its normal derivative of the boundary:

$$U(\mathbf{R})=U_0(\mathbf{R})+\int_S dS\left\{U(\mathbf{R}_S)\frac{\partial}{\partial n_S}G(\mathbf{R},\mathbf{R}_S)-G(\mathbf{R},\mathbf{R}_S)\frac{\partial}{\partial n_S}U(\mathbf{R}_S)\right\}.$$

Here $U_0(\mathbf{R}) = \int d\,\mathbf{R}_0\rho(\mathbf{R}_0)G(\mathbf{R},\mathbf{R}_0)$ is the field of the incident wave, $\rho(\mathbf{R}_0)$ is the density of field sources,

$$\frac{\partial}{\partial n_S}=\left[\frac{\partial}{\partial z}-\nabla\eta(\mathbf{r})\frac{\partial}{\partial \mathbf{r}}\right][1+(\nabla\eta(\mathbf{r}))^2]^{-1/2}$$

is the derivative along the normal to the surface,

$$dS=(1+(\nabla\eta(\mathbf{r}))^2)^{1/2}d\mathbf{r}$$

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is a surface area element, and $G(\mathbf{R},\mathbf{R}_0)$ is the free-space Green's function, given by

$$G(\mathbf{R},\mathbf{R}_0)=\frac{1}{4\pi}\frac{\exp\{iK|\mathbf{R}-\mathbf{R}_0|\}}{|\mathbf{R}-\mathbf{R}_0|},$$

$$K=\frac{\omega}{c},\quad |\mathbf{R}-\mathbf{R}_0|=[(\mathbf{r}-\mathbf{r}_0)^2+(z-z_0)^2]^{1/2}.$$

In the case of an absolutely soft surface, the field vanishes at the boundary, i.e., $U(\mathbf{r},z=\eta(r))=0$. In the case of a hard surface, the derivative along the normal to the boundary vanishes, i.e., $(\partial/\partial n)U(\mathbf{r},z=\eta(\mathbf{r}))=0$. The expressions for the fields in the half-space are

$$U_D(\mathbf{r},z)=U_0(\mathbf{r},z)-\frac{i}{2(2\pi)^2}\int\frac{d\mathbf{k}}{\nu}\int d\mathbf{r}'$$
$$\times\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')+i\nu|z-\eta(\mathbf{r}')|]$$
$$\times\left[\frac{\partial}{\partial z'}-\nabla\eta(\mathbf{r}')\frac{\partial}{\partial\mathbf{r}'}\right]\bigg|_{z'=\eta(\mathbf{r}')}U_D(\mathbf{r}',z'),\quad (1a)$$

for a soft surface and

$$U_N(\mathbf{r},z)=U_0(\mathbf{r},z)+\frac{1}{2(2\pi)^2}\int d\mathbf{k}\int d\mathbf{r}'$$
$$\times\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')+i\nu|z-\eta(\mathbf{r}')|]$$
$$\times\left[\operatorname{sign}(z-\eta(\mathbf{r}'))-\frac{\mathbf{k}\nabla\eta(\mathbf{r}')}{\nu}\right]U_N(\mathbf{r}',\eta(\mathbf{r}'))$$
$$+\frac{1}{2}\delta_{z,\eta(\mathbf{r})}U_N(\mathbf{r},\eta(\mathbf{r}))\quad (1b)$$

for a hard one. Here

$$\operatorname{sign}(0)=0,\quad \delta_{z,\eta}=\begin{cases}1, & z=\eta\\ 0, & z\neq\eta\end{cases}.$$

Here we have represented the Green's function as a Weyl expansion of spherical waves in plane waves,[18]

$$G(\mathbf{r},z)=\frac{i}{2(2\pi)^2}\int\frac{d\mathbf{k}}{\nu}\exp(i\mathbf{k}\mathbf{r}+i\nu|z|),\quad (2)$$
$$\nu=(K^2-k^2)^{1/2}\ \text{for}\ K>k,\ \nu=i(k^2-K^2)^{1/2}\ \text{for}\ K<k.$$

The asymptotic behavior of the field far from the scattering part of the surface $[z>\max\eta(\mathbf{r})]$ is of interest in several problems. In this case the total field can be written as the sum of incident and reflected plane waves:

$$U(\mathbf{r},z)=U_0(\mathbf{r},z)+\int d\mathbf{k}A(\mathbf{k})\exp(i\mathbf{k}\mathbf{r}+i\nu z),$$

where $A(\mathbf{K})$ is the scattering amplitude. It follows from (1a) and (1b) that the scattering amplitudes for an absolutely rough surface, A_D, and an absolutely hard one, A_N, are given by

$$\begin{Bmatrix}A_D(\mathbf{k})\\ A_N(\mathbf{k})\end{Bmatrix}=\frac{1}{2(2\pi)^2}\int\frac{d\mathbf{r}}{\nu}\exp[-i\mathbf{k}\mathbf{r}-i\nu\eta(\mathbf{r})]$$
$$\times\begin{Bmatrix}-i\left(\frac{\partial}{\partial z}-\nabla\eta(\mathbf{r})\frac{\partial}{\partial\mathbf{r}}\right)\Big|_{z=\eta(\mathbf{r})}U_D(\mathbf{r},z)\\ (\nu-\mathbf{k}\nabla\eta(\mathbf{r}))U_N(\mathbf{r},\eta(\mathbf{r}))\end{Bmatrix}.\quad (3)$$

We wish to stress that the scattering amplitude can be utilized to calculate the asymptotic behavior of the scattered field far from the scattering surface. The use of the representation (3) near the surface and actually at the surface (the latter use is equivalent to the Rayleigh assumption) is thus incorrect. For the functions $U_N(\mathbf{r},\eta(\mathbf{r}))$ and

$$V_D=\left[\frac{\partial}{\partial z}-\nabla\eta(\mathbf{r})\frac{\partial}{\partial\mathbf{r}}\right]\bigg|_{z=\eta(\mathbf{r})}U_D(\mathbf{r},z)$$

one finds the following equations from (1a) and (1b):

$$\begin{Bmatrix}V_D(\mathbf{r},\eta(\mathbf{r}))\\ U_N(\mathbf{r},\eta(\mathbf{r}))\end{Bmatrix}=2\exp[i\mathbf{k}_0\mathbf{r}-i\nu_0\eta(\mathbf{r})]\begin{Bmatrix}-i[\nu_0+\mathbf{k}_0\nabla\eta(\mathbf{r})]\\ 1\end{Bmatrix}+$$
$$+\frac{1}{(2\pi)^2}\int d\mathbf{k}\int d\mathbf{r}'\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')+i\nu|\eta(\mathbf{r})-\eta(\mathbf{r}')|]$$
$$\times\begin{Bmatrix}V_D(\mathbf{r}',\eta(\mathbf{r}'))[\operatorname{sign}(\eta(\mathbf{r})-\eta(\mathbf{r}'))-\mathbf{k}\nabla\eta(\mathbf{r})/\nu]\\ U_N(\mathbf{r}',\eta(\mathbf{r}'))[\operatorname{sign}(\eta(\mathbf{r})-\eta(\mathbf{r}'))-\mathbf{k}\nabla\eta(\mathbf{r}')/\nu]\end{Bmatrix}.\quad (4)$$

In the case $\eta(\mathbf{r})=$ const, i.e., in the case of a plane surface, the integral terms in (4) vanish, and one finds the known boundary conditions at absolutely reflecting level surfaces, specifically, a doubling of the field or of its derivative along the normal at the boundary.

It can be shown that a corresponding result is found when the surface is a sloping plane, $\eta(\mathbf{r})=\mathbf{n}\mathbf{r}$. For this purpose, we integrate the terms containing the signature by parts (over the variable $\mathbf{k}$) in (4). The integral terms in (4) then takes the form

$$\frac{1}{(2\pi)^2}\int d\mathbf{k}\int d\mathbf{r}'\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')+i\nu|\eta(\mathbf{r})-\eta(\mathbf{r}')|]$$
$$\times\begin{Bmatrix}V_D(\mathbf{r}',\eta(\mathbf{r}'))\\ U_N(\mathbf{r}',\eta(\mathbf{r}'))\end{Bmatrix}\frac{\mathbf{k}}{\nu}\left[\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\frac{(\eta(\mathbf{r})-\eta(\mathbf{r}'))}{|\mathbf{r}-\mathbf{r}'|}-\begin{Bmatrix}\nabla\eta(\mathbf{r})\\ \nabla\eta(\mathbf{r}')\end{Bmatrix}\right]$$
$$=\frac{-i}{(2\pi)}\int d\mathbf{r}'\begin{Bmatrix}V_D(\mathbf{r}',\eta(\mathbf{r}'))\\ U_N(\mathbf{r}',\eta(\mathbf{r}'))\end{Bmatrix}$$
$$\times\left[\eta(\mathbf{r})-\eta(\mathbf{r}')-(\mathbf{r}-\mathbf{r}')\begin{Bmatrix}\nabla\eta(\mathbf{r})\\ \nabla\eta(\mathbf{r}')\end{Bmatrix}\right]$$
$$\times\frac{1}{R}\frac{\partial}{\partial R}\frac{\exp(iKR)}{R},\quad (4a)$$

where $R=[(\mathbf{r}-\mathbf{r}')^2+(\eta(\mathbf{r})-\eta(\mathbf{r}'))^2]^{1/2}$. It follows from the representation (4a) that for $\eta(\mathbf{r})=\mathbf{n}\mathbf{r}$ the integral term vanishes. After (4) is substituted into (3), we find the following expression for the scattering amplitude:

$$\begin{Bmatrix}A_D(\mathbf{k})\\ A_N(\mathbf{k})\end{Bmatrix}=\mp\delta(\mathbf{k}_0-\mathbf{k}-(\nu_0+\nu)\mathbf{n})\begin{Bmatrix}\nu_0/\nu+\mathbf{k}_0\mathbf{n}/\nu\\ 1-\mathbf{k}\mathbf{n}/\nu\end{Bmatrix}.$$

This expression describes the field specularly reflected from the sloping plane.

Expressions (3) and (4) can also be written in a form which has the same structure as was chosen in Ref. 11 on the basis of the transformation properties of the scattering amplitude. Specifically, from the quantities V_D and U_N we factor out the phase corresponding to the incident wave:

$$\begin{Bmatrix}V_D(\mathbf{r},\eta(\mathbf{r}))\\ U_N(\mathbf{r},\eta(\mathbf{r}))\end{Bmatrix}=2\exp[i\mathbf{k}_0\mathbf{r}-i\nu_0\eta(\mathbf{r})]\begin{Bmatrix}-i\nu_0 v(\mathbf{r})\\ u(\mathbf{r})\end{Bmatrix}$$

For the scattering amplitude we then find the representation

$$\left\{\begin{matrix} A_D(\mathbf{k}) \\ A_N(\mathbf{k}) \end{matrix}\right\} = \frac{1}{(2\pi)^2}\int d\mathbf{r}\exp[i(\mathbf{k}_0-\mathbf{k})\mathbf{r} -i(\nu+\nu_0)\eta(\mathbf{r})]\left\{\begin{matrix} (\nu_0/\nu)v(\mathbf{r}) \\ (1-\mathbf{k}\nabla\eta(\mathbf{r})/\nu)u(\mathbf{r}) \end{matrix}\right\}, \quad (3a)$$

where $v(\mathbf{r})$ and $u(\mathbf{r})$ satisfy the equations

$$\left\{\begin{matrix} v(\mathbf{r}) \\ u(\mathbf{r}) \end{matrix}\right\} = \left\{\begin{matrix} 1+(\mathbf{k}_0/\nu_0)\nabla\eta(\mathbf{r}) \\ 1 \end{matrix}\right\} + \frac{1}{(2\pi)^2}\int d\mathbf{k}\int d\mathbf{r}'\exp[i(\mathbf{k}-\mathbf{k}_0)\mathbf{r}] \times\exp[i(\mathbf{k}_0-\mathbf{k})\mathbf{r}]\left\{\begin{matrix} v(\mathbf{r}) \\ u(\mathbf{r}) \end{matrix}\right\}\exp[i\nu_0(\eta(\mathbf{r}) -\eta(\mathbf{r}'))+i\nu|\eta(\mathbf{r})-\eta(\mathbf{r}')|] \times\frac{\mathbf{k}}{\nu}\left[\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\frac{(\eta(\mathbf{r})-\eta(\mathbf{r}'))}{|\mathbf{r}-\mathbf{r}'|}-\left\{\begin{matrix} \nabla\eta(\mathbf{r}) \\ \nabla\eta(\mathbf{r}') \end{matrix}\right\}\right]. \quad (4b)$$

Since the kernel of Eq. (4b) depends on only the difference $\eta(\mathbf{r}) - \eta(\mathbf{r}')$ and $\nabla\eta(\mathbf{r})$, a solution of Eq. (4b) for surfaces of sufficiently gentle slope can be found through a power-series expansion in the surface slopes, as was postulated in Ref. 11.

Equations (4) are integral Fredholm equations of the second kind; applicable solution methods have been worked out in detail. Here we will use an iterative procedure to solve Eqs. (4). The formal solution found as a result can be written in the form

$$U(\mathbf{r}_0,\eta(\mathbf{r}_0)) = \sum_{n=0}^{\infty}\frac{1}{(2\pi)^{2n}}\left\{\prod_{i=0}^{n}\int d\mathbf{r}_i\int d\mathbf{k}_i \times\exp[i\mathbf{k}_i(\mathbf{r}_{i-1}-\mathbf{r}_i)]\exp[i\nu_i|\eta(\mathbf{r}_{i-1}) -\eta(\mathbf{r}_i)|]O(\mathbf{r}_{i-1},\mathbf{r}_i,\mathbf{k}_i)\right\}U_0(\mathbf{r}_n,\eta(\mathbf{r}_n)), \quad (5)$$

where

$$U=\left\{\begin{matrix} V_D \\ U_N \end{matrix}\right\},\quad U_0=\left\{\begin{matrix} V_{D0} \\ U_{N0} \end{matrix}\right\} =\left\{\begin{matrix} -2i[\nu_0+\mathbf{k}_0\nabla\eta(\mathbf{r})] \\ 2 \end{matrix}\right\}\exp[i\mathbf{k}_0\mathbf{r}-i\nu_0\eta(\mathbf{r})],$$

$$O(\mathbf{r}_{i-1},\mathbf{r}_i,\mathbf{k}_i)=\left\{\operatorname{sign}(\eta(\mathbf{r}_{i-1})-\eta(\mathbf{r}_i))-\frac{\mathbf{k}_i}{\nu_i}\left\{\begin{matrix} \nabla\eta(\mathbf{r}_{i-1}) \\ \nabla\eta(\mathbf{r}_i) \end{matrix}\right\}\right\}.$$

Representation (5) is an expansion in the number of times the waves are scattered by the surface, as can be seen by rewriting expression (5) in the equivalent form

$$U(\mathbf{r}_0,\eta(\mathbf{r}_0)) = \sum_{n=0}^{\infty}\left(\frac{-i}{2\pi}\right)^n\left\{\prod_{i=0}^{n}\int d\mathbf{r}_i O(\mathbf{r}_{i-1},\mathbf{r}_i) \times G(\mathbf{r}_{i-1}-\mathbf{r}_i,\eta(\mathbf{r}_{i-1})-\eta(\mathbf{r}_i))\right\}U_0(\mathbf{r}_n,\eta(\mathbf{r}_n)), \quad (6)$$

where the integrals are to be understood in the principal-value sense, and the local reflection operators O are of the form

$$O(\mathbf{r}_{i-1},\mathbf{r}_i) = \mp i\left[\frac{\partial}{\partial\eta(\mathbf{r})}-\nabla\eta(\mathbf{r})\frac{\partial}{\partial\mathbf{r}}\right]\Bigg|_{\mathbf{r}=\left\{\begin{matrix} \mathbf{r}_{i-1} \\ \mathbf{r}_i \end{matrix}\right\}}.$$

Using representation (6), we can draw the following picture of the field propagation: The field which is incident on the boundary at the point $\mathbf{R}_n$ subsequently propagates freely to the next surface point, $\mathbf{R}_i$, where it is reflected from the surface (this is the effect of the operator O). The process then repeats itself in such a way that the field is reflected n times along the path from the point $\mathbf{R}_n$ to the point $\mathbf{R}_0$. The total field at $\mathbf{R}_0$ is written as a sum of fields with different numbers of reflections and with all possible initial and intermediate points of reflection from the surface. A finite number of terms in the sum in (5) thus reflects the part of the total field due to waves which are scattered no more than a given number of times.

A detailed study of the solution of Eq. (4) found by iterative procedure (6) in the limit $K\to\infty$ was carried out by Liska and McCoy,[10] who treated the case of $2D$ roughness. They showed that the component of the first term of the iterative series which stems from the region which is a neighborhood of the point $\mathbf{r}$ with radius $1/K$ results in the incorporation of diffraction corrections for the surface curvature. This curvature is small under the condition $(KR\cos^3\chi)^{-1}\ll 1$. The component due to the external region is related to the existence of stationary points determined by the equation

$$\frac{\mathbf{r}'}{[1+(\nabla\eta(\mathbf{r}'))^2]^{1/2}}\left\{\mathbf{k}_0-\nu_0\nabla\eta(\mathbf{r}') -iK\frac{(\mathbf{r}-\mathbf{r}')+[\eta(\mathbf{r})-\eta(\mathbf{r}')]\nabla\eta(\mathbf{r}')}{[(\mathbf{r}-\mathbf{r}')^2+(\eta(\mathbf{r})-\eta(\mathbf{r}'))^2]^{1/2}}\right\} \equiv(\mathbf{n}_0+\mathbf{e}_{R'})\boldsymbol{\tau}'=0,$$

where $\mathbf{n}_0$ is a unit vector along the direction of the incident wave, r' is the tangent to the surface at the point $\mathbf{r}'$, and $\mathbf{e}_{R'}$ is a unit vector along the direction connecting the surface points with coordinates $\mathbf{r}$ and $\mathbf{r}'$. If the condition $\mathbf{n}_0\boldsymbol{\tau}' = -\mathbf{e}_{R'}\boldsymbol{\tau}'$ holds, the stationary point is the point where repeated specular reflection occurs. If the condition $\mathbf{n}_0 = -\mathbf{e}_{R'}$ holds, the stationary point is in a shadow. In the limit $K\to\infty$, the contribution from the neighborhood of the stationary points can be found through a power-series expansion in the parameter $(KR)^{-1}$. This expansion contains contributions from K-independent terms. As was shown in Ref. 10, the K-independent contribution from the stationary points in shadows is canceled out exactly, in the first iteration, by corresponding contributions from iterative terms of higher orders. Overall, waves scattered specularly from the surface (possibly more than once) thus completely determine the K-independent contribution.

If there are no stationary points (specular or shadowed), i.e., if the condition $\tan\psi > 2\max|\nabla\eta|$ holds, where $\psi = 90° - \theta$ is the glancing angle, then the diffraction correction to the approximation of single wave scattering, $U(\mathbf{r}) = 2U_0(\mathbf{r})$, found by retaining the first iterative term in

(6), is[10] $2iU_0(\mathbf{r})/(2KR\cos^3\chi)$. As was stated above, this correction is small because of the small parameter $(KR\cos^3\chi)^{-1}\ll 1$.

These comments regarding the behavior of the solution of Eq. (4) in the limit $K\to\infty$ are necessary for a correct interpretation of the expressions derived below for mean values.

3. CALCULATION OF SCATTERING AMPLITUDES

The representation (5) can be used to calculate both the field itself and the scattering amplitude. In a statistical formulation of the problem, in which the realizations of the surface $z=\eta(\mathbf{r})$ are random, we are interested in calculating mean values. If only the $n=0$ term in sum (5) is taken into account, we find the approximation of singly scattered waves. This approximation is equivalent to the tangent-plane approximation, in which the surface or its derivative along the normal is twice its value in the incident wave.[1,6] When these values are substituted into integral representation (3) for the scattering amplitude, and an average is then taken over the ensemble of realizations of the surface $z=\eta(\mathbf{r})$, one finds the well-known expressions.[1,6] For a Gaussian process of random realizations of the surface, for example, the amplitude of the scattered field is given by

$$\langle A_{DN}{}^0(\mathbf{k})\rangle=\mp\delta(\mathbf{k}-\mathbf{k}_0)\exp(-2\sigma^2\nu_0^2), \tag{7}$$

where $\sigma^2=\langle\eta^2(\mathbf{r})\rangle$ is the mean square roughness height, and (here and in similar cases) the upper sign on the right side (the minus sign in this case) corresponds to the subscript D, while the lower one (the plus sign in this case) corresponds to N.

We now consider the contribution from doubly scattered waves [the $n=1$ term in (5)]. Substituting the $n=1$ term from (5) into (3), we find

$$\begin{Bmatrix} A_D{}^1(\mathbf{k}') \\ A_N{}^1(\mathbf{k}') \end{Bmatrix} = \mp\frac{1}{(2\pi)^4}\int d\mathbf{r}_0\int d\mathbf{r}_1\int d\mathbf{k}_1 \times\exp[i(\mathbf{k}_1-\mathbf{k}')\mathbf{r}_0+i(\mathbf{k}_0-\mathbf{k}_1)\mathbf{r}_1]$$
$$\times\begin{Bmatrix} \left[\operatorname{sign}(\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1))-\dfrac{\mathbf{k}_1\nabla\eta(\mathbf{r}_0)}{\nu_1}\right]\left[\dfrac{\nu_0}{\nu'}+\dfrac{\mathbf{k}_0\nabla\eta(\mathbf{r}_1)}{\nu'}\right] \\ \left[\operatorname{sign}(\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1))-\dfrac{\mathbf{k}_1\nabla\eta(\mathbf{r}_1)}{\nu_1}\right]\left[1-\dfrac{\mathbf{k}'\nabla\eta(\mathbf{r}_0)}{\nu'}\right] \end{Bmatrix}$$
$$\times\exp[-i\nu'\eta(\mathbf{r}_0)-i\nu_0\eta(\mathbf{r}_1)+i\nu_1|\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)|]. \tag{8}$$

Taking an average of this expression over the random realizations of the surface (the corresponding expressions for the mean values are given in the Appendix), and assuming that the roughness correlation function

$$W(\mathbf{r}_0-\mathbf{r}_1)=\langle\eta(\mathbf{r}_0)\eta(\mathbf{r}_1)\rangle/\sigma^2$$

depends on only the difference $\rho=\mathbf{r}_0-\mathbf{r}_1$, by virtue of the uniformity on the average over the surface, we find the following expressions for the mean scattering amplitudes [the integration over the variable $\mathbf{R}=\mathbf{r}_0+\mathbf{r}_1$, after the change of variables $(\mathbf{r}_0,\mathbf{r}_1)\to(\mathbf{R},\rho)$, generates a δ-function $(2\pi)^2\delta(\mathbf{k}'-\mathbf{k}_0)$:

$$\langle A_{D,N}^{1}(\mathbf{k}')\rangle=-\delta(\mathbf{k}'-\mathbf{k}_0)\frac{1}{(2\pi)^2}\int d\rho\int d\mathbf{k}\exp[i(\mathbf{k}-\mathbf{k}_0)\rho]$$
$$\times\exp\{-\nu_0^2\sigma^2[1+W(\rho)]\}\exp\{-\nu^2\sigma^2[1-W(\rho)]\}$$
$$\times\operatorname{erfc}\{-i\nu\sigma[1-W(\rho)]^{1/2}\}\left\{i\sigma^2\frac{\nu_0}{\nu}\mathbf{k}\nabla W(\rho)-i\sigma^2\frac{\nu}{\nu_0}\mathbf{k}_0\nabla W(\rho)\right.$$
$$\left.+\sigma^2\frac{(\mathbf{k}\nabla)(\mathbf{k}_0\nabla)W(\rho)}{\nu\nu_0}-\sigma^4\frac{\nu_0}{\nu}(\mathbf{k}\nabla W(\rho))(\mathbf{k}_0\nabla W(\rho))\right\}. \tag{9}$$

Expression (9) can be put in a more convenient form. Using the Efros theorem[19] to evaluate the integral over k (to do this, we need to switch from k_x,k_y to the new variables $p_x=ik_x,p_y=ik_y$, we find an expression for $\langle A^{1}_{D,N}(\mathbf{k}')\rangle$:

$$\langle A_{D,N}^{1}(\mathbf{k}')\rangle=\delta(\mathbf{k}'-\mathbf{k}_0)\frac{i}{(2\pi)}\int d\rho\exp(-i\mathbf{k}_0\rho)$$
$$\times\exp\{-\nu_0^2\sigma^2[1+W(\rho)]\}\int_0^\infty d\zeta Q(\zeta)\left\{\sigma^2\nu_0\left(\nabla W(\rho)\frac{\partial}{\partial\rho}\right)\right.$$
$$+i\frac{\sigma^2}{\nu_0}(\mathbf{k}_0\nabla W(\rho))\frac{\partial^2}{\partial\zeta^2}+i\frac{\sigma^2}{\nu_0}(\mathbf{k}_0\nabla)\left(\nabla W(\rho)\frac{\partial}{\partial\rho}\right)$$
$$\left.+i\sigma^4\nu_0(\mathbf{k}_0\nabla W(\rho))\left(\nabla W(\rho)\frac{\partial}{\partial\rho}\right)\right\}\frac{\exp[iK(\rho^2+\zeta^2)^{1/2}]}{(\rho^2+\zeta^2)^{1/2}}, \tag{10}$$

where

$$Q(\zeta)=\frac{1}{2\pi}\int dk\exp(ik\zeta)\exp\{-k^2\sigma^2$$
$$\times[1-W(\rho)]\}\operatorname{erfc}\{ik\sigma[1-W(\rho)]^{1/2}\}$$
$$=\frac{2}{\{4\pi\sigma^2[1-W(\rho)]\}^{1/2}}\exp\left\{-\frac{\zeta^2}{4\sigma^2[1-W(\rho)]}\right\}. \tag{11}$$

Since the last factor in (10) is the free-space Green's function and therefore satisfies the wave equation

$$\left[\Delta\rho+\frac{\partial^2}{\partial\zeta^2}+K^2\right]G(\rho,\zeta)=-\delta(\rho)\delta(\zeta),$$

we can find an expression for the second derivative $\partial^2G(\rho,\zeta)\partial\zeta^2$ from this equation. Noting that the term with the δ-function vanishes, we can rewrite (10) as

$$\langle A_{D,N}^{1}(\mathbf{k}')\rangle=\delta(\mathbf{k}'-\mathbf{k}_0)\frac{1}{(2\pi)}\int d\rho\exp(-i\mathbf{k}_0\rho)$$
$$\times\exp\{-\sigma^2\nu_0^2[1+W(\rho)]\}\left\{i\sigma^2\nu_0\left(\nabla W(\rho)\frac{\partial}{\partial\rho}\right)\right.$$
$$+\frac{\sigma^2}{\nu_0}(\mathbf{k}_0\nabla W(\rho))(\Delta_\rho+K^2)+\frac{\sigma^2}{\nu_0}(\mathbf{k}_0\nabla)\left(\nabla W(\rho)\frac{\partial}{\partial\rho}\right)$$
$$\left.-\sigma^4\nu_0(\mathbf{k}_0\nabla W(\rho))\left(\nabla W(\rho)\frac{\partial}{\partial\rho}\right)\right\}\int_0^\infty d\zeta\frac{2}{\{4\pi\sigma^2[1-W(\rho)]\}^{1/2}}$$
$$\times\exp\left\{-\frac{\zeta^2}{4\sigma^2[1-W(\rho)]}\right\}\frac{\exp[iK(\rho^2+\zeta^2)^{1/2}]}{(\rho^2+\zeta^2)^{1/2}}. \tag{12}$$

If the correlation function is isotropic, i.e., if $W(\rho)=W(\rho)$, expression (12) can be simplified further, through integration over the angular variable $\varphi (d\rho = \rho d\rho d\varphi)$ and a switch to the dimensionless variables $y = \rho/a, z = \zeta/a$ (a is the correlation radius). We find

$$\langle A^1_{D,N}(\mathbf{k}')\rangle = \delta(\mathbf{k}'-\mathbf{k}_0)\, i\sigma^2 a \int_0^\infty dy\, y\cdot$$

$$\times \exp\{-\sigma^2\nu_0^2[1+W(y)]\}\Big\{\frac{\nu_0}{a^2}W'(y)J_0(k_0ay)\frac{\partial}{\partial y}$$

$$-J_1(k_0ay)\Big[\frac{k_0}{\nu_0 a^3}W'(y)\Big(\frac{1}{y}\frac{\partial}{\partial y}y\frac{\partial}{\partial y}+1\Big)$$

$$+\frac{k_0}{\nu_0 a^3}\frac{\partial}{\partial y}W'(y)\frac{\partial}{\partial y}-\frac{\sigma^2\nu_0 k_0}{a^3}W'^2(y)\frac{\partial}{\partial y}\Big]\Big\}$$

$$\times\int_0^\infty dz\frac{2}{\{4\pi\sigma^2[1-W(y)]\}^{1/2}}\exp\Big\{-\frac{z^2a^2}{4\sigma^2[1-W(y)]}\Big\}$$

$$\times\frac{\exp[iKa(y^2+z^2)^{1/2}]}{(y^2+z^2)^{1/2}}, \qquad (13)$$

where $J_0(x)$ and $J_1(x)$ are Bessel functions.

After the change of variables $u^2 = z^2 + y^2$, th integral over z becomes

$$\exp\Big\{\frac{a^2y^2}{4\sigma^2[1-W(y)]}\Big\}\{\pi\sigma^2[1-W(y)]\}^{-1/2}$$

$$\times\int_y^\infty\frac{du}{(u^2-y^2)^{1/2}}\exp\Big\{-\frac{u^2a^2}{4\sigma^2[1-W(y)]}+iKau\Big\}. \qquad (14)$$

This integral is dominated by values of u near the lower limit, so we can assume $(u^2 - y^2)^{1/2} \approx [2y(u-y)]^{1/2}$. The integral found as a result can be evaluated exactly:[20]

$$\frac{1}{2}\frac{\{y-2iKa(\sigma^2/a^2)[1-W(y)]\}^{1/2}}{\{\pi y\sigma^2[1-W(y)]\}^{1/2}}$$

$$\times\exp\Big\{\frac{\sigma^2[1-W(y)]}{2a^2}\Big(\frac{a^2y}{2\sigma^2[1-W(y)]}+iKa\Big)^2\Big\}$$

$$\times K_{1/4}\Big(\frac{\sigma^2[1-W(y)]}{2a^2}\Big[\frac{a^2y}{2\sigma^2[1-W(y)]}-iKa\Big]^2\Big), \qquad (15)$$

where $K_{14}(z)$ is the modified Bessel function.

Under the two conditions

$$\Big|\frac{\sigma^2[1-W]}{2a^2}\Big(\frac{a^2y}{2\sigma^2[1-W]}-iKa\Big)^2\Big|\gg 1,\qquad \frac{ya^2}{2\sigma^2[1-W]}\gg Ka, \qquad (16)$$

expression (15) simplifies dramatically, becoming $\exp(iKay)/y$. The first of the inequalities (16) can be rewritten as

$$y^2\gg 8\frac{\sigma^2}{a^2}[1-W]-K^2a^2\Big(2\frac{\sigma^2}{a^2}[1-W]\Big)^2,$$

or, when we use the second inequality in (16),

$$y^2\gg 8\frac{\sigma^2}{a^2}[1-W(y)].$$

This inequality always holds for $y \gg \sigma/a$, while for $y \lesssim \sigma/a$ it may hold if $y \lesssim \sigma/a \lesssim 1$ (a is the roughness correlation radius) and $W''(0)(\sigma^2/a^2) \ll 1$ [we are making use of the fact that the expansion of the correlation function near its maximum is of the form $W(y) \approx 1 - \frac{1}{2}W''(0)y^2)$]. The first condition thus holds for arbitrary y if $\sigma \lesssim a$ and $W''(0)(\sigma^2/a^2) \ll 1$. The second of conditions (16) always holds for $y \gg K\sigma^2/a$; for $y \lesssim K\sigma^2/a$ it may hold if

$$y\leqslant K\sigma^2/a\leqslant 1,\qquad y\ll a/(K\sigma^2W''(0)).$$

In other words, for arbitrary y the second condition in (16) holds if the following inequalities hold:

$$K\sigma^2/a\leqslant 1,\qquad W''(0)(K\sigma^2/a)^2\ll 1.$$

From the physical standpoint, the meaning of these inequalities is that the mean slope angles of the roughness are small, $\sigma/a \ll 1$, and we are incorporating the fact that under the condition $K\sigma^2/a \ll 1$ the field of a doubly reflected wave is dominated by the waves calculated in the Fraunhofer approximation.

Analysis of (13) in the opposite limit, $K\sigma^2/a \gg 1$, $\sigma/a \lesssim 1$, i.e., in the geometric-optics limit, leads to an expression which shows that $\langle A^1\rangle$ is dominated by regions $y \gtrsim 1$ (the component of $\langle A^1\rangle$ which comes from these regions may be greater than the component of $\langle A\rangle$ resulting from the single-scattering approximation). It is clear from physical considerations that a systematic account of the shadowing for multiply reflected waves should have the result that such regions make a small contribution. This contradiction can be explained as follows: When we take an average with a Gaussian probability density of realizations of the random surface, we encounter, albeit with an exponentially small probability, realizations with arbitrarily large surface slopes. In this case, as was mentioned above, the number of shadowed points can be arbitrarily high, and since the contribution from each such point contains a K-independent part we find that although the relative number of realizations with large surface slopes is small their contribution may be large. We should emphasize that, as was pointed out in Ref. 10, the K-independent contribution from the shadowed points cancels out when the multiply scattered waves are taken into account. To take that approach, however, we need to examine a fairly large number of terms in the iterative series (5), i.e., deal with a very involved problem. Physically, a better approach to the solution of Eq. (4) is to exclude the contributions of the shadowed points at the outset, through an appropriate transformation of this equation. The derivation of a systematic theory incorporating shadowing in the multiple scattering of waves by a surface is a matter for the future, however.

In the present paper we consider the case in which the shadowed points are not important, and the iterative procedure (5) is correct in the sense that the correction terms calculated for the scattering amplitude in the Kirchoff approximation are small.

Assuming that the conditions

$$\sigma/a\ll 1,\qquad K\sigma^2/a\ll 1 \qquad (17)$$

hold, so that the integral is dominated by small values of y, we find an expression for the scattering amplitude in which double reflection is taken into account:

$$\langle A_{D,N}(\mathbf{k}')\rangle=\delta(\mathbf{k}'-\mathbf{k}_0)\exp(-2\nu_0^2\sigma^2)$$

$$\times\left[\mp1+i\frac{\sigma^2}{a^2}\int_0^\infty dy\, y\exp\left(-\frac{\sigma^2\nu_0^2}{2K^2a^2}y^2\right)\right.$$

$$\times\left\{\frac{\nu_0}{K}yJ_0(y\sin\theta)\frac{\partial}{\partial y}-J_1(y\sin\theta)\left[2\frac{k_0}{\nu_0}y\left(\frac{1}{y}\frac{\partial}{\partial y}y\frac{\partial}{\partial y}+Ka\right)\right.\right.$$

$$\left.\left.\left.+2\frac{k_0}{\nu_0}\frac{\partial}{\partial y}y\frac{\partial}{\partial y}-4\frac{\sigma^2}{a^2}\frac{\nu_0k_0}{K^2}y^2\frac{\partial}{\partial y}\right]\right\}\frac{\exp(iy)}{y}.$$

The requirement that double scattering contribute little to the scattering amplitude is satisfied if the conditions $\sigma^2/a^2\ll1$, $K\sigma^2/a\ll1$, $a^2/\sigma^2\gg\tan\theta$; hold. The last of these conditions is equivalent to the absence of any shadowing of the rough surface for the incident wave (our original assumption was that the entire surface was "illuminated" by the incident wave).

Note that when the Fresnel parameter $K\sigma^2/a$ is small one can derive a simplified expression for the scattering amplitude even before the averaging procedure, which becomes considerably more complicated when waves reflected larger numbers of times are taken into account. When the parameters σ/a and $K\sigma^2/a$ are small, the quantity $\nabla\eta(\mathbf{r})$ is small (the slope angles of the roughness are small), as is the combination $\nu_1|\eta(\mathbf{r})-\eta(\mathbf{r}')|$, which is equal to $K\sigma^2/a$ by virtue of the estimates $|\eta(\mathbf{r})-\eta(\mathbf{r}')|\lesssim\sigma$, $\nu_1=K\sin\theta_1\lesssim K\sigma/a$ (θ_1 is the angle between the $z=0$ plane and the direction of the rescattered wave). Physically, a small value of the parameter $\nu_1|\eta(\mathbf{r})-\eta(\mathbf{r}')|$ means that the integral in (8) is dominated by intermediate waves which have undergone repeated reflection in the first Fresnel zone with respect to the point of the preceding scattering. The contributions of the other waves cancel out. As the points of successive scattering become further and further apart, waves progressively closer to the horizontal contribute to the scattered field.

Expanding the exponential function containing the absolute value in (8), and retaining small terms of up to second order collectively in the parameters $\nabla\eta(\mathbf{r})$ and $\nu_1|\eta(\mathbf{r})-\eta(\mathbf{r}')|$, we find the following expression for the scattering amplitude $A^1_{D,N}(\mathbf{k}')$:

$$A^1_{D,N}(\mathbf{k}')=\mp\frac{1}{(2\pi)^4}\int d\mathbf{r}_0\int d\mathbf{r}_1\int d\mathbf{k}_1\exp[i(\mathbf{k}_1-\mathbf{k}')\mathbf{r}_0$$
$$+i(\mathbf{k}_0-\mathbf{k}_1)\mathbf{r}_1]$$
$$\times\exp[-i\nu'\eta(\mathbf{r}_0)-i\nu_0\eta(\mathbf{r}_1)]B^1_{D,N}(\mathbf{k}_0,\mathbf{k}_1,\mathbf{r}_0,\mathbf{r}_1),\qquad(18)$$

where

$$B_D^1(\mathbf{k}_0,\mathbf{k}_1,\mathbf{r}_0,\mathbf{r}_1)=\frac{\nu_0}{\nu'}\Big\{\operatorname{sign}[\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)]+i\nu_1[\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)]$$
$$+\frac{(i\nu_1)^2}{2}[\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)]|\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)|\Big\}-\frac{\nu_0}{\nu'}\frac{\mathbf{k}_1\nabla\eta(\mathbf{r}_0)}{\nu_1}$$
$$-i\frac{\nu_0}{\nu'}\mathbf{k}_1\nabla\eta(\mathbf{r}_0)|\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)|-\frac{\mathbf{k}_1\nabla\eta(\mathbf{r}_0)}{\nu_1}\frac{\mathbf{k}_0\nabla\eta(\mathbf{r}_1)}{\nu'},\qquad(19)$$

$$B_N^1(\mathbf{k}_0,\mathbf{k}_1,\mathbf{r}_0,\mathbf{r}_1)=\operatorname{sign}[\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)]+i\nu_1[\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)]$$
$$+\frac{(i\nu_1)^2}{2}[\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)]|\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)|-\frac{\mathbf{k}_1\nabla\eta(\mathbf{r}_1)}{\nu_1}$$
$$-i\mathbf{k}_1\nabla\eta(\mathbf{r}_1)|\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)|+\frac{\mathbf{k}_1\nabla\eta(\mathbf{r}_0)}{\nu'}\frac{\mathbf{k}_1\nabla\eta(\mathbf{r}_1)}{\nu_1}.$$

Taking an average (see the appendix), we find an expression for the mean scattering amplitudes:

$$\langle A^1_{D,N}(\mathbf{k}')\rangle=\delta(\mathbf{k}_0-\mathbf{k}')\frac{1}{(2\pi)^2}\int d\boldsymbol{\rho}\int d\mathbf{k}_1\exp[i(\mathbf{k}_1-\mathbf{k}_0)\boldsymbol{\rho}]$$
$$\times\left\{\frac{2}{\sqrt{\pi}}\sigma^3\nu_0(\mathbf{k}_1\nabla[1-W(\boldsymbol{\rho})]^{1/2})[1-W(\boldsymbol{\rho})]+i\sigma^2\frac{\nu_0}{\nu_1}\mathbf{k}_1\nabla W(\boldsymbol{\rho})\right.$$
$$\left.-i\sigma^2\frac{\nu_1}{\nu_0}\mathbf{k}_0\nabla W(\boldsymbol{\rho})-\frac{(\mathbf{k}_0\nabla)(\mathbf{k}_1\nabla)}{\nu_0^3\nu_1}\right\}\exp\{-\sigma^2\nu_0^2[1+W(\boldsymbol{\rho})]\}.\qquad(20)$$

Except for the first term, expression (20) is the same as (9), if only the first term in the expansion of the function

$$\exp\{-\nu^2\sigma^2[1-W(\boldsymbol{\rho})]\}\operatorname{erfc}[-i\nu\sigma(1-W(\boldsymbol{\rho}))^{1/2}]$$

in the parameter $\nu\sigma[1-W(\rho)]^{1/2}\ll1$, which is identical to $K\sigma^2/a\ll1$ is taken into account in that earlier expression. The first term in (20) corresponds to the incorporation of the next term in the expansion in this parameter; it can be ignored in this approximation. We see that when the conditions (17) hold a preliminary expansion of the exponential functions containing absolute values leads to the same result which we would find if we systematically carried out the averaging and further simplifications. This calculation method makes it possible to find the contribution to the scattering amplitude made by triply scattered waves of terms up to second order in the small parameters (17). The contribution from waves scattered a higher number of times is of a higher than second order in these small parameters. The contribution of triply scattered waves to the scattering amplitude is thus given, at this accuracy level, by

$$A^2_{D,N}(\mathbf{k}')=\mp\frac{1}{(2\pi)^6}\int d\mathbf{r}_0\int d\mathbf{r}_1\int d\mathbf{r}_2\int d\mathbf{k}_1\int d\mathbf{k}_2$$
$$\times\exp[i(\mathbf{k}_1-\mathbf{k}')\mathbf{r}_0+i(\mathbf{k}_2-\mathbf{k}_1)\mathbf{r}_1+i(\mathbf{k}_0-\mathbf{k}_2)\mathbf{r}_2]\qquad(21)$$
$$\times\exp[-i\nu'\eta(\mathbf{r}_0)-i\nu_0\eta(\mathbf{r}_2)]B^2_{D,N}(\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2,\mathbf{r}_0,\mathbf{r}_1,\mathbf{r}_2),$$

where

$$B_D^2(\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2,\mathbf{r}_0,\mathbf{r}_1,\mathbf{r}_2)=\frac{\nu_0}{\nu'}\Big\{-\nu_1\nu_2[\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)]$$
$$\times[\eta(\mathbf{r}_1)-\eta(\mathbf{r}_2)]-i\frac{\nu_1}{\nu_2}\mathbf{k}_2\nabla\eta(\mathbf{r}_1)[\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)]$$
$$-i\frac{\nu_2}{\nu_1}\mathbf{k}_1\nabla\eta(\mathbf{r}_0)[\eta(\mathbf{r}_1)-\eta(\mathbf{r}_0)]+\frac{\mathbf{k}_1\nabla\eta(\mathbf{r}_0)}{\nu_1}\frac{\mathbf{k}_2\nabla\eta(\mathbf{r}_1)}{\nu_2}\Big\},$$

$$B_N^2(\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2,\mathbf{r}_0,\mathbf{r}_1,\mathbf{r}_2)=-\nu_1\nu_2[\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)][\eta(\mathbf{r}_1)-\eta(\mathbf{r}_2)]$$
$$-i\frac{\nu_1}{\nu_2}\mathbf{k}_2\nabla\eta(\mathbf{r}_2)[\eta(\mathbf{r}_0)-\eta(\mathbf{r}_1)]-i\frac{\nu_2}{\nu_1}\mathbf{k}_1\nabla\eta(\mathbf{r}_1)[\eta(\mathbf{r}_1)-\eta(\mathbf{r}_2)]$$
$$+\frac{\mathbf{k}_1\nabla\eta(\mathbf{r}_1)}{\nu_1}\frac{\mathbf{k}_2\nabla\eta(\mathbf{r}_2)}{\nu_2}.$$

Taking the average of expression (21) is a lengthy process, but no difficulties arise. Making use of the uniformity on the average of the surface roughness, which means that the correlation functions depend on only the coordinate difference, $\langle \eta(\mathbf{r}_i)\eta(\mathbf{r}_j)\rangle = \sigma^2 W(\mathbf{r}_i - \mathbf{r}_j)$, and transforming to the new coordinate system $(\mathbf{r}_0,\mathbf{r}_1,\mathbf{r}_2) \to (\mathbf{R}, \boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$, where $\mathbf{R} = \mathbf{r}_0 + \mathbf{r}_1 + \mathbf{r}_2$, $\boldsymbol{\rho}_1 = \mathbf{r}_0 - \mathbf{r}_1$, $\boldsymbol{\rho}_2 = \mathbf{r}_1 - \mathbf{r}_2$, we find the following expression for the contribution of triply scattered waves to the amplitude:

$$\langle A^2_{D,N}(\mathbf{k}')\rangle = \mp\delta(\mathbf{k}_0-\mathbf{k}')\frac{1}{(2\pi)^4} \times \int d\boldsymbol{\rho}_1 \int d\boldsymbol{\rho}_2 \int d\mathbf{k}_1 \int d\mathbf{k}_2 \exp[i(\mathbf{k}_1-\mathbf{k}_0)\boldsymbol{\rho}_1] \times \exp[i(\mathbf{k}_2-\mathbf{k}_0)\boldsymbol{\rho}_2]\exp\{-\sigma^2\nu_0^2[1+W(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)]\} \times C_{D,N}(\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2), \qquad (22)$$

where

$$C_D = \sigma^2\nu_1\nu_2 D(1-\sigma^2\nu_0^2 D) + i\sigma^2\frac{\nu}{\nu_1}\mathbf{k}_1(\nabla_1 W_{1+2} - \nabla W_1 - \sigma^2\nu_0^2\nabla_1 W_{1+2}D) + i\sigma^2\frac{\nu_1}{\nu_2}\mathbf{k}_2 \times [\nabla W_1 + \sigma^2\nu_0^2(\nabla W_2 - \nabla W_1)D] - \frac{\sigma^2}{\nu_1\nu_2}[(\mathbf{k}_1\nabla)(\mathbf{k}_2\nabla)W_1 + \sigma^2\nu_0^2(\mathbf{k}_1\nabla_1 W_{1+2})(\mathbf{k}_2(\nabla W_2 - \nabla W_1))],$$

$$C_N = \sigma^2\nu_1\nu_2 D(1-\sigma^2\nu_0^2 D) + i\sigma^2\frac{\nu_2}{\nu_1}\mathbf{k}_1[\nabla W_2 + \sigma^2\nu_0^2(\nabla W_2 - \nabla W_1)D] + i\sigma^2\frac{\nu_1}{\nu_2}\mathbf{k}_2(\nabla_2 W_{1+2} - \nabla W_2 - \sigma^2\nu_0^2\nabla_2 W_{1+2}D) - \frac{\sigma^2}{\nu_1\nu_2}[(\mathbf{k}_1\nabla)(\mathbf{k}_2\nabla)W_2 + \sigma^2\nu_0^2(\mathbf{k}_1(\nabla W_2 - \nabla W_1))(\mathbf{k}_2\nabla_2 W_{1+2})]$$

and where we have introduced $W_1 = W(\boldsymbol{\rho}_1)$, $W_2 = W(\boldsymbol{\rho}_2)$, $W_{1+2} = W(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)$, and $D = 1 + W_{1+2} - W_1 - W_2$. A subscript on ∇ specifies the variable on which this operator is acting. Expressions (7), (9), and (22) determine the scattering amplitude up to and including quantities of second order in the small parameters (17). For a small roughness, $\sigma\nu_0 \ll 1$, we find the following expressions, retaining the terms to within $\sigma^2\nu_0^2$ inclusively in expressions (7), (9), and (22):

$$\langle A_{D,N}(\mathbf{k}')\rangle = \mp\delta(\mathbf{k}'-\mathbf{k}_0)\Big[\sigma^2\nu_0^2 + \frac{\sigma^2}{(2\pi)^2}\int d\boldsymbol{\rho}\int d\mathbf{k} \times \exp[i(\mathbf{k}-\mathbf{k}_0)\boldsymbol{\rho}]\Big\{(\nu^2-2\nu_0\nu)W(\boldsymbol{\rho}) + i\mathbf{k}\nabla W(\boldsymbol{\rho}) - i\frac{\nu_0}{\nu}\mathbf{k}\nabla W(\boldsymbol{\rho}) + i\frac{\nu}{\nu_0}\mathbf{k}_0\nabla W(\boldsymbol{\rho}) - \frac{(\mathbf{k}\nabla)(\mathbf{k}_0\nabla)W(\boldsymbol{\rho})}{\nu\nu_0}\Big\}\Big] \qquad (23)$$

$$\langle A^0_{D,N}(\mathbf{k}')\rangle = \mp\delta(\mathbf{k}'-\mathbf{k}_0)[1-2\sigma^2\nu_0^2],$$

$$\langle A^1_{D,N}(\mathbf{k}')\rangle = -\delta(\mathbf{k}'-\mathbf{k}_0)\frac{\sigma^2}{(2\pi)^2}\int d\boldsymbol{\rho}\int d\mathbf{k}\exp[i(\mathbf{k}-\mathbf{k}_0)\boldsymbol{\rho}], \times\Big[i\frac{\nu_0}{\nu}\mathbf{k}\nabla W(\boldsymbol{\rho}) - i\frac{\nu}{\nu_0}\mathbf{k}_0\nabla W(\boldsymbol{\rho}) + \frac{(\mathbf{k}\nabla)(\mathbf{k}_0\nabla)W(\boldsymbol{\rho})}{\nu\nu_0}\Big],$$

Correspondingly, we have the following expressions for the total scattering amplitudes for scattering by rough surfaces:

$$\langle A_D(\mathbf{k}')\rangle = -\delta(\mathbf{k}'-\mathbf{k}_0)\Big[1-\sigma^2\nu_0^2 + \frac{\sigma^2}{(2\pi)^2}\int d\boldsymbol{\rho}\int d\mathbf{k} \times \exp[i(\mathbf{k}-\mathbf{k}_0)\boldsymbol{\rho}]\{(\nu^2-2\nu_0\nu)W(\boldsymbol{\rho}) + i\mathbf{k}\nabla W(\boldsymbol{\rho})\}\Big], \qquad (24a)$$

$$\langle A_N(\mathbf{k}')\rangle = \delta(\mathbf{k}'-\mathbf{k}_0)\Big[1-\sigma^2\nu_0^2 + \frac{\sigma^2}{(2\pi)^2}\int d\boldsymbol{\rho}\int d\mathbf{k} \times \exp[i(\mathbf{k}-\mathbf{k}_0)\boldsymbol{\rho}]\Big\{(\nu^2-2\nu_0\nu)W(\boldsymbol{\rho}) + i\mathbf{k}\nabla W(\boldsymbol{\rho}) - 2i\frac{\nu_0}{\nu}\mathbf{k}\nabla W(\boldsymbol{\rho}) + 2i\frac{\nu}{\nu_0}\mathbf{k}_0\nabla W(\boldsymbol{\rho}) - 2\frac{(\mathbf{k}\nabla)(\mathbf{k}_0\nabla)W(\boldsymbol{\rho})}{\nu\nu_0}\Big\}\Big]. \qquad (24b)$$

Expressions (24a) and (24b) can be simplified by integrating by parts the terms containing gradients of the correlation function:

$$\langle A_D(\mathbf{k}')\rangle = -\delta(\mathbf{k}_0-\mathbf{k}')\Big[1 - \frac{2\nu_0\sigma^2}{(2\pi)^2}\int d\mathbf{k}\,\nu W(\mathbf{k}-\mathbf{k}_0)\Big], \qquad (25a)$$

$$\langle A_N(\mathbf{k}')\rangle = \delta(\mathbf{k}_0-\mathbf{k}')\Big[1 - \frac{2\sigma^2}{(2\pi)^2}\int d\mathbf{k}\frac{(K^2-\mathbf{k}_0\mathbf{k})^2}{\nu_0\nu}W(\mathbf{k}-\mathbf{k}_0)\Big], \qquad (25b)$$

where

$$W(\mathbf{k}-\mathbf{k}_0) = \int d\boldsymbol{\rho}\exp[i(\mathbf{k}-\mathbf{k}_0)\boldsymbol{\rho}]W(\boldsymbol{\rho})$$

is the Fourier spectrum of the correlation function, and where we have used the equation

$$\sigma^2\nu_0^2 - \frac{\sigma^2}{(2\pi)^2}\int d\boldsymbol{\rho}\int d\mathbf{k}\exp[i(\mathbf{k}-\mathbf{k}_0)\boldsymbol{\rho}]W(\boldsymbol{\rho})(K^2-\mathbf{k}_0\mathbf{k}) = 0.$$

Expressions (25a) and (25b) are completely identical to the results found through the use of standard perturbation theory.[1] In particular, using the chain of equalities

$$\frac{i}{2\pi}\int d\mathbf{k}\,\nu\exp(i\mathbf{k}\boldsymbol{\rho}) = \lim_{z\to 0}\frac{i}{(2\pi)}\int d\mathbf{k}\,\nu\exp(i\mathbf{k}\boldsymbol{\rho}+i\nu z) = -\lim_{z\to 0}\frac{i}{(2\pi)}\frac{\partial^2}{\partial z^2}\int d\mathbf{k}\frac{\exp(i\mathbf{k}\boldsymbol{\rho}+i\nu z)}{\nu} = -\lim_{z\to 0}\frac{\partial^2}{\partial z^2}\frac{\exp[iK(\rho^2+z^2)^{1/2}]}{(\rho^2+z^2)^{1/2}},$$

we can easily put expression (25a) in the form

$$\langle A_D(\mathbf{k}')\rangle = -\delta(\mathbf{k}_0-\mathbf{k}')\Big\{1 - \frac{2i\sigma^2\nu_0}{(2\pi)}\lim_{z\to 0}\frac{\partial^2}{\partial z^2} \times \int d\boldsymbol{\rho}\exp(-i\mathbf{k}_0\boldsymbol{\rho})W(\boldsymbol{\rho})\frac{\exp[iK(\rho^2+z^2)^{1/2}]}{(\rho^2+z^2)^{1/2}}\Big\}.$$

Further manipulations of this expression are carried out as in Ref. 1; they lead to the following expression in the case of an isotropic correlation function:

$$\langle A_D(\mathbf{k}')\rangle=-\delta(\mathbf{k}_0-\mathbf{k}')\Big\{1-2K\nu_0\sigma^2+2i\sigma^2\nu_0\int_0^\infty d\rho\frac{e^{iK\rho}}{\rho}\frac{\partial}{\partial\rho}[J_0(k_0\rho)W(\rho)]\Big\}$$

a detailed analysis of the latter integral for various limiting cases has been carried out by Bass and Fuks.[1]

4. MEAN FIELD

The structure of the mean field is of considerable interest. Substituting the expresions for the fields at the surface in the single-scattering approximation into the exact expressions (1a) and (1b) for the field, we find the following expression for the mean field:

$$\begin{Bmatrix}\langle U_D(\mathbf{r},z)\rangle\\ \langle U_N(\mathbf{r},z)\rangle\end{Bmatrix}=\exp(i\mathbf{k}_0\mathbf{r}-i\nu_0 z)\mp\exp(i\mathbf{k}_0\mathbf{r})\times\Big\langle\Big[\begin{Bmatrix}1\\ \operatorname{sign}(z-\eta(\mathbf{r}'))\end{Bmatrix}\pm\frac{\mathbf{k}_0\nabla\eta(\mathbf{r}')}{\nu_0}\Big]\times\exp[-i\nu_0\eta(\mathbf{r}')+i\nu_0|z-\eta(\mathbf{r}')|]\Big\rangle.$$

Away from the boundary $[z<\min\eta(\mathbf{r})]$, the mean field is zero. This result is in complete accordance with the physics of the process. If the random realizations of the surface are Gaussian, we find the following expression, making use of the expressions for the mean values given in the Appendix:

$$\langle U_{D,N}(\mathbf{r},z)\rangle=\exp(i\mathbf{k}_0\mathbf{r}-i\nu_0 z)\mp\frac{1}{2}\exp(i\mathbf{k}_0\mathbf{r}-i\nu_0 z)\exp\Big(-\frac{z^2}{2\sigma^2}\Big)\times\Big\{\exp\Big[\frac{(z+2i\sigma^2\nu_0)^2}{2\sigma^2}\Big]\operatorname{erfc}\Big[-\frac{z+2i\sigma^2\nu_0}{(2\sigma^2)^{1/2}}\Big]\times\exp\Big(\frac{z^2}{2\sigma^2}\Big)\operatorname{erfc}\Big[\frac{z}{(2\sigma^2)^{1/2}}\Big]\Big\}. \quad (26)$$

Under the conditions $z/\sigma\gg1$ and $z/\sigma\gg\sigma\nu_0$ (this is actually the Fraunhofer approximation) we can use the asymptotic expressions $\operatorname{erfc}(z)=\exp(-z^2)/(z\sqrt{\pi})$ as $z\to\infty$ and $\operatorname{erfc}(z)=2-\exp(-z^2)/(-z\sqrt{\pi})$ as $z\to-\infty$. We find the standard expression for the mean field in the Kirchoff approximation for positive z:

$$\langle U_{D,N}(\mathbf{r},z)\rangle=\exp(i\mathbf{k}_0\mathbf{r}-i\nu_0 z)\mp\exp(i\mathbf{k}_0\mathbf{r}+i\nu_0 z)\exp(-2\sigma^2\nu_0^2).$$

For large negative values of z we find from (26)

$$\langle U_{D,N}(\mathbf{r},z)\rangle=\Big(\frac{2}{\pi}\Big)^{1/2}\frac{\sigma}{(-z)}\exp\Big(-\frac{z^2}{2\sigma^2}\Big)\exp(i\mathbf{k}_0\mathbf{r}-i\nu_0 z).$$

In other words, at large negative values of z the mean field is the incident wave, damped exponentially over depth. With a Gaussian distribution of roughness heights we have $\min\eta(\mathbf{r})\to-\infty$, so this result does not contradict the vanishing of the mean field at $z<\min\eta(\mathbf{r})$.

Near the mean plane, $z=0$, and under the conditions $z\ll\sigma$ and $z\ll\sigma^2\nu_0$, the structure of the mean field is given by

$$\langle U_{D,N}(\mathbf{r},z)\rangle={}^1/_2\exp(i\mathbf{k}_0\mathbf{r}-i\nu_0 z)\mp{}^1/_2\exp(i\mathbf{k}_0\mathbf{r}+i\nu_0 z)\exp(-2\sigma^2\nu_0^2)\operatorname{erfc}(-2^{1/2}i\sigma\nu_0). \quad (27)$$

It is interesting to note here that the incident wave has half the amplitude, and the amplitude of the specularly reflected wave is determined by the factor $\frac{1}{2}\exp(-2\sigma^2\nu_0^2)\operatorname{erfc}(-2^{1/2}i\sigma\nu_0)$. It can also be seen from (27) that the mean reflected field is formed by both the secondary incident waves (reflected from higher parts of the surface) and reflected waves. According to the Rayleigh assumption, one would seek this contribution in the form of reflected waves alone.

5. CONCLUSION

Summarizing, we have systematically calculated the scattering amplitude of the mean field reflected from absolutely hard and soft surfaces without using the Rayleigh assumption. The expressions derived here contain as limiting cases the results of the Kirchoff method and the small-perturbation method. The ranges of applicability of the expressions derived here have been found. These ranges are determined by the condition that the mean slope angles of the roughness be small, $\sigma/a\ll1$, and the condition that the Fresnel parameter be small, $K\sigma^2/a\ll1$. No restriction is imposed on the Rayleigh parameter $\sigma^2\nu_0^2$. Accordingly, the basic restriction on the properties of the roughness is that the slopes be gentle. Several results found through this approximation agree with results derived in Refs. 11–13.

In the limit $K\to\infty$, $K\sigma^2/a\gg1$, which corresponds to the geometric-optics approximation, the only calculation which would be valid without consideration of shadowing is a calculation of singly reflected waves. (Calculations for waves reflected more than once absolutely must include a shadowing factor: The small value of the glancing angle of the rereflected waves, $\theta_1\sim\sigma/a\ll1$, has the consequence that in this limit regions far from the surface point near which the field is being calculated dominate, while it is clear from physical considerations that such a situation would be unlikely.)

Another result of this study which we regard as important comes from the analysis of the behavior of the mean field near the boundary. It has been shown that the mean field has a structure more complex than simply a superposition of the incident and specularly reflected waves. The assumption that this simple superposition is valid was used in Ref. 1, for example, in a renormalization of the reflection coefficient and in a study of the possibility that surface waves could exist at rough surfaces. A consideration of the actual structure of the mean field may lead to some revision of the results in that area.

APPENDIX

The nonanalytic expressions which arose in the text proper can be averaged exactly in the case of a Gaussian probability density of random realizations of the surface. The resulting formulas for certain mean values are

1. $$\Big\langle\exp\Big(i\sum_{j=1}^{n}\nu_j\eta(\mathbf{r}_j)\Big)\Big\rangle=\exp\Big[-\frac{\sigma^2}{2}\sum_{\substack{l=1\\ j=1}}^{n}\nu_j\nu_l W(\mathbf{r}_j-\mathbf{r}_l)\Big]$$

2. $$\langle\operatorname{sign}[\eta(\mathbf{r})-\eta(\mathbf{r}')]\exp[-i\nu\eta(\mathbf{r})-i\nu'\eta(\mathbf{r}')]\rangle=\exp\{-{}^1/_4(\nu+\nu')^2\sigma^2[1+W(\rho)]\}\exp\{-{}^1/_4(\nu-\nu')^2\sigma^2\times[1-W(\rho)]\}\operatorname{erfc}\{{}^1/_2 i(\nu-\nu')\sigma[1-W(\rho)]^{1/2}\}.$$

3.

$$\left\langle\left\{\begin{matrix}1\\ \operatorname{sign}[\eta(\mathbf{r})-\eta(\mathbf{r}')]\end{matrix}\right\}\exp[-i\nu_0(\eta(\mathbf{r})+\eta(\mathbf{r}'))+i\nu|\eta(\mathbf{r})-\eta(\mathbf{r}')|]\right\rangle=\left\{\begin{matrix}1\\0\end{matrix}\right\}\exp\{-\nu_0^2\sigma^2[1+W(\rho)]\}$$

$$\times\exp\{-\nu^2\sigma^2[1-W(\rho)]\}\operatorname{erfc}\{-i\nu\sigma[1-W(\rho)]^{1/2}\}.$$

4.

$$\left\langle\left\{\begin{matrix}1\\ \operatorname{sign}[z-\eta(\mathbf{r})]\end{matrix}\right\}\exp[-i\nu_0\eta(\mathbf{r})+i\nu|z-\eta(\mathbf{r})|]\right\rangle$$

$$=\frac{1}{2}\exp(-i\nu_0 z)\exp\left(-\frac{z^2}{2\sigma^2}\right)\left[\exp\left\{\frac{[z+i\sigma^2(\nu+\nu_0)]^2}{2\sigma^2}\right\}\times\operatorname{erfc}\left[-\frac{z+i\sigma^2(\nu+\nu_0)}{(2\sigma^2)^{1/2}}\right]\pm\exp\left\{\frac{[z+i\sigma^2(\nu_0-\nu)]^2}{2\sigma^2}\right\}\times\operatorname{erfc}\left[\frac{z+i\sigma^2(\nu_0-\nu)}{(2\sigma^2)^{1/2}}\right].$$

5.

$$\left\langle\mathbf{k}\nabla\eta(\mathbf{r}')\left\{\begin{matrix}1\\ \operatorname{sign}[\eta(\mathbf{r})-\eta(\mathbf{r}')]\end{matrix}\right\}\exp\{-i\nu_0[\eta(\mathbf{r})+\eta(\mathbf{r}')]+i\nu_1|\eta(\mathbf{r})-\eta(\mathbf{r}')|\}\right\rangle$$

$$=i\sigma^2\mathbf{k}\nabla W(\rho)\left\{\begin{matrix}-\nu_0\\ \nu_1\end{matrix}\right\}\exp\{-\nu_0^2\sigma^2[1+W(\rho)]\}\cdot$$

$$\times\exp\{-\nu_1^2\sigma^2[1-W(\rho)\}\operatorname{erfc}\{-i\nu_1\sigma[1-W(\rho)]^{1/2}\}$$

$$-\left\{\begin{matrix}0\\1\end{matrix}\right\}(2\sigma/\pi)\exp\{-\nu_0^2\sigma^2[1+W(\rho)]\}\mathbf{k}\nabla[1-W(\rho)]^{1/2}.$$

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