Control of ultrarelativistic electron beam instability in a nonlinear plasma

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(Submitted 1 January 12 1990; resubmitted 28 March 1990)
Zh. Eksp. Teor. Fiz. **98**, 419–433 (August 1990)

The instability of a modulated ultrarelativistic electron beam in a dense plasma is studied in the case in which phase oscillations in the beam are suppressed by the relativistic increase in the electron mass, and the nonlinear mechanism by which the oscillation amplitude reaches saturation is described by a functional dependence of the dielectric constant of the plasma on the field amplitude. External modulation of the beam is an effective tool for controlling the instability process. Under certain relations among the modulation frequency, the plasma frequency of the plasma, and the growth rate, this modulation prevents the system from entering the turbulent regime. There is thus the possibility in principle of beam transport over large distance through a plasma. The discrete set of modulation frequencies for which there exists a solitary-pulse solution and for which energy is returned to the beam is found by numerical and analytic methods.

A preliminary modulation of an electron beam as it enters a plasma delays the transition of the system to a turbulent regime and promotes conversion of beam energy into the energy of a monochromatic wave, with a frequency equal to the modulation frequency.[1] For nonrelativistic beams of low density $\nu = n_b/n_p \ll 1$ (n_b and n_p are the beam density and the plasma density, respectively), the growth of the wave amplitude is stabilized by a trapping of resonant electrons by the wave and by a bunching of the beam.[2,3] The same effect limits the wave amplitude in the case of a "slightly relativistic" beam, with $\gamma\nu^{1/3} \ll 1$, where γ is the relativistic factor.[4]

At ultrarelativistic energies, $\gamma\nu^{1/3} \gg 1$, the nature of the instability changes, since the phase oscillations are suppressed by the relativistic increase in the mass of the beam electrons (the frequency of the phase oscillations, Ω_{ph}, is smaller than the growth rate δ), and the field amplitude reaches saturation because the dielectric constant of the plasma depends on the amplitude and the phase velocity of the wave decreases.[5] In the latter case, the wave amplitude in the beam is small ($|\tilde{n}_b| \ll n_b$), and conditions become favorable for the transport of intense electron beams along a strong magnetic field through a plasma without any substantial changes in the beam properties.[1)]

The equation for the complex field amplitude in a plasma with an ultrarelativistic electron beam is[5]

$$2i\frac{d^3y}{d\tau^3} - \frac{d^2}{d\tau^2}[(\alpha+|y|^2)y] - y = 0,$$

$$y = \frac{E}{E_p}, \quad E_p^2 = \frac{32\pi}{3}\frac{\nu^{1/3}}{\gamma}n_p mc^2, \quad \alpha = \frac{\gamma\varepsilon}{\nu^{1/3}},$$

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}, \quad \tau = \nu^{1/3}\omega_p t/\gamma. \tag{1}$$

This equation describes undamped nonlinear waves[6] which grown in time in the linear stage of the instability, with a growth rate which depends on the parameter α, for an arbitrary relation between the plasma frequency ω_p and the modulation frequency ω (Fig. 1).

The existence of such waves in a plasma should clearly be accompanied by trapping of resonant electrons by a wave over a time on the order of Ω_{ph}^{-1} and by significant deviations from the original values of the beam properties. It is thus worthwhile to study beam modulation regimes under condition such that field energy is returned to the beam, and the properties of the beam are unable to undergo irreversible changes as a result of phase mixing. In other words, we seek solutions of Eq. (1) which satisfy the condition $d^n y/d\tau^n = 0$ ($n = 1, 2, \ldots$) at $\tau = \pm\infty$.

As is shown by the numerical study reported below, solutions of this type exist for a discrete set α_n of parameter values. The ratio of the period of the nonlinear field oscillations to the period of the beam density oscillations is plotted along the ordinate in Fig. 2a. An analytic solution has been found in the adiabatic approximation for the case $|\alpha| \gg 1$, in which Eq. (1) becomes an equation in which the highest derivative is multiplied by a small parameter.[5] The results area illustrated by Fig. 2a, which compares the adiabatic theory with the numerical simulation.

In contrast with the linear boundary-value problems of quantum mechanics, the problem at hand is nonlinear; by solving it one finds eigenvalues and eigenfunctions of a nonlinear operator. The reason is that the change of scales $y_1 = |\alpha|^{-1/2}y$ and $\tau_1 = |\alpha|\tau$ puts Eq. (1) in the form

$$\hat{F}(y_1)y_1 = \Lambda y_1,$$

$$\hat{F}(y_1) = 2i\frac{d^3}{d\tau_1^3} - \frac{d^2}{d\tau_1^2}(\text{sign}\,\alpha + |y_1|^2), \quad \Lambda = |\alpha|^{-3}.$$

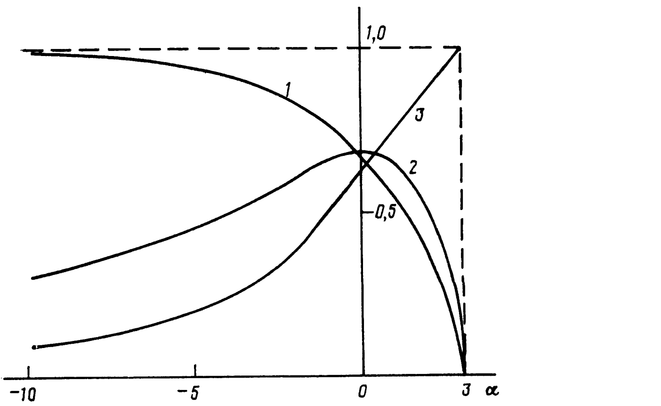

FIG. 1. Linear theory. Plots of several properties versus the parameter α. *1*—The initial difference phase of the beam; *2*—the growth rate; *3*—the initial ratio of the field amplitude a to the beam density modulation N.

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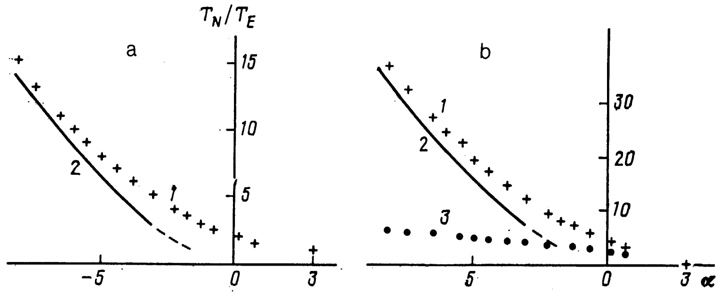

FIG. 2. a: *1*—Ratio of the period of the nonlinear density oscillations, T_N, to the period of the nonlinear field oscillations, T_E, versus the parameter α; *2*—curve calculated in the adiabatic approximation. b: α dependence of (1,2) the density maximum N_m found through a numerical solution and in the adiabatic approximation, respectively, and (3) the largest of the field amplitude maxima $a_{m,m}$.

On the practical side, this study leads to the conclusion that there exists a discrete set ω_n, $\alpha(\omega_n)=\alpha_n$, of modulation frequencies for which an ultrarelativistic beam has nonlinear stability in a plasma.

1. SYSTEM OF NONLINEAR EQUATIONS

According to Ref. 5, the system of hydrodynamic equations describing the oscillations of a nonliner plasma with an ultrarelativistic beam can be transformed through the change of variables

$$E(t,x)=i\left(\frac{32\pi}{3}\frac{v^{1/3}}{\gamma}n_p mc^2\right)^{1/2} a(t)\exp[i(\Phi-\theta)],$$

$$\tilde{n}_b(t,x)=n_b\left(\frac{8}{3v\gamma^3}\right)^{1/2} N(t)\exp[i(\Phi-\varphi)]$$

[$\Phi=\omega(t-x/v)$, where v is the beam velocity] to a system of equations for the "slow" amplitudes and phases of the field, a, θ, and of the beam density modulation, N, φ:

$$\dot{a}=-\frac{N}{2}\sin(\theta-\varphi),\qquad \ddot{N}-\dot{\varphi}^2N=-a\cos(\theta-\varphi),$$

$$\dot{\theta}=\frac{1}{2}(\alpha+a^2)-\frac{N}{2a}\cos(\theta-\varphi),\qquad \frac{d}{d\tau}(N^2\dot{\varphi})=-aN\sin(\theta-\varphi),\tag{2}$$

This system of equations is equivalent to Eq. (1) (the superior dot means the derivative $d/d\tau$).

From Eqs. (2) we find the integrals of motion

$$H=\frac{\dot{N}^2}{4}-\frac{a^2}{4}\left(\alpha+\frac{a^2}{2}\right)+\frac{1}{2}aN\cos(\theta-\varphi)+\frac{(P-a^2)^2}{4N^2},$$

$$P=a^2-N^2\dot{\varphi},\tag{3}$$

which reflect energy and momentum conservation in a plasma with a beam. If there are no waves in the plasma ($H=P=0$), we can lower the order of Eqs. (2) by transforming to the difference phase $\eta=\theta-\varphi$ and using (3)[2]:

$$\dot{a}=-\frac{N}{2}\sin\eta,$$

$$\dot{\eta}=\frac{1}{2}(\alpha+a^2)-\frac{N}{2a}\cos\eta-\frac{a^2}{N^2},$$

$$\dot{N}=\left[a^2\left(\alpha+\frac{a^2}{2}\right)-2aN\cos\eta-\frac{a^4}{N^2}\right]^{1/2}.\tag{4}$$

Working in the linear approximation, and omitting the terms $a^2/2$ and $a^4/2$ (which reflect the nonlinear dependence of the dielectric constant of the plasma on the field amplitude) from the second and third equations of system (4), we find the following results for an exponentially growing solution, $\propto\exp(\delta\tau)$, with $\delta>0$:

$$\delta=3^{1/2}(h_+^2-h_-^2),\qquad \cos\eta=\tfrac{1}{2}[\alpha(h_++h_-)-1],$$

$$\frac{a}{N}=h_++h_-,\qquad h_\pm=\frac{1}{2}\left\{1\pm\left[1-\left(\frac{\alpha}{3}\right)^3\right]^{1/2}\right\}^{1/3}.\tag{5}$$

These functions of the parameter α are shown in Fig. 1. The instability occurs after a threshold is reached, and it occurs under the condition $\alpha<3$. The growth rate reaches a maximum of $\delta_m=3^{1/2}\cdot2^{-4/3}$ at $\alpha=0$ and decreases, $\delta\approx(-\alpha)^{-1/2}$, as $\alpha\to-\infty$ (Ref. 10).

2. NUMERICAL SIMULATION

For numerical calculations it is convenient to put Eqs. (2) in the form of canonical equations

$$\dot{p}=\frac{\partial H}{\partial q},\qquad \dot{q}=-\frac{\partial H}{\partial p},\qquad \dot{p}_N=\frac{\partial H}{\partial N},\qquad \dot{N}=-\frac{\partial H}{\partial p_N}\tag{6}$$

for the two pairs of canonical variables $p=a\sin\eta$, $q=a\times\cos\eta$ and $p_N=1/2\dot{N}$, N with the Hamiltonian

$$H(p_N,N;p,q)=p_N^2-\frac{a^2}{4}\left(\alpha+\frac{a^2}{2}-\frac{a^2}{N^2}\right)+\frac{1}{2}qN.\tag{7}$$

Since the function $H(p_N,N;p,q)$ is an even function of the variables p_N and p, Eqs. (6) are symmetric under the transformations $p\to-p$, $p_N\to-p_N$, $\tau\to-\tau$. If a phase trajectory which develops in a 3D region $H(p_N,N;p,q)=0$ of phase space intersects the curve

$$H(p_N=0,\ N;\ \ p=0,\ q)=0\tag{8}$$

at some time $\tau=0$, its evolution will therefore be symmetric with respect to this event. Accordingly, a trajectory of this sort, which emerges at $\tau\to-\infty$ from a point of an unstable equilibrium position $p=p_N=q=N=0$, returns to it as $\tau\to+\infty$. Numerical calculations show that symmetric solutions of this sort do indeed exist.

Equations (6) with initial conditions (5) were integrated over a wide range of values of the parameter α. The points on the curves in Fig. 2 show symmetric solutions for the discrete set of values $\alpha_n=0.7362$, 0.09298, -0.7307, -1.2716, -1.7771, -2.2661, -3.08056, -3.7901, -4.4267, -5.0115, -5.5571, -6.0711, -6.543, -7.478, and -8.2112. These points are seen to conform to smooth curves. Each value of α on curve 1 (Fig. 2a) corresponds to nonlinear oscillations with a given ratio of the

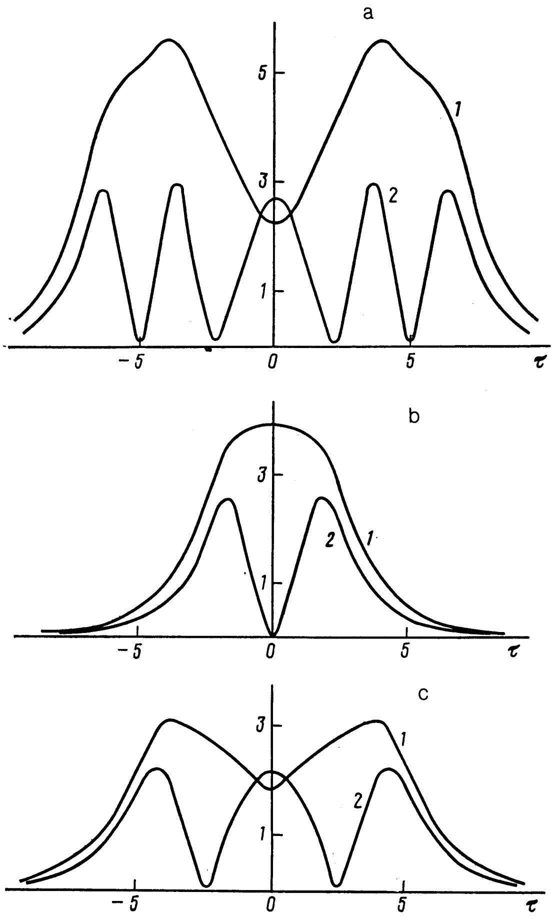

FIG. 3. Symmetric solutions for various values of α and n_E/n_N. *1*—Oscillations of the density N; *2*—oscillations of the field a. a: $\alpha = -0.7307$, $n_E n_N = 5/2$; b: $\alpha = 0.09298$, $n_E/n_N = 2/1$; c: $\alpha = 0.73625$, $n_E/n_N = 3/2$.

oscillation periods of the field, T_E, and the density, T_N. If this ratio is a rational number n_E/n_N, Eqs. (6) have a symmetric solution, with n_E field maxima and n_N density maxima, which corresponds to a return regime of the beam-plasma instability. Corresponding to these values of the parameter α_n are the ratios $n_E/n_N = 3/2$, 2/1, 5/2, 3/1, 7/2, 4/1, 5/1, 6/1, 7/1, 8/1, 9/1, 10/1, 11/1, 13/1, and 15/1.

Figure 3 shows examples of symmetric solutions in the immediate vicinity of the resonance. Figure 4 shows projections of a phase trajectory onto the N, $\dot N$ and q, p planes for the case $\alpha = 0.09298$.

3. SOLITON SOLUTION NEAR THE INSTABILITY THRESHOLD

Since the instability is stopped by a nonlinearity at a low field amplitude near the threshold growth rate, $1 - \alpha/3 \ll 1$, one can expand Eq. (4) in powers of $|\eta| \ll 1$,

$$\dot a = -\frac{N}{2}\eta, \quad \dot\eta = \frac{1}{2}(\alpha + a^2) - \frac{N}{2a}\left(1 - \frac{\eta^2}{2}\right) - \frac{a^2}{N^2},$$
$$\dot N = \left[\left(\alpha + \frac{a^2}{2}\right)a^2 - 2aN\left(1 - \frac{\eta^2}{2}\right) - \frac{a^4}{N^2}\right]^{1/2} \tag{9}$$

and use the linear asymptotic behavior in (5) as an initial condition,

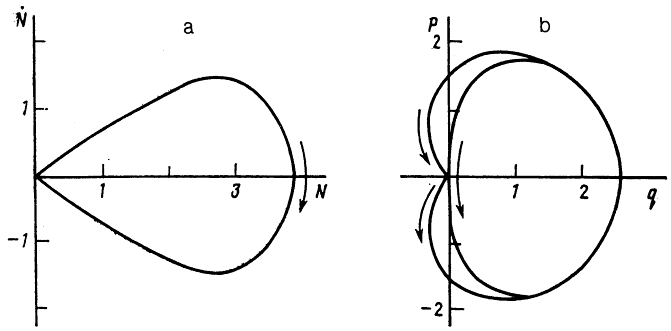

FIG. 4. Projections of the phase trajectory of a symmetric solution for $n_E = 2$, $n_N = 1$, and $\alpha = 0.09298$ onto the (a) N, $\dot N$ plane and (b) the q,p plane.

$$\eta = -2\delta, \quad \frac{a}{N} = 1 - \frac{\delta^2}{3}, \quad \delta = \left(1 - \frac{\alpha}{3}\right)^{1/2}. \tag{10}$$

After some simple calculations, the system (9) simplifies to

$$\dot a = a\left(\delta^2 - \frac{a^2}{6}\right)^{1/2}, \quad \eta = -2\left(\delta^2 - \frac{a^2}{6}\right)^{1/2} \tag{11}$$

and has the soliton solution

$$a = 6^{1/2}\delta\,\mathrm{ch}^{-1}\psi, \quad \eta = 2\delta\,\mathrm{th}\,\psi,$$
$$\psi = \delta\tau - \mathrm{arcch}(6^{1/2}\delta/a_0), \tag{12}$$

where α_0 is the initial perturbation, and the function $N(\tau)$ is found from the last of Eqs. (9).

From (1) and (12) we find the maximum field energy density and the maximum amplitude of the beam density modulation:

$$\left\langle\frac{E^2}{4\pi}\right\rangle_{max} = \frac{8\nu^{1/2}\delta^2}{\gamma} n_p mc^2, \quad |\tilde n_b|_{max} = \frac{4\delta}{(\nu\gamma^3)^{1/2}} n_b. \tag{13}$$

A mechanical analogy is useful in explaining the appearance of a threshold frequency for the stability of a beam-plasma system and in explaining the nonlinear stabilization of the growth of the wave amplitude. Above the instability threshold,

$$\omega > \omega_c = \omega_p(1 - 3\nu^{1/3}/\gamma)^{-1/2}, \ \alpha(\omega_c) = 3$$

the two oscillators—the plasma and the beam—are coupled only weakly, since their resonant frequencies are quite far apart. As the parameter α decreases, these frequencies move closer together, and the system becomes unstable when the plasma frequency in the frame of reference of the beam,

$$\Omega_p = \gamma(kv - \omega_p) \approx \frac{\gamma\omega\varepsilon}{2}$$

approaches the frequency of the plasma waves of the beam in the plasma,

$$\Omega_b = \gamma\omega_b/\varepsilon^{1/2}, \quad \omega_b = (\nu/\gamma^3)^{1/3}\omega_p.$$

We thus find, in order of magnitude, $\alpha_c \approx (\Omega_p/\Omega_b)^{2/3} \approx 1$. In the nonlinear stage of the instability, the growth of the field amplitude is accompanied by a change in the dielectric constant of the plasma $\varepsilon_{\rm NL} = \varepsilon + \nu^{1/3}a^2/\gamma$, and a shift of the threshold frequency of the system. For a fixed perturbation frequency ω, the instability thus saturates when the field amplitude reaches the value $a_m = 6^{1/2}\delta$, and we have $\alpha_{\rm NL}(a_m) = 3$. Since the phase of the oscillations shifts simultaneously, the beam goes into a region of retarding phases of the field at the time $\tau_m = \delta^{-1}\,\mathrm{arccosh}(a_m/a_0)$, and the wave growth gives way to damping. As $\tau \to \infty$, the beam-plasma system returns to its original (unperturbed) state, as the phase approaches its asymptotic value $\eta_\infty = 2\delta$.

As α decreases as $\omega \to \omega_p$, the growth rate increases to its maximum value $\delta_m = 3^{1/2}/2^{4/3}$, and the necessary condition for the use of the approximation of a weak nonlinearity, $\delta \ll 1$, which we used above, is violated.[3)] Numerical integration of Eq. (4) for arbitrary values $\alpha \sim 1$ shows that the phase η increases without bound as time elapses, and complex nonlinear undamped waves characterized by two scales arise in a plasma with a beam. Asymptotic straight lines running parallel to the abscissa correspond to symmetric solutions (solitary pulses) as $\tau \to \infty$ (Fig. 5).

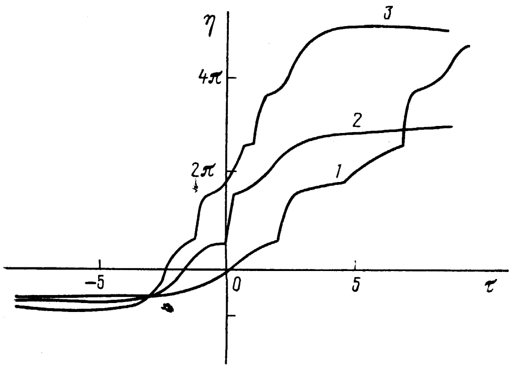

FIG. 5. The oscillation phase $\eta(\tau)$ for various values of α: *1*—$\alpha = 1$; *2*—0.093; *3*—-1.27.

4. ADIABATIC APPROXIMATION

As we move away from the resonance into the frequency region $\omega < \omega_p$, and the damping rate in (5) decreases to[4)]

$$\delta \approx |\alpha|^{-1/2}, \qquad \alpha < 0, \qquad |\alpha| = \frac{|\varepsilon|\gamma}{\nu^{1/3}} \gg 1 \tag{14}$$

a change occurs in the nature of the nonlinear oscillations (Fig. 6). The reason is that a large number of periods of the field amplitude, $T_E \approx |\alpha|^{-1}$, fit into one period of the modulation of the beam density, $T_N \approx \delta^{-1}$:

$$T_N/T_E \approx |\alpha|^{3/2} \gg 1.$$

The problem can be solved in the adiabatic approximation, in which the density is assumed to remain constant over the field period.[11]

Following Ref. 5, we renormalize system (4),

$$A = |E|/E_m, \qquad E_m^2 = \frac{32\pi}{3}|\varepsilon| n_p mc^2, \qquad \rho = \frac{4\pi e v}{\omega_p |\varepsilon| E_m}|\tilde{n}_b|$$

and write it in the form

$$\begin{aligned} \mu A' &= -\rho \sin\eta, \\ \mu\eta' &= A^2 - 1 - \frac{\rho}{A}\cos\eta, \\ \rho' &= \left[\left(\frac{A^2}{2} - 1\right)A^2 - 2A\rho\cos\eta\right]^{1/2}, \end{aligned} \tag{15}$$

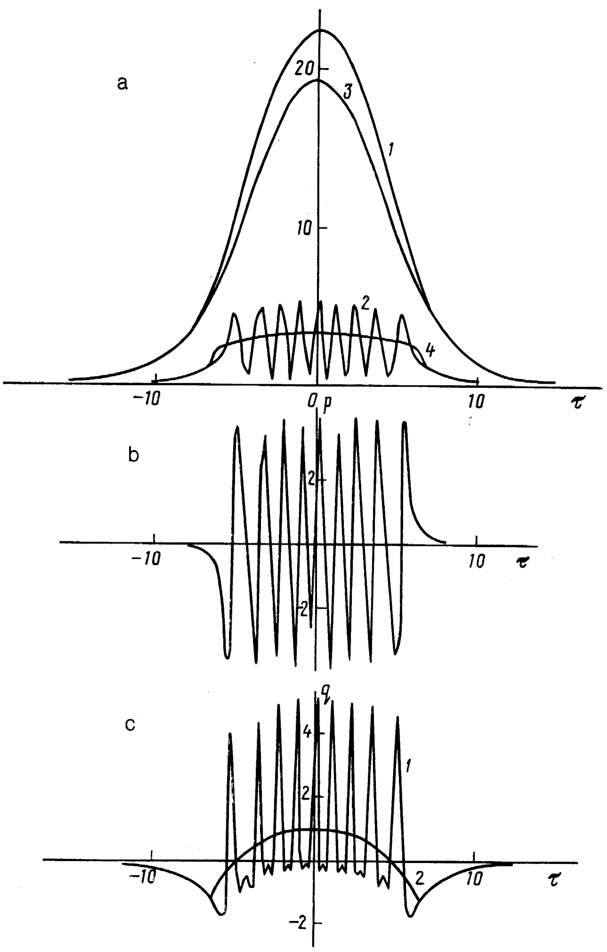

FIG. 6. Solitary pulse with $\alpha = -5.5571$ ($\mu = 0.15267$). a: The amplitudes N (curves *1* and *3*) and a (*2* and *4*). *1,2*—Numerical solution; *3,4*—analytical solution ($\langle N\rangle, \langle a^2\rangle^{1/2}$). b: $p(\tau)$. c: *1*—$q(\tau)$; *2*—$\langle q\rangle$.

where the prime means the derivative with respect to $x = \tilde{\delta}t$, and where we have identified the small parameter of the problem:

$$\mu = \frac{2\tilde{\delta}}{\omega_p|\varepsilon|} = 2|\alpha|^{-3/2} \ll 1.$$

With $\mu = 0$, we find the following equation from (15):

$$A' = A\frac{(1 - 3A^2/2)^{1/2}}{1 - 3A^2}, \qquad \rho = A(A^2 - 1). \qquad \eta = 0, \tag{16}$$

A solution of this equation is

$$-\mathrm{arcch}(2/3A^2)^{1/2} + 2(1 - 3A^2/2)^{1/2} = \pm(x - x_0) + 2^{1/2} - \ln(1 + 2^{1/2}). \tag{17}$$

The $\pm$ correspond to the wings of a cusp soliton which has a singularity in its derivative at the point x_0, $A(x_0) = 3^{-1/2}$ (Refs. 5 and 11). At $\mu > 0$, this singularity disappears, but a numerical integration (Fig. 6) reveals that Eq. (17) describes only the tails of a solitary field pulse, since fast oscillations with a period $T_E \ll T_N$ arise when the singularity is crossed (see Ref. 11, where a similar equation was studied).

To study the solution near the singularity x_0 we introduce the new variables $A_1 = A - 3^{-1/2}$, $|A_1| \ll 3^{-1/2}$ and

$\xi = x - x_0$, $|\xi| \ll x_0$. In the limit $\mu\eta' \ll 1$, one finds[5] from (15)

$$\mu\eta' = 3A_1^2 - \eta^2/3 + \xi/2^{1/3}, \quad \mu A_1' = 2\eta/3^{3/2}. \tag{18}$$

To the left of the singularity, the asymptotic form of (18),

$$A_1 = -[-\xi/(3\cdot 2^{1/3})]^{1/2}, \quad -\xi \gg \mu^{4/5} \tag{19}$$

is the same as the asymptotic behavior of solution (17).

In the region $|\xi| \lesssim \mu^{4/5}$ it is convenient to introduce the new variables $u = \mu^{-2/5}/A_1$ and $\zeta = \mu^{-4/5}\xi$:

$$\frac{d^2u}{d\zeta^2} = \frac{2}{3^{1/2}} u^2 + \left(\frac{2}{27}\right)^{1/3} \zeta. \tag{20}$$

The small term $\eta^2 \approx \mu^{-6/5}$ was discarded in the transformation from (18) to (20).

A numerical solution of (20) reveals (Fig. 7) that the function increases monotonically near $\zeta = 0$, coincides with (19) at $-\zeta \sim 1$, and has a singularity at $\zeta_\infty \approx 4.3$:

$$u(\zeta) = 3^{3/2}(\zeta_\infty - \zeta)^{-2}, \qquad \eta(\zeta) = [3\mu^{1/5}(\zeta_\infty - \zeta)^{-1}]^3. \tag{21}$$

It follows from (21) that the width of the transition region is, in order of magnitude, $\Delta\zeta \sim 1$ and $\Delta\xi \sim \mu^{4/5}$.

There is another solution, $\eta' = -1/(4\mu)$, near the singularity (Fig. 7), for which a' changes sign at the point $\xi = 0$ (Ref. 5). However, a numerical calculation shows that that solution does not join with (19), and a cusp soliton[5] is not realized.

At finite values of μ, the singularity in the derivative at the point x_0 thus disappears. When we move into the region $x > x_0$, however, the singularity in (21) appears; near it we have $u \gg 1$, and the approximation $|A_1| \ll 1$, used in the derivation of Eq. (18), is not valid. Furthermore, the nature of the solution changes to the right of the transition region, $x - x_0 \gtrsim \mu^{4/5}$, according to the numerical calculation (Fig. 6).

The appearance of fast oscillations in the field amplitude, while the amplitude of the beam density varies monotonically, makes it possible to integrate Eqs. (15) at constant values of ρ and ρ'. Eliminating the phase η from the first equation with the help of the third, we find

$$\mu^2 w'^2 + U(w) = 0, \quad w = A^2 - 1, \qquad U(w) = {}^1/_4(w^2 - 1 - 2\rho'^2)^2 - 4\rho^2(1+w). \tag{22}$$

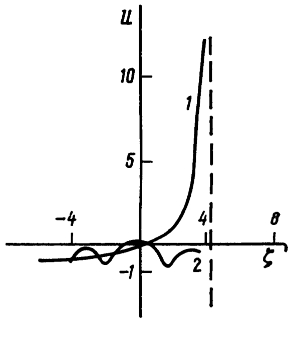

FIG. 7. Solution near the singular point of the field amplitude $A(x_0)$. *1*—$U(\zeta)$; *2*—corresponding solution according to the equations of Ref. 5 ($\mu \approx 0.15267$).

Integrating (22), we find (cf. Ref. 11)

$$w = \frac{w_1 + w_2}{2} - \frac{w_1 - w_2}{2} \frac{\lambda - \mathrm{cn}[(fg)^{1/2}\xi/2\mu, k]}{1 - \lambda\,\mathrm{cn}[(fg)^{1/2}\xi/2\mu, k]},$$

$$\lambda = \frac{f-g}{f+g}; \quad f, g = [(w_{1,2} - b)^2 + d^2]^{1/2}, \tag{23}$$

$$k^2 = \frac{1}{2}\left[1 - \frac{(w_1 - b)(w_2 - b) + d^2}{fg}\right],$$

where $cn(x,k)$ is the elliptic cosine, $\xi = x - x_0$, and $w_1 > w_2$ and $b \pm id$ are the roots of the fourth-degree equation $U(w) = 0$ (Appendix 1). The period of the nonlinear oscillations of the field amplitude is

$$T_E = \frac{8\mu}{(fg)^{1/2}} K(k), \tag{24}$$

where $K(k)$ is the complete elliptic integral of the first kind.[12]

In the central part of the pulse we have $f \approx g \approx 1$, and the oscillation period is $T_E \approx \mu$. As we approach the singularity $\rho_c = 2\cdot 3^{-3/2}$ and $\rho_c' = 6^{-1/2}$, however, the oscillation period increases (Appendix 2):

$$T_E \approx \mu\xi^{-1/4}, \quad \xi \to 0 \tag{25}$$

The condition for adiabatic behavior, $T_E' \ll 1$, is violated. At the limit of the range of validity of the approximation, $T_E' \lesssim 1$, at which expression (24) is still valid qualitatively, we find $T_E \approx \mu^{4/5}$, which agrees in order of magnitude with the width of the transition region in (21).

Near the singularity, the asymptotic form of (23) is bell-shaped,

$$A = (1+w)^{1/2} = \left[3\frac{1 + (2\xi/9\mu)^2}{1 + (2\xi/3\mu)^2}\right]^{1/2} \tag{26}$$

and it describes an extreme peak in the field amplitude (Fig. 6). For $-\xi \gg \mu$, the asymptotic behavior in (26) is the same as that in (21) (to within $\Delta x \approx \mu^{4/5}$).

Solutions which are averaged over the fast oscillations and which describe the monotonic variation of the beam density are of interest:

$$\langle\rho\rangle = T_E^{-1} \int_\xi^{T_E + \xi} \rho(\xi')\,d\xi'.$$

For the calculations it is convenient to use the equation $\rho'' = -A\cos\eta$, which follows from (15). Averaging it over the nonlinear period (Appendix 3), we find

$$\rho'' = -\langle A\cos\eta\rangle = \frac{1}{4\rho}\left[1 + 2\rho'^2 + \frac{w_1^2 g - w_2^2 f}{f-g} - fg\frac{E(k)}{K(k)}\right], \tag{27}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals,[12] and $\rho = \langle\rho\rangle$.

Equation (27) can be integrated by quadratures. Because of the complexity of the final expressions, however, we have carried out a numerical integration. The results are shown in Fig. 6a (curve 3); they agree well with the results of a numerical solution of the original system of equations, (15).

Also plotted in Fig. 6 are the quantities

$$\langle q\rangle=(-\alpha)^{1/2}\langle A\cos\eta\rangle,$$
$$\langle a^2\rangle^{1/2}=(-\alpha)^{1/2}\langle A^2\rangle^{1/2},$$

where $\langle A\cos\eta\rangle$ is given by (27), and $\langle A^2\rangle$ is calculated in a corresponding way:

$$\langle A^2\rangle=1+\frac{w_2f+w_1g}{f+g}+\frac{\pi(fg)^{1/2}}{2K(k)}$$
$$-\frac{fg(f-g)}{(w_1-w_2)(f+g)}\left[1+\frac{(w_1-b)(w_2-b)+d^2}{fg}\right]\frac{\Pi(n,k)}{K(k)},$$
$$n=k^2\frac{(f+g)^2}{(w_1-w_2)^2}, \tag{28}$$

where $\Pi(n,k)$ is the complete elliptic integral of the third kind.[12] The other quantities are defined in (23).

By analogy with mechanics, Eq. (22) can be interpreted as the motion of a particle with a mass of $2\mu^2$ in a potential well $U(w)$. The characteristics of this well evolve slowly in time (Fig. 8) as the functions ρ and ρ' described by Eq. (27) vary. At the initial time, with $\rho=\rho'=0$, the particle is at the bottom of the well, at the point $w=-1$ (Fig. 8a). With increasing ρ, the depth of the left-hand minimum decreases (Fig. 8b), and at the singularity, with $\rho_c=2\times3^{-3/2}$ and $\rho'_c=6^{-1/2}$, the curve has only a single minimum. The particle thus "rolls down" into the well on the right and becomes a nonlinear oscillator (Fig. 8, c and d). After the density $\rho(x)$ has reached it's maximum, the well evolves in the opposite direction, and the system returns to its original state as $x\to\infty$.

Finally, we can derive the asymptotic behavior of the spectrum of values μ_n corresponding to soliton solutions

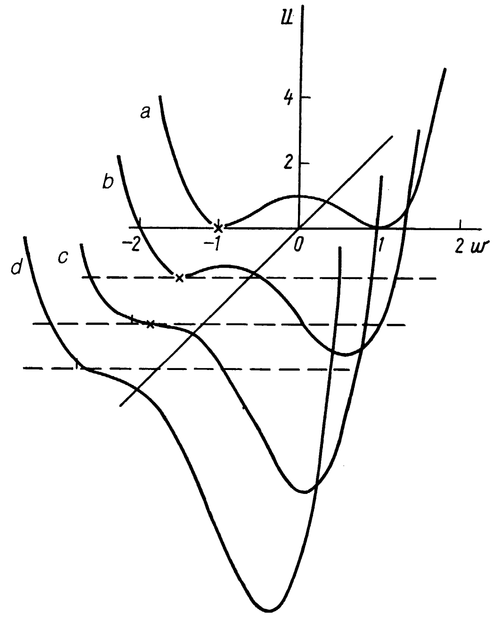

FIG. 8. Plot of the potential energy $U(w)$. a—$\rho=0, \rho'=0$; b—$\rho\approx0.273$, $\rho'\approx0.279$; c—$\rho=2\cdot3^{-3/2}\approx0.385$, $\rho'=6^{-1/2}\approx0.408$; d—$\rho'\approx0.451$, $\rho'\approx0.475$ (the crosses on curves a–c show the position of the particle).

with $n\gg1$ maxima of the field amplitude and a single density maximum (Figs. 1 and 6). For this purpose we note that a solitary pulse corresponds to an integral number of periods of the field amplitude oscillation, $\Delta x=nT_E$, between the two singular points on the $\rho(x)$ curve ($\rho_c=2\cdot3^{-3/2}$, $\rho'_c=\pm6^{-1/2}$), as can easily be verified with the help of the mechanical model introduced above: When the solution returns to the slow branch, the position of the particle on the potential curve should be the same as that at the time at which the fast oscillations appeared (Fig. 8c). This situation corresponds to the condition specified. A more rigorous perturbation theory[13] leads to an expression which differs from (23) by the replacement

$$x-x_0=\xi\to T_E\int_0^{\xi}\frac{d\xi'}{T_E},$$

in the argument of the elliptic cosine. Using this expression and (24), we find the following from the relation $\Delta x=nT_E$:

$$\mu_n=\frac{G}{n},\qquad G=\frac{1}{8}\int_0^{\xi_c}\frac{(fg)^{1/2}d\xi}{K(k)}\approx1.17, \tag{29}$$

where the points 0 and ξ_c correspond to the critical values $\rho'_c=6^{-1/2}$ and $\rho'_c=-6^{-1/2}$ ($\rho_c=2\cdot3^{-3/2}$). The integration has been carried out numerically. The corresponding spectrum of values of $\alpha=(2/\mu)^{2/3}$,

$$\alpha_n=-\left(\frac{2}{G}\right)^{2/3}n^{2/3}\approx-1.43n^{2/3},$$

is shown by curve 2 in Fig. 2, for comparison with the results of the numerical calculations.

Let us examine the mechanism for the nonlinear saturation of the instability at frequencies $\omega<\omega_p$; this mechanism is particularly obvious far from the plasma resonance, $|\varepsilon|\gg\nu^{1/3}/\gamma$. In this case the presence of an external modulation singles out a beam mode $\omega=kv$, and the plasma mode is not excited in the initial stage of the instability. With increasing field amplitude, however, the frequency of the natural waves of the plasma,

$$\omega_{NL}^2=\omega_p^2(1-|E|^2/E_p^2),\qquad E_p^2=\frac{32\pi}{3}n_pmc^2$$

decreases, and the beam wave branch and the plasma wave branch[10] come close together. At a field amplitude $|E_c|=3^{-1/3}|\varepsilon|^{1/2}E_p$, the beam mode and the plasma mode couple nonlinearly. It is important to note that in this stage of the instability the dielectric constant is $\varepsilon_{\rm NL}=(2/3)\varepsilon<0$, the plasma remains opaque to the growing waves, and energy continues to build up in the beam mode. Later on, however, the behavior of the system changes radically, and the rapid growth of the wave amplitude is accompanied by an abrupt transition (abrupt on the scale of the small parameter μ) of the plasma to a state with $\varepsilon>0$. Beyond this point, the nonlinear interaction of the beam mode and the plasma mode is accompanied by a nonlinear process in which the beam and the wave go out of phase, and energy is returned to the beam because the two oscillators—the plasma and the beam—go out of phase (Sec. 3). An important distinction from the threshold (resonant) instability, however, is that waves appear at the frequency $\omega_p - \omega$ (which is considerably higher than the rate $\tilde{\delta} = \omega_b/|\varepsilon|^{1/2}$). These waves are accompanied by small-scale jumps in the wave phase velocity, like those described in Ref. 5. In contrast with a cusp soliton,[5] formed by growing and decaying beam wave branches, the solutions found above support the assertion that energy is returned to the beam only during the excitation of the plasma mode.

5. MAIN RESULTS

In a plasma with an ultrarelativistic electron beam, $\gamma\nu^{1/3} \gg 1$, the nonlinear saturation of the instability stems from the dependence of the dielectric constant of the plasma on the field amplitude:

$$\varepsilon_{NL}=\varepsilon+|E|^2/E_p^2.$$

This dependence leads to a nonlinear change in the phase velocity

$$v_{NL}=v(1-|E|^2/E_p^2)^{1/2}$$

with respect to the beam velocity v; a further consequence is a deviation from the phase resonance of the beam with the wave. The electrons go into accelerating phases of the field, and energy is returned to the beam. Since the period of the nonlinear oscillations (which is on the order of the reciprocal of the growth rate) is small in comparison with the time scale for trapping of electrons by the wave (in comparison with the reciprocal of the phase oscillation frequency), the beam as a whole is displaced with respect to the wave, without breaking up into bunches.

Near the instability threshold, where the growth rate vanishes ($\delta \to 0$) and the relaxation time of the phase velocity is quite long ($T_E \sim \delta^{-1} \to 0$), the beam electrons undergo a smooth transition into a region of accelerating phases of the field, and the solution assumes a soliton form. With increasing growth rate, the relaxation time of the phase velocity decreases. Since the process is inertial, the beam slips through the region of accelerating phases, enters a region of retarding phases, etc. The number of these nonlinear cycles is finite for the values of the parameter α found above (Fig. 2).

At modulation frequencies $\omega < \omega_p$, far from resonance, $|\alpha| \gg 1$, the nature of the instability changes, since the plasma is initially opaque to the perturbations which arise in the beam, and the role of the plasma electrons reduces to one of screening these perturbations. As the field amplitude grows, however, the dielectric constant increases and the plasma is bleached when ε_{NL} changes sign.[5)] The beam subsequently interacts with the plasma wave, and the nonlinear process in which the phase matching is lost occurs as described above, over a large number of nonlinear cycles. For modulation frequencies close to the resonant frequency, $|\alpha| \ll 1$, the sign of ε is essentially irrelevant (Fig. 2), since the plasma bleaching time $T_N \approx \tilde{\delta}^{-1}$ is comparable in magnitude to the time scale of the nonlinear process in which the plasma matching is lost, $\tilde{T}_E \approx (|\varepsilon|\omega_p)^{-1}$.

A necessary condition for the transport of the beam over a large distance in the plasma is a modulation at frequencies close to the resonance, $\alpha_2 = 0.093$ and $\alpha_3 = 1.271$; this modulation results in a rapid return of wave energy to the beam (the return regime of the instability). The optimum instability regime for the excitation of waves with a high energy density is found at small growth rates, at $\omega < \omega_p$, where the prolongation of the linear state of the instability makes possible the buildup of a significant amount of energy in the beam mode (Fig. 2b).[5]

It is physically obvious that the transport of a beam of relativistic electrons through a plasma results in a heating of the plasma, so the dissipative effects which are seen at a finite temperature must be taken into account. The condition for the collisionless approximation holds if the collision rate in the plasma, ν_{ei}, is small in comparison with the instability growth rate δ (Ref. 9).

Furthermore, the appearance of a low-velocity electron beam as a result of the appearance of a return current $j/e = n_p u \approx n_b c$ is accompanied by the excitation of ion acoustic waves, with a growth rate[14]

$$\delta_i=\left(\frac{\pi}{8}\frac{m}{M}\right)^{1/2}\frac{k}{Q}\left(u-\frac{v_s}{Q}\right),\quad Q=(1+k^2\lambda_D^2)^{1/2},$$

$$v_s=(T_e/M)^{1/2},\ \lambda_D=(T_e/m\omega_p^2)^{1/2},$$

if $u > v_s$ (Ref. 9). Consequently, the condition under which our approximation is valid, $\delta \gg \delta_i$ [with $\delta \approx (\nu^{1/3}/\gamma)\omega_p$ and $k \approx \lambda_D^{-1}$)], is the inequality

$$\nu^{1/3} \gg \gamma\frac{m}{M}\frac{u}{v_s}.$$

APPENDIX 1

The roots of the fourth-degree equation

$$U(w)={}^1/_4(w^2-h)^2-4\rho^2(1+w)=0, \tag{A1.1}$$

$$h=1+2\rho'^2$$

are[15]

$$w_{1,2}=z_1^{1/2}\pm 2\,\mathrm{Re}\,z_2^{1/2},\quad b=-z_1^{1/2},\quad d=2\,\mathrm{Im}\,z_2^{1/2}, \tag{A1.2}$$

where $z_{1,2}$ are given by

$$z_1=h/3+R+S,\quad z_2=h/3-(R+S)/2+3^{1/2}i(R-S)/2,$$
$$R,\ S=(h^3/27-2\rho^2h/3+2\rho^4\pm Q^{1/2})^{1/3}, \tag{A1.3}$$
$$Q=4\rho^4(\rho^4+16\rho^2/27-2\rho^2h/3+h^3/27-h^2/27)>0.$$

APPENDIX 2

To determine the asymptotic behavior of T_E near the critical point, $\rho \to 2\cdot 3^{-3/2}$, $\rho' \to 6^{-1/2}$, we seek a solution of (27) in the form

$$\rho\approx 2\cdot 3^{-3/2}+6^{-1/2}\xi+L\xi^2/2+M\xi^{5/2}, \tag{A2.1}$$

where the coefficients L and M are to be determined. Substituting (A2.1) into the right side of (27), and using (A1.3), and (23), we find $L = 3^{-1/2}$. For M we find the transcendental equation

$$M=\frac{3^{1/2}}{5}\{R_0+S_0+3^{1/2}(R_0^2+S_0^2+R_0S_0)^{1/2}[1-2E(k_0)/K(k_0)]\},$$

$$k_0^2=\frac{1}{2}\left[1-\frac{3^{1/2}(R_0+S_0)}{2(R_0^2+S_0^2+R_0S_0)^{1/2}}\right],$$

$$R_0,\ S_0=5^{1/3}\cdot 2^{7/6}\cdot 3^{-3/2}[M\pm(M^2+2^{7/3}\cdot 5^{-2}\cdot 3^{-3})^{1/2}]^{1/3}, \tag{A2.2}$$

from which we find $M \approx -0.12$. Substitution of (A2.1) into (23) leads to

$$T_E \approx \frac{4K(k_0)}{[3(R_0^2+S_0^2+R_0S_0)]^{1/4}}\mu\xi^{-1/4} \approx 9{,}72\mu\xi^{-1/4}, \qquad \text{(A2.3)}$$

which yields estimate (25).

APPENDIX 3

A convenient way to evaluate the integral on the right side of (27) is to eliminate $\cos\eta$ and to transform to the variable w with the help of (16), (22), and the relation $d\xi = dw/w'$, where w' is given by (22). As a result we find

$$\frac{1}{T_E}\int_{\xi}^{T_E+\xi} A\cos\eta\, d\xi' = \int_{w_1}^{w_2}\left(\frac{\rho'^2}{2}+\frac{1-w^2}{4}\right)\frac{dw}{[-U(w)]^{1/2}} \times\left\{\int_{w_1}^{w_2}\frac{dw}{[-U(w)]^{1/2}}\right\}^{-1}. \qquad \text{(A3.1)}$$

The trigonometric substitution[15]

$$w = \frac{w_1+w_2}{2} - \frac{w_1-w_2}{2}\,\frac{\lambda-\cos\varphi}{1-\lambda\cos\varphi} \qquad \text{(A3.2)}$$

[see (23)] puts the integral in a standard form.[16] This integral is expression (27).

[1)] In an unmagnetized beam, under the condition $\gamma\nu^{1/3} \gg 1$, transverse oscillations arise, and the instability becomes a kinetic instability.[7-9]
[2)] In the region $|\alpha| \gg 1$, the terms a^2/N^2 and a^4/N^2 can be discarded, and system (4) becomes the same as that found in Ref. 5.
[3)] Even for $\delta \approx 1$, the modulation of the beam density in (13) remains lower than the background density $|\tilde{n}_b|_{\max} \ll n_b$, because of the parameter $(\nu\gamma^3)^{1/2} \gg 1$ in the denominator.
[4)] In dimensional variables, we have $= \tilde{\delta}(\nu/\gamma^3|\varepsilon|)^{1/2}\omega_p \ll |\varepsilon|\omega_p$.
[5)] This effect is analogous to the nonlinear bleaching which occurs in laser systems with optical shutters.

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Translated by D. Parsons