Stationary vortex structures in an inviscid fluid

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New classes of axisymmetric formations have been found that develop in steady inviscid flows. Particular attention is paid to toroidal and periodic vortex structures. The group classification has been carried out on the equations for the stream function. Several structures are derived which arise in plane-parallel motions of stratified fluid and in axisymmetric plasma flows.

1. In the past fifteen years vortex structures, which are of interest in their own right, have attracted special attention from researchers in connection with coherent structures in fluids and plasmas. As is well known, however, there are very few explicit solutions for them. The present work describes a number of structures which arise in steady axisymmetric flows. Particular attention is devoted to the formation of toroidal vortices. Special cases include (a) an exponentially decaying vortex screened by two walls; (b) an inviscid analog of Taylor vortex columns; and (c) a periodic "loop street," as well as a number of other flows. The group classification of the corresponding differential equations is carried out. Steady planar motion of a stratified fluid is also considered. Several magnetic vortex structures[1] are found in plasma.

2. In classical hydrodynamics[2] the equation for the stream function,

$$\psi_{rr}+\psi_{zz}-\frac{1}{r}\psi_r=r^2G(\psi)+F(\psi) \qquad (1)$$

with $G=H_\psi$ and $F=-\Gamma\Gamma_\psi$, where H and Γ are arbitrary functions of ψ and the subscript denotes differentiation with respect to the corresponding variable, describes the class of steady axisymmetric (swirling) motions of an inviscid fluid. In plasma physics this equation is customarily referred to as the Grad–Shafranov equation.

As is well known,[3] steady planar flow of an inviscid stratified fluid can be described by means of the equation

$$\varphi_{zz}+\varphi_{yy}=zG+F, \qquad (2)$$

where G and F are arbitrary functions of φ. In what follows we assume that at least one of the functions G and F in Eqs. (1) and (2) is nonlinear, and that G is not equal to zero.

The group classification[4] of Eq. (2) with vanishing G permits us to specialize to an equation which admits an infinite-dimensional group–Liouville's equation. We can convert[5] Liouville's equation into the Laplace equation and obtain a rich variety of exact solutions. In the general case Eqs. (1) and (2) are invariant only under translations. Infinitesimal operators other than the translation operators are called supplementary.[4] The result of the group classification of Eqs. (1) and (2) can be formulated as follows.

The nonlinear equation (1) admits supplementary infinitesimal operators only for two types of terms on the right-hand side:

$$G=A\exp(\psi),\quad F=B\exp(\psi/2),\qquad G=A\psi^n,\ F=B\psi^{(n+1)/2}, \qquad (3)$$

with $A, B\in R$. In the first case this operator is

$$r\partial_r+z\partial_z-4\partial_\psi,$$

and in the second it is

$$r\partial_r+z\partial_z-\frac{4}{n-1}\psi\partial_\psi,$$

while for $n=-7$ an additional operator is admitted:

$$rz\partial_r+\frac{z^2-r^2}{2}\partial_z+\frac{z\psi}{2}\partial_\psi.$$

Supplementary operators for Eq. (2) appear only for the following right-hand sides:

$$G=A\varphi^n,\ F=B\varphi^{(2n+1)/3},$$

$$G=A\exp(\varphi),\ F=B\exp(^2/_3\varphi),$$

with $A, B\in R$. The first pair of functions corresponds to the dilatation operator

$$z\partial_z+y\partial_y-\frac{3}{n-1}\varphi\partial_\varphi,$$

and the second to the two operators

$$M_1=z\partial_z+y\partial_y-3\partial_\varphi,\qquad M_2=zy\partial_z+\frac{y^2-z^2}{2}\partial_y-3y\partial_\varphi.$$

Thus linearization of even one of these equations seems unlikely.

We introduce two illustrative examples. The solution of Eq. (1) with functions G, F given by (3) is invariant under dilatation, and for $n=-3$ has the form $\psi=rV(w)$, where $w=z/r$. The function V satisfies a second-order differential equation. Under the substitution $\xi=\ln|w+(w^2+1)^{1/2}|$ it goes over to a form in which the independent variable and the first derivative do not appear:

$$U''=U+AU^{-3}+BU^{-1}.$$

The constants A, B can be chosen (e.g., $A=1$, $B=3$) so that this equation has periodic solutions or solutions that tend to a constant as $|\xi|\to\infty$. But in that case none of the contour lines of the function ψ are closed.

The solution of Eq. (2) with an exponential right-hand side which is invariant under the operator M_2 can be represented as

$$\varphi=-3\ln|z|+P(w),$$

where

$$w=z+y^2z^{-1}.$$

After the change of variables $\xi=\ln|w|$ the equation for the function P is transformed into

$$U''+U'=-3+A\exp(U)+B\exp(^2/_3U).$$

The only singular point of this equation (for $B=0$, $A>0$), namely, $U'=0$, $U=\ln(3/A)$, is a saddle point. As a solution of this equation we can take the function corresponding

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to a phase trajectory coming into this singular point. Then it follows from the representation of the function φ that for sufficiently large values of z the streamlines $R^2(y,z)$ in the plane of the flow are close to the line z = const. Letting one of the streamlines represent a solid wall, we obtain the flow of a stratified fluid over an "almost flat" bottom.

3. More nontrivial solutions of Eqs. (1) and (2) can be found by the method of separation of variables. A number of solutions of Eq. (2) with vanishing G, obtained by generalized separation of variables, are given by Kaptsov.[6]

Among all nonlinear equations of the form (1), only the equation

$$\psi_{zz}+\psi_{rr}-\frac{1}{r}\psi_r=Ar^2\psi+B\psi\ln|\psi|, \quad (4)$$

with A, $B\in R$, admits the separation $\psi=f(z)g(r)$. To the right-hand side of (4) we can add a term of the form $c\psi$ which is annihilated by the dilatation transformation of the function ψ. In that case, if the constant A is nonvanishing, we can assume without loss of generality that it has unit absolute value. The functions f, g must satisfy the equations

$$g''=\frac{g'}{r}+g(Ar^2-s+B\ln|g|), \quad (5)$$

$$f''=f(B\ln|f|+s), \quad (6)$$

where s is a separation constant. A nonlinearity of the form $\psi\ln\psi$ probably first appeared in the work of Rosen,[7] in a field-theory application.

The qualitative behavior of the solutions of Eq. (6) is given by the potential function.[8]

$$\Pi=f^2(B/2-s-B\ln|f|).$$

Bifurcation occurs for $B=0$.

Suppose the constant B is negative. Depending on the initial conditions, f then has one of the following forms: (a) periodic positive- (or negative-) definite; (b) periodically changing signs; (c) having a single extremum and vanishing as $|z|\to\infty$ In the last case there is an explicit expression for f:

$$|f|=\exp\left(\frac{B}{4}z^2-\frac{s}{B}+\frac{1}{2}\right).$$

In what follows, solutions of the form (a), (b), and (c) will be denoted by 1_f, 2_f, and 3_f, respectively. We will not consider negtive-definite 1_f and 3_f solutions.

Equation (5) also has a solution expressible in terms of elementary functions:

$$g=\exp(kr^2+s/B),$$

where

$$k=\frac{B\pm(B^2+16A)^{1/2}}{8}.$$

Other solutions can be found by applying numerical methods such as Runge–Kutta. The only problem lies in choosing constants A, B, and s and initial conditions for (5) so that adequately physical and interesting streamline plots are produced. The existence of solutions of Eqs. (5) is guaranteed by Wintner's theorem[9] for $r>r_0>0$.

For $A=-1$, $B=-10$, and $s=0$ the solution of Eq. (5) satisfying the initial conditions $g(r_0)=Ar_0^4/8$, $g'(r_0)=Ar_0^3/2$ (where $r_0=0.1$), is an oscillatory function taking on positive and negative values alternately. In every interval within which it has a definite sign it has a single extremum. But if we set $A=1$ and leave the initial data and the constants B and s unchanged, then we get a solution that grows on the interval $[r_0,r_1)$ and decreases on $(r_1,r_2]$ (with $r_1=2.05$ and $r_2=2.95$), then grows again and diverges as $r\to\infty$. The local minimum g_{min} is equal to 0.79 and the local maximum g_{max} is equal to 2.05. These three solutions will be denoted by 1_g, 2_g, and 3_g, respectively.

Every pair of solutions i_f, j_g $(1\leqslant i,j\leqslant 3)$ which allows us to reconstruct a stream function is called a combination of solutions and is denoted by i_f+j_g.

In order to discern the subtleties of the streamline pictures obtained by means of computer graphics, it is desirable to know the qualitative behavior of the contours of the function $\psi(y,z)$. To draw the contours we need to find the critical points of the function ψ, i.e., the points at which the gradient of ψ vanishes. According to Morse theory,[10] the nature of the nondegenerate critical points can be determined from the Hessian of ψ. In the present instance every nondegenerate critical point is either a saddle point or a local extremum. Below we will refer to the set of all the stream-function contours as the flow portrait.

Consider a 2_f+2_g combination of solutions. The points at which the functions f and g vanish correspond to straight lines in the $R^2(r,z)$ plane, parallel to the r or the z axis. We thus obtain a partition of the flow portrait into rectangular cells. Inside each cell there is a critical point (where $f'=g'=0$ holds), i.e., an extremal point of the function ψ. All contours lying within cells are closed. Closed contours correspond to toroidal stream tubes.

The choice of constants $A=-2$, $B=-10$, $s=1$ also can give rise to a 2_f+2_g combination of solutions (see Fig. 1; in all figures the z axis is directed upward). For this we need to specify initial data, e.g., by taking $f(0)=10$ and $f'(0)=0$ and by setting the function g equal to 0.5 and its derivative equal to 3.0 at $r_0=3.8$. Then f has a half-period of 0.7 and g is positive on the interval (R_1,R_2), vanishing at the end points, where $R_1=3.66$ and $R_2=4.25$. If we treat the cylindrical stream tubes with radii $r=R_1$ and $r=R_2$ as having solid boundaries, then there are 17 torroidal vortices distributed between the two cylinders with height $H=20.2d$ (where $d=R_2-R_1$ and $R_1/R_2=0.86$. This solution qualitatively resembles toroidal Taylor vortices. The ratio of the

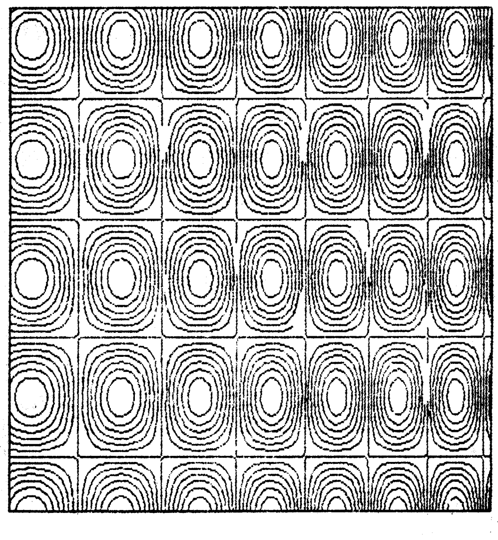

FIG. 1.

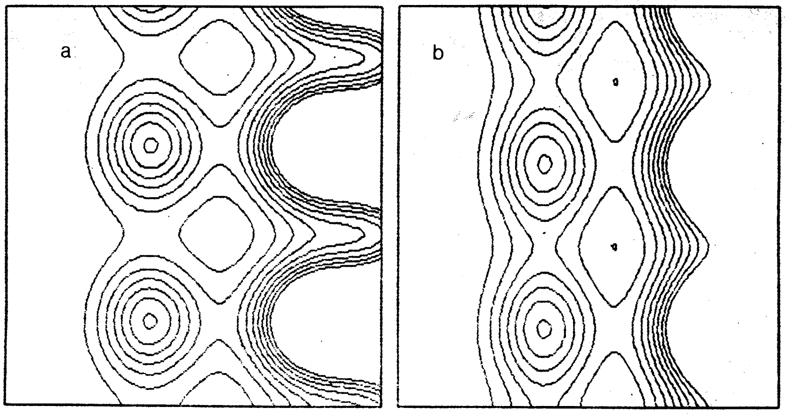

FIG. 2.

two radii and the ratio H/d of the cylinder height to the separation, as well as the number of vortices, agree with the experimental data of Ref. 11. The azimuthal component of velocity in this solution vanishes at the solid walls. Hence this solution is more like the situation in which the inner cylinder is first spun in order to create Taylor vortices, then stopped.

The structure shown in Fig. 2a, which is periodic in z, arises from a $1_f + 3_g$ combination of solutions. The initial data for f are specified as follows: $f(0) = 0.3$, $f'(0) = 0$; the maximum value of the function is $f_{max} = 1.52$; and the minimum value is $f_{min} = 0.3$. This structure will be referred to as a "loop street." It contains two "paths" consisting of loops inside which there are vortices. The paths are separated by a band consisting of open unbounded contour lines. On the right and left of this pair of paths are located wavelike contours. When the initial conditions for f change [for example, to $f(0) = 0.6$], this structure may become two periodic vortex chains arranged in checkerboard fashion (Fig. 2b). In both cases the stream function has four critical values: $c_1 = f_{max} g_{max}$, $c_2 = f_{max} g_{min}$, $c_3 = f_{min} g_{max}$, $c_4 = f_{min} g_{min}$ (the values of g_{min} and g_{max} were given above). Which one of these structures appears depends on which of the numbers c_2 and c_3 is bigger. If $c_3 < c_2$ holds, then a loop street results; in the opposite case vortex chains result.

The $3_f + 2_g$ and $1_f + 2_g$ combinations of solutions give rise to the flow portraits shown in Fig. 3. Every isolated vortex situated between two neighboring cylindrical stream tubes dies away exponentially as $|z| \to \infty$ (Fig. 3a). In this sense it resembles the vortices found in Ref. 1. Figures 3b displays a "cat's-eye" structure which is periodic in z. In all probability, elements of this structure would be observed if one filled the space between two coaxial cylinders with liquid and displaced them paraxially in opposite directions.

The localized vortex structure shown in Fig. 4 corresponds to the $3_f + 3_g$ combination of solutions. The open contour lines proceed to infinity as $|z| \to \infty$. If we take some bounded stream tube to be a solid wall, then the resulting solution can be interpreted as a flow of fluid or plasma in a torus. Every trajectory is now found to be "wound" about the corresponding toroidal surface.

The solutions in Eq. (4) expressible in elementary functions (for $A = -B = 2$) are

$$\psi = \exp\left(-r^2 - \frac{z^2}{2} + \frac{1}{2}\right)$$

and

$$\psi = \exp\left(-\frac{z^2}{2} + \frac{r^2}{2} + \frac{1}{2}\right).$$

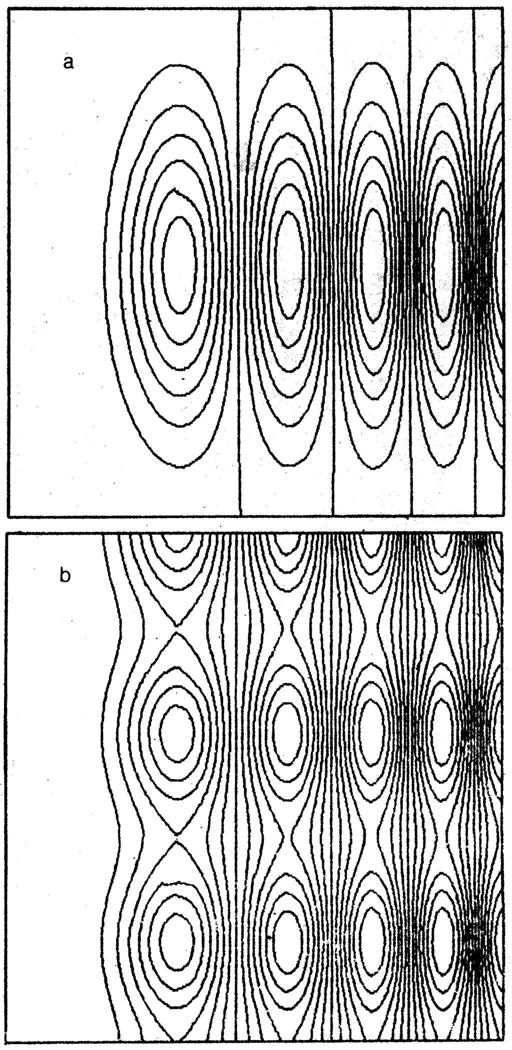

FIG. 3.

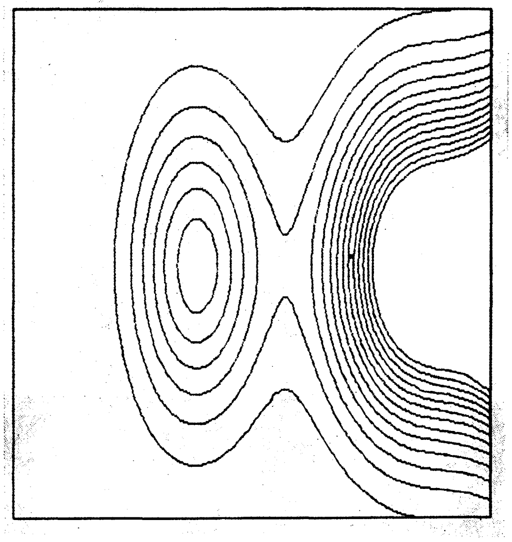

FIG. 4.

The vortex formations shown in Figs. 1–4 do not exhaust the possible structures inherent in Eqs. (5) and (6). This is because Eq. (5), in addition to the 1_g–3_g solutions described above, has other, qualitatively different solutions.

If we add a term $cr^4\psi$ to the right-hand side of (4), an equation results which may be used to describe dynamic and magnetic vortices (in the terminology of Ref. 1) in plasmas. This equation also allows separation of variables in the form $\psi = f(z)g(r)$, the equation for f being identical with (6). To get the equation for g it suffices to add to the right-hand side of (5) the term cr^4g, which may be regarded as a perturbation. In addition to solutions of the forms 2_g and 3_g, the perturbed equation has a solution (for $c = -0.2$, $A = 3$, $B = -10$, and $s = 0$) which is positive in the interval $[r_0, r_1)$ (with $r_0 = 0.1$ and $r_2 = 4.2$), has two maxima and one minimum, and vanishes at the point r_1. For $r > r_1$ this solution is oscillatory. This solution enables us to exhibit new magnetohydrodynamic structures. One of this is displayed in Fig. 5. A figure-eight-shaped vortex pair is separated from single vortices along a straight line and dies away exponentially as $|z| \to \infty$. In the present example we have $f = \exp(-2.5z^2 + 0.5)$. If for the initial conditions in Eq. (6) we take $f(0) = 0.5$ and $f'(0) = 0$ or $f(0) = 0.8$ and $f'(0) = 0$, then 1_f solutions result, which give rise to the flows shown in Figs. 6a, b. It turns out that the topology of these structures is extremely sensitive to the choice of the initial conditions for Eq. (6), so that we can obtain in addition to these structures others that are "close" to them.

4. The only nonlinear solution of the form (2) that allows the separation of variables $\varphi(y,z) = r(y)h(z)$ in the following:

$$\varphi_{zz} + \varphi_{yy} = Az\varphi + B\varphi \ln|\varphi|, \tag{7}$$

with $A, B \in R$. Then f satisfies Eq. (6) and h satisfies

$$h'' = h(Az + B\ln|h| - s). \tag{8}$$

One solution of this equation can be expressed in terms of elementary functions:

$$h = \exp\left(-\frac{A}{B}z + \frac{A^2}{B^3} + \frac{s}{B}\right).$$

Numerical methods can be used to find other solutions. Choosing the constants $B = -2$, $A = 2$, and $s = -1$ and the initial conditions $h(-1.8) = 0.15$ and $h'(-1.8) = 2.3$ yields a solution which is positive-definite for $z > z_0 = -1.8$, has two extrema, and diverges as $z \to \infty$. This solution, extended to the region $R^z = \{z : z < z_0\}$, is an oscillatory function in R^z.

A completely different solution of Eq. (8) results from the following choice of the constants A, B, and s: $A = -3$, $B = -8$, $s = 1$, $h(-6.2) = 9.5$, and $h'(-6.2) = -2.8$. This solution is positive definite and has six extrema in the interval $-6.2 < z < 1.9$, and for $z > 1.9$ it oscillates with alternating signs. These three types of solution are denoted 1_h, 2_h, and 3_h, respectively.

In order to construct a picture of the streamlines it suffices to plot the contours of constant φ. But to find the fluid density it is necessary to go from Eq. (7) back to the Dubreil–Jacotin equation[3] for the stream function. In the present case any function of the form

$$\varphi = \beta(\psi) = a\exp(c\psi) + b\exp(-c\psi),$$

with $a, b, c \in R$, can be used to carry out the transformation. The density is expressed in terms of the function β by $\rho = k\beta'$, and the magnitude of the attractive force is determined by

$$g = \frac{|A|}{2c^2}.$$

For $A > 0$ the attractive force is in the positive z direction and for $A < 0$ it points in the opposite direction. Since the sign of the density distribution is not specified by (7), there is a certain arbitrariness in the choice of the velocity field.

Equation (7) has a solution expressible in terms of elementary functions:

$$\varphi = \exp\left(-\frac{A}{B}z + \frac{A^2}{B^3} + \frac{B}{4}y^2 + \frac{1}{2}\right).$$

For this case the paths of the fluid elements are parabolas.

The solutions 2_h (for $z > -1.8$) and 3_g (for $r > 0.1$) have the same qualitative behavior. It follows that planar stratified flows of inviscid fluids can also exhibit "loop streets" (Fig. 2) and localized vortex structures (Fig. 4).

Choosing the combination of solutions $1_f + 1_h$ gives

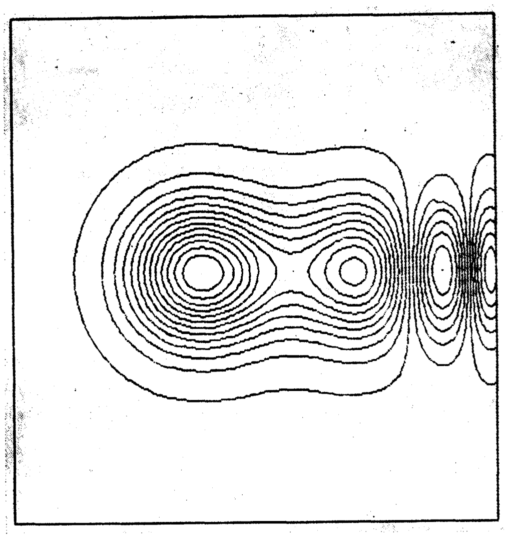

FIG. 5.

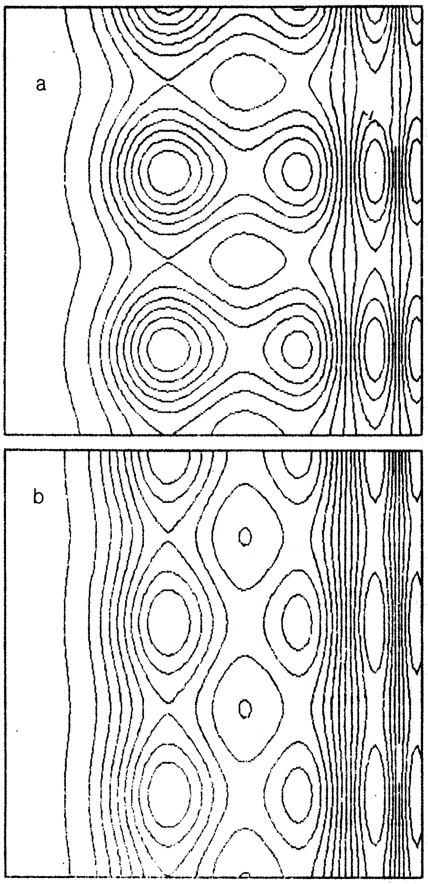

FIG. 6.

rise to streamlines that are periodic in y. When the attractive force is in the negative z direction, the streamlines tend toward straight lines parallel to the y axis in the limit $z \to \infty$. If we regard one trajectory as a solid wall, the result is a description of an inhomogeneous fluid flowing over a periodic bottom.

Finally, Fig. 7 displays the streamlines associated with a 3_h solution [initial conditions $f(0) = 0.5, f'(0) = 0$]. It constitutes an unusual example of the "biocenosis"[1)] of structures encountered earlier. The streamline topology can be changed by changing the initial conditions for f.

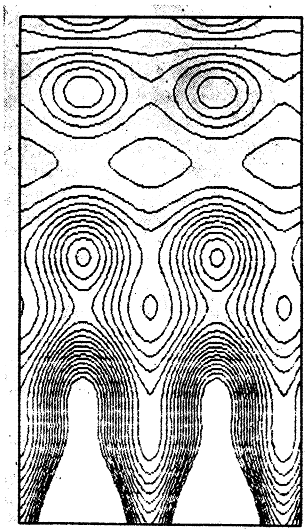

FIG. 7.

[1)]"An assemblage of diverse organisms inhabiting a common biotope: a biotic community," Webster's Third New International Dictionary. [Transl. Ed.]

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