Angular characteristics of radiation from relativistic electrons in matter

N. V. Laskin and A. I. Zhukov

Khar'kov Physics and Engineering Institute, Ukrainian Academy of Sciences
(Submitted 14 January 1990; resubmitted 16 April 1990)
Zh. Eksp. Teor. Fiz. **98**, 571–583 (August 1990)

Multiple scattering is treated in a new setting as Brownian motion in the space of scattering angles. We employ path integrals to derive new equations that describe the effects of scattering on the differential spectrum and polarization characteristics of the radiation. The results are of sufficient generality that we can derive as limiting cases both the known equations for an amorphous medium (nonoriented crystal) and new equations for an ideal crystal at $T = 0$ K. We compare numerical predictions with the spectral and polarization properties of electrons at $E = 40$ GeV in tungsten.

1. INTRODUCTION

The recent upsurge in interest in the effects of multiple scattering on the radiation from high energy charged particles in matter is due to our newfound ability to experimentally investigate the passage of relativistic electrons and positrons through crystals. Experiments[1] have been carried out at CERN, for example, confirming the theoretically predicted effects[2] of multiple scattering on the spectrum of radiation emitted by high energy particles in crystals.

Since the differential spectral characteristics of that radiation carry the most detailed information about the multiple scattering process, theoretical and experimental investigations of those properties are of most interest.

It was pointed out in Ref. 3 that when one actually gets down to prescribing initial conditions for the problem of multiple-scattering effects on the radiation from a relativistic particle in an amorphous layer, the natural questions that arise have to do with the degree of polarization and the angular distribution of bremsstrahlung emission. The method used in Ref. 3 to answer those questions was proposed by Migdal, and has been widely employed in studies of multiple-scattering effects on electrodynamic processes in condensed media.[4]

In the present paper, we describe a new treatment of multiple scattering as Brownian motion. We take a path integral approach to obtain new equations that describe the effects of multiple scattering on the polarization and differential spectral characteristics of the radiation. The generality of our results enables us to derive from various limiting cases both the known equations for an amorphous medium (nonoriented crystal)[3] and new equations for an ideal crystal at $T = 0$ K. We also compare numerical predictions with angular and polarization characteristics for these cases, obtained in experiments with tungsten.

2. BROWNIAN MOTION AND PATH INTEGRALS

We shall assume that the progress of high energy charged particle through matter constitutes random motion. For an amorphous medium, this is so because the random locations of the individual scattering centers (atoms of the medium) ensure that an infinitesimal change in the initial conditions will lead to substantially different final trajectories. In other words, the random nature of motion in an amorphous medium derives from its structure.

On the other hand, random motion in a periodic structure—a crystal—is due to the specifics of particle dynamics. Under certain conditions (which are reviewed in detail in Ref. 5), the motion of a fast charged particle becomes unstable, and dynamical chaos sets in. One of the hallmarks of the latter is that a small change in initial conditions leads to trajectories that differ by so much that the motion may be considered to be random.[1)]

This sort of motion ensues when a fast-moving charged particle is incident upon a crystal at a small angle ψ to one of the crystallographic axes (z). In that event, the scatterers are not individual atoms, but chains of atoms parallel to the z axis. In order for scattering chains not to "coalesce" into a scattering plane (in which case particle motion would be regular), the angle between the initial particle velocity and the close-packed crystallographic planes must not be less than some characteristic angle α.[5] Under those circumstances, multiple scattering takes place in a crystal in much the same way as if the crystal were in fact an "amorphous" medium consisting of parallel filamentary scatterers separated from one another by random distances.[5]

The multiple-scattering induced velocity variation of a particle in a medium will be described here in terms of the deflection angle $\boldsymbol{\vartheta}(t)$ of the particle velocity vector $\mathbf{v}(t)$, measured from the direction $\mathbf{n}$ of the emitted photon:[6]

$$\mathbf{v}=v(1-{}^1\!/_2\vartheta^2)\mathbf{n}+v\boldsymbol{\vartheta},\quad \mathbf{n}\boldsymbol{\vartheta}=0,\quad |\vartheta|\ll 1. \tag{2.1}$$

Multiple scattering involves the overall effect of a large number of small, independent, random changes in the two-dimensional vector $\boldsymbol{\vartheta}$, so by virtue of the central limit theorem,[7,8] we can assume that $\boldsymbol{\vartheta}(t)$ is a Gaussian random process.[2)] Furthermore, the behavior of $\boldsymbol{\vartheta}(t)$ can be considered to be statistically independent in non overlapping time intervals, since it is governed by independent scattering events; the random process $\boldsymbol{\vartheta}(t)$ can therefore be approximated by a Wiener process, or as Brownian motion. In other words, we will describe the multiple scattering of a relativistic charged particle in matter as Brownian motion in the space of the two-dimensional vectors $\boldsymbol{\vartheta}$.

We can go from consideration of a random trajectory in the scattering-angle space to a statistical description in terms of an ensemble of trajectories in that same space, in which case Brownian motion can be described by the diffusion equation for the distribution function $f(\boldsymbol{\vartheta},t)$ of the random quantity $\boldsymbol{\vartheta}$ at time t:

 0038-5646/90/080318-07$03.00

$$\frac{\partial f}{\partial t}=\frac{1}{2}D_{ij}\frac{\partial^2 f}{\partial\vartheta_i\partial\vartheta_j},\quad i,j=1.2. \tag{2.2}$$

Below we shall assume that the diffusion tensor satisfies $D_{ij}=\sigma_i\delta_{ij}$, where σ_i is the rms scattering angle per unit time. By virtue of the isotropy of an amorphous medium, $\sigma_1=\sigma_2$; for an ideal crystal at $T=0$ K, particle dynamics are such that $\sigma_1=0$, $\sigma_2=\sigma_c$. Intermediate cases can be interpreted as if they were imperfect crystals at finite temperature. We can write the Green's function for Eq. (2.2) in the form

$$P(\boldsymbol{\vartheta}_0,t_0;\boldsymbol{\vartheta},t)=p(\vartheta_{01},t_0;\vartheta_1,t)p(\vartheta_{02},t_0;\vartheta_2,t),\ t>t_0, \tag{2.3}$$

$$p(\vartheta_{0i},t_0;\vartheta_i,t)=(2\pi\sigma_i(t-t_0))^{-1/2}\exp\left[-\frac{(\vartheta_i-\vartheta_{0i})^2}{2\sigma_i(t-t_0)}\right]. \tag{2.4}$$

The probability density $p(\vartheta_{0i},t_0;\vartheta_i,t)$ yields the probability that the state of the process at time t is ϑ_i, given that it was ϑ_{0i} at time t_0. The conditional probability density $p(\vartheta_{0i},t_0;\vartheta_i,t)$ depends solely on the state of the process at the preceding instant of time, not on its entire previous history; this of course is the identifying characteristic of a Markov random process. For such a process, the probability density[3)] $p(\vartheta_{i(1)},t_1;...;\vartheta_{i(n)},t_n)$ for the process to start out from the initial state $\vartheta_{(0)}=0$ at time $t=t_{(0)}=0$ and wind up in the states $\vartheta_{(1)},...,\vartheta_{(n)}$ is given by the product

$$p(\vartheta_{(1)},t_1;\ldots;\vartheta_{(n)},t_n)$$
$$=p(\vartheta_{(0)},t_0;\vartheta_{(1)},t_1)\ldots p(\vartheta_{(n-1)},t_{n-1};\vartheta_{(n)},t_n). \tag{2.5}$$

With this expression as a starting point, following Einstein and Smoluchowski (see, e.g., Ref. 9), we find the probability distribution for realizations of such trajectories $\vartheta(t)$ when the process progresses through n intervals subsequent to its initial state; those intervals bound the trajectory such that $\vartheta_{(k)}\leqslant\vartheta(t_k)\leqslant\vartheta'_{(k)}$, $k=1,...,n$, where $\vartheta_{(k)}$ and $\vartheta'_{(k)}$ are certain fixed values. The distribution takes the form

$$P_n\{\vartheta_{(1)}\leqslant\vartheta(t_1)\leqslant\vartheta'_{(1)},\ldots,\ \vartheta_{(n)}\leqslant\vartheta(t_n)\leqslant\vartheta'_{(n)}\}$$
$$=\int_{\vartheta_{(1)}}^{\vartheta'_{(1)}}d\vartheta_1\ldots\int_{\vartheta_{(n)}}^{\vartheta'_{(n)}}d\vartheta_n p(\vartheta_{(0)},t_0;\vartheta_{(1)},t_1)$$
$$\ldots p(\vartheta_{(n-1)},t_{n-1};\vartheta_{(n)},t_n).$$

The system of probability distributions thus obtained,

$$P_1\{\vartheta_{(1)}\leqslant\vartheta(t_1)\leqslant\vartheta_{(1)}\},$$
$$P_2\{\vartheta_{(1)}\leqslant\vartheta(t_1)\leqslant\vartheta'_{(1)},\vartheta_{(2)}\leqslant\vartheta(t_2)\leqslant\vartheta'_{(2)}\}\ldots \tag{2.6}$$

is consistent, i.e., the distribution P_n is the partial distribution for P_{n+1}; specifically,

$$P_n=\int_{-\infty}^{\infty}d\vartheta_{n+1}P_{n+1}$$

by virtue of the fact that $p(\vartheta_{(1)},t_1;\ \vartheta_{(2)},t_2)$, as a conditional probability for a Markov process, satisfies the Chapman–Kolmogorov equation

$$p(\vartheta_{(1)},t_1;\vartheta_{(2)},t_2)=\int_{-\infty}^{\infty}d\vartheta'\,p(\vartheta_{(1)},t_1;\vartheta',t')p(\vartheta',t';\vartheta_{(2)},t_2).$$

According to a theorem due to Kolmogorov,[7] the family of probability distributions (2.6) can be uniquely continued with respect to a probability measure over the set of continuous, nondifferentiable functions $\vartheta(t)$, which vanish at $t=0$. That probability measure is known as the Wiener measure. This sort of mathematical construction enables one to average over Brownian trajectories; in particular, for a broad class of functionals $J\{\vartheta(t)\}$ in the space of functions continuous over the interval $[0,t]$, it is possible to calculate path integrals over Brownian trajectories in the following manner. The trajectories $\vartheta(t)$ are made up of straight-line segments $\vartheta_n(t)$ joined at the points $\vartheta(0)=0$, $\vartheta(t_1)=\vartheta_{(1)},...,\vartheta(t_n)=\vartheta_{(n)}$, so that the points $t_1,...,t_{n-1}$ divide the interval $[0,t]$ into n equal parts. The integral over paths is then given by[10]

$$\int D\vartheta(t)J\{\vartheta(t)\}=\lim_{n\to\infty}\int_{-\infty}^{\infty}d\vartheta_{(1)}\ldots\int_{-\infty}^{\infty}d\vartheta_{(n)}$$
$$\times\prod_{k=0}^{n-1}p\left[\vartheta_{(k)},\frac{kt}{n};\vartheta_{(k+1)},\frac{(k+1)t}{n}\right]$$
$$\times J(\vartheta_{(1)},\ldots,\vartheta_{(n)}), \tag{2.7}$$

where $\int D\vartheta(t)$ signifies integration with respect to the Wiener measure, and the function $J(\vartheta_{(1)},...,\vartheta_{(n)})$ of n variables is the value of the functional $J\{\vartheta(t)\}$ at the breakpoint $\vartheta_{(n)}(t)$.

In the next section, we show how the functionals $J\{\vartheta(t)\}$ arise in the problem of the radiation emitted by relativistic charged particles in matter, and how, making use of the definition (2.7), one can average these functionals over trajectories of the process $\vartheta(t)$.

3. DIFFERENTIAL SPECTRUM OF THE RADIATION

Let a relativistic charged particle be incident with velocity $\mathbf{v}_0$ at time $t=0$ upon some condensed medium, and let it move within that medium for some time T. In dealing with the specific initial condition $\mathbf{v}(t=0)=\mathbf{v}_0$, as remarked in Ref. 3, one also has to deal with the degree of polarization and the angular distribution of bremsstrahlung. We now show that the proposed approach solves these problems. In macroscopic electrodynamics, the differential spectrum of polarized radiation within a solid angle $d\Omega$ and a frequency interval $d\omega$ is a functional of the classical particle trajectory[11] $\{\mathbf{r}(t),\mathbf{v}(t)\}$:

$$J_{\mathbf{n},\omega}^{(i)}=\frac{e^2\omega^2}{4\pi^2c^3}\left|\int_0^T dt\,\mathbf{e}_{(i)}^*\mathbf{v}(t)\exp[i(\omega t-\mathbf{kr}(t))]\right|^2, \tag{3.1}$$

where $\mathbf{k}=\omega\mathbf{n}$ is the wave vector, and the polarization vectors $\mathbf{e}_i$ $(i=1,2)$, which in general are complex quantities, satisfy

$$\mathbf{ne}_{(i)}=0,\quad \mathbf{e}_{(i)}^*\mathbf{e}_{(j)}=\delta_{ij}. \tag{3.2}$$

We transform Eq. (3.1) to the form

$$J_{\mathbf{n},\omega}^{(i)}=\frac{e^2\omega^2}{2\pi^2c^3}\mathrm{Re}\int_0^T dt\int_0^{T-t}d\tau(\mathbf{e}_{(i)}^*\mathbf{v}(t))(\mathbf{e}_{(i)}\mathbf{v}(t+\tau))$$
$$\times\exp[-i\omega\tau+i\mathbf{k}(\mathbf{r}(t+\tau)-\mathbf{r}(t))]. \tag{3.3}$$

We adopt for the $\mathbf{e}_i$ the real vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, specifying the linearly independent components of the radiation, without referring them as yet to any special directions in the plane perpendicular to $\mathbf{n}$.

The trajectory of a relativistic charged particle in matter is a realization of a random process; it is therefore necessary to average Eq. (3.3) over all possible trajectories. With this in mind, we take advantage of the results obtained in the previous section, and making use of Eqs. (2.1) and (3.2), we write (3.3) in the form

$$J_{\mathbf{n},\omega}^{(i)}=\frac{e^2\omega^2}{2\pi^2c}\operatorname{Re}\int_0^T dt\int_0^{T-t} d\tau\,\vartheta_{(i)}(t)\vartheta_{(i)}(t+\tau)$$

$$\times\exp\Big[-i\frac{\omega\tau}{2\gamma^2}-i\frac{\omega}{2}\int_t^{t+\tau}d\tau'\,\vartheta^2(\tau')\Big] \qquad (3.4)$$

where $\vartheta_i(t)=\mathbf{e}_i\boldsymbol{\vartheta}(t)$, $\vartheta^2(t)=\vartheta_1^2(t)+\vartheta_2^2(t)$, $\gamma^2=(1-v^2/c^2)^{-1}$. In averaging this expression by integration over paths, we also take advantage of the fact that we need only integrate over those trajectories that pass through the point $\vartheta_i(t=0)=\vartheta_{0i}$ at $t=0$. Averaging Eq. (3.4) yields

$$\langle J_{\mathbf{n},\omega}^{(1,2)}\rangle=\frac{e^2\omega^2}{2\pi^2c}\operatorname{Re}\int_0^T dt\int_0^{T-t} d\tau\exp\Big(-i\frac{\omega\tau}{2\gamma^2}\Big)$$

$$\times\Big[Q^{(2,1)}(\mu)\frac{\partial}{\partial\mu}Q^{(1,2)}(\mu)\Big]\Big|_{\mu=0}, \qquad (3.5)$$

where

$$Q^{(i)}(\mu)\equiv\int D\vartheta(\tau)\exp\Big\{\mu\vartheta_{(i)}(t)\vartheta_{(i)}(t+\tau)$$

$$-i\frac{\omega}{2}\int_t^{t+\tau}d\tau'\,\vartheta_{(i)}^2(\tau')\Big\}. \qquad (3.6)$$

Since the functional in (3.6) is a Gaussian, the integral over paths can be evaluated analytically. The result (see the Appendix) is

$$Q^{(i)}(0)=[(1+r_it\,\mathrm{th}\,r_i\tau)\,\mathrm{ch}\,r_i\tau]^{-1/2}\exp\Big[-\frac{\vartheta_{0i}^2}{2\sigma_i}\frac{r_i\,\mathrm{th}\,r_i\tau}{1+r_it\,\mathrm{th}\,r_i\tau}\Big], \qquad (3.7)$$

$$\frac{\partial}{\partial\mu}Q^{(i)}|_{\mu=0}=\frac{\sigma_it}{\mathrm{ch}\,r_i\tau(1+r_it\,\mathrm{th}\,r_i\tau)}$$

$$\times\Big[1+\frac{\vartheta_{0i}^2}{\sigma_it(1+r_it\,\mathrm{th}\,r_i\tau)}\Big]Q^{(i)}(0), \qquad (3.8)$$

where $r_i=(i\omega\sigma_i)^{1/2}$. With these expressions, Eq. (3.5) becomes

$$\langle J_{\mathbf{n},\omega}^{(i)}\rangle=\frac{e^2\omega^2}{2\pi^2c}\operatorname{Re}\int_0^T dt\int_0^{T-t}d\tau\frac{1}{\mathrm{ch}\,r_i\tau(1+r_it\,\mathrm{th}\,r_i\tau)}$$

$$\times\Big[\sigma_it+\frac{\vartheta_{0i}^2}{(1+r_it\,\mathrm{th}\,r_i\tau)}\Big]$$

$$\times\prod_{j=1}^{2}[\mathrm{ch}\,r_j\tau(1+r_jt\,\mathrm{th}\,r_j\tau)]^{-1/2}\exp\Big[-i\frac{\omega\tau}{2\gamma^2}\Big]$$

$$\times\exp\Big[-\sum_{j=1}^{2}\frac{\vartheta_{0j}^2}{2\sigma_j}\frac{r_j\,\mathrm{th}\,r_j\tau}{1+r_jt\,\mathrm{th}\,r_j\tau}\Big]. \qquad (3.9)$$

This equation is the general solution for the effect of multiple scattering on the differential spectral characteristics of radiation from a relativistic charged particle in matter, and it enables one to investigate the polarization of that radiation. The degree of polarization of the bremsstrahlung is

$$P_{\mathbf{n},\omega}=\frac{\langle J_{\mathbf{n},\omega}^{(1)}\rangle-\langle J_{\mathbf{n},\omega}^{(2)}\rangle}{\langle J_{\mathbf{n},\omega}^{(1)}\rangle+\langle J_{\mathbf{n},\omega}^{(2)}\rangle}.$$

The angular spectrum of unpolarized radiation $\langle J_{\mathbf{n},\omega}\rangle$ is given by the sum $\langle J_{\mathbf{n},\omega}^{(1)}\rangle+\langle J_{\mathbf{n},\omega}^{(2)}\rangle$.

Using Eq. (3.9), we can derive the well-known results for the effects of multiple scattering on the differential spectrum of bremsstrahlung from a relativistic particle in an amorphous target. To do so, we put $\sigma_1=\sigma_2=\sigma$, where 2σ is the rms scattering angle of the particle per unit time, and we transform from the system of vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ to the physically interesting system $\mathbf{e}_\parallel,\mathbf{e}_\perp$, where $\mathbf{e}_\parallel$ lies in the plane of propagation, which is defined by the initial velocity vector $\mathbf{v}_0$ and the direction of the wave vector $\mathbf{n}$, and $\mathbf{e}_\perp$ is orthogonal to both $\mathbf{e}_\parallel$ and $\mathbf{n}$. Equation (3.9) then goes into Eq. (13) of Ref. 3, which specifies the average angular spectral density of bremsstrahlung polarized either in the plane of propagation or perpendicular to it.

4. MULTIPLE SCATTERING EFFECTS IN THE DIFFERENTIAL SPECTRUM OF COHERENT RADIATION FROM RELATIVISTIC PARTICLES IN CRYSTALS

We now show that (3.9) can be used to investigate the effects of multiple scattering on the differential spectrum of the radiation emitted by relativistic charges particles in crystals.

To that end, we consider a situation in which a particle of energy E is incident upon a crystal at a small angle ψ to one of the crystallographic axes (z). In that event, the particle will interact not with the individual atoms in the lattice, but with chains of atoms parallel to the z axis. The requirement for the atomic chain approximation to hold is $\psi\ll R/a$, where R is the screening radius, and a is the distance between atoms in the z direction. If the scattering chains do not constitute a scattering (close-packed) plane, then particle motion will be essentially random,[5] a condition produced by the dynamics of the problem. This makes it possible to treat the crystal as a condensed medium consisting of parallel chains of atoms that are randomly spaced with respect to one another.

In the continuous field produced by a chain, the component of particle momentum parallel to the chain's axis (the z axis) is conserved; the particle can only be scattered azimuthally, through an angle φ, in the plane orthogonal to the z axis (see Fig. 1). That angle is determined by the particle's transverse energy $E_\perp=E\psi^2/2$ and the impact parameter b:

$$\varphi(b)=\pi-2b\int_{\rho_0}^{\infty}\frac{d\rho}{\rho^2}(1-U(\rho)/E_\perp-b^2/\rho^2)^{-1/2}, \qquad (4.1)$$

where $U(\rho)=(cT)^{-1}\int dz\,u(r)$, $u(r)$ is the potential energy of interaction between the particle and an individual atom

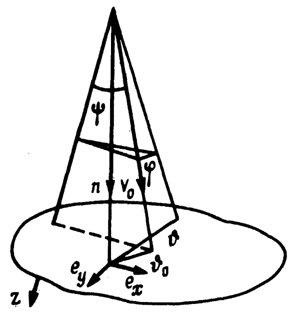

FIG. 1.

in the chain, ρ is the radius vector in the plane orthogonal to the z axis, cT is the thickness of the crystal, and ρ_0 is the distance of closest approach of the particle to the axis of the chain.

The angle $|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_0|$ through which the particle is scattered by the chain is related to the azimuthal angle φ by (see Fig. 1)

$$|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_0|=2\psi\sin(\varphi/2). \tag{4.2}$$

Multiple scattering by different chains leads to a redistribution of particles in φ. The process is described by the kinetic equation for the particle distribution function $f(\varphi,z)$ over angle φ at depth z:[5]

$$\frac{\partial f(\varphi,z)}{\partial z}=na\psi\int db\,\{f(\varphi+\varphi(b),z)-f(\varphi,z)\}, \tag{4.3}$$

where n is the density of atoms in the crystal.

In the general case, the solution of this equation with boundary condition $f(\varphi,0)=\delta(\varphi)$ is a complicated function of φ and z (Ref. 5). It can be simplified considerably if $\psi\gg\psi_c$, where $\psi_c=(4Ze^2/Ea)^{1/2}$ is the critical angle for axial channeling, and $Z|e|$ is the nuclear charge on each atom in the crystal. For angles ψ in that range, Eqs. (4.1) and (4.2) can be expanded in the small parameter $U/E_\perp\sim\psi_c^2/\psi^2$. To first order, we obtain

$$|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_0|\approx\psi\varphi(b),\quad \varphi(b)\approx\frac{1}{2}E_\perp^{-1}\frac{\partial}{\partial b}\int dx\,U((x^2+b^2)^{1/2}). \tag{4.4}$$

Typical values of the azimuthal scattering angle φ with $\psi\gg\psi_c$ are small compared with unity, a circumstance that makes it possible to bring Eq. (4.3) to the form

$$\frac{\partial f(\varphi,z)}{\partial z}=\frac{1}{2}\,\overline{\varphi}^2\frac{\partial^2}{\partial\varphi^2}f(\varphi,z),\quad \overline{\varphi}^2=na\psi\int db\,\varphi^2(b). \tag{4.5}$$

Equations (4.4) and (4.5) demonstrate that for angles of incidence in the range $\psi_c\ll\psi\ll R/a$, the distribution of incident particles over the angles $|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_0|$ inside the crystal is Gaussian:

$$f(|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_0|,t)=(2\pi\sigma_c)^{-1/2}\exp\left(-\frac{|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_0|^2}{2\sigma_c t}\right),$$

where $\sigma_c=\psi^2\overline{\varphi}^2$ is the rms scattering angle per unit time. If we assume each atom in the crystal to produce a screened Coulomb potential $u(r)=Ze^2/r\exp(-r/R)$, we obtain[2,5]

$$\sigma_c=4\pi^2Z^2e^4cnE^{-2}R/(\psi a). \tag{4.6}$$

It is well known[6] that for an amorphous medium,

$$\sigma=4\pi Z^2e^4cnE^{-2}\ln(183Z^{-1/3}). \tag{4.7}$$

These equations imply that the rms angle for multiple scattering of a relativistic particle from chains of atoms in a crystal is significantly larger than the same quantity in an amorphous medium. This fact, plus the nature of particle scattering in a crystal, leads to markedly different angular spectral distributions of the radiation emitted in a crystal as compared with that emitted in an amorphous medium.

In order to show how to go from the general equations (3.9) to a form applicable to particle scattering in a crystal, we transform the system of vectors $\mathbf{e}_i$ to the $\mathbf{e}_x,\mathbf{e}_y$ system shown in Fig. 1. In that figure, $\mathbf{e}_x$ is parallel to the intersection of the plane containing $\mathbf{e}_x$ and $\mathbf{e}_y$ with the plane crystallographic axis z and the initial velocity vector $\mathbf{v}_0$. Renaming indices $1\to x, 2\to y$ in (3.9) and putting $\sigma_x=0$, $\sigma_y=\sigma_c$, we find that the aforementioned properties of particle scattering in a crystal imply that $\langle J^{(x)}_{\mathbf{n},\omega}\rangle$, the distribution over energy and angle of radiation polarized in the plane formed by $\mathbf{n}$ and $\mathbf{e}_x$, and $\langle J^{(y)}_{\mathbf{n},\omega}\rangle$, the distribution for orthogonally polarized radiation, are given by

$$\langle J^{(x)}_{\mathbf{n},\omega}\rangle=\frac{e^2\omega^2}{2\pi^2c}\operatorname{Re}\int_0^T dt\int_0^{T-t}d\tau\,\vartheta_{0x}^2[(1+r_ct\,\mathrm{th}\,r_c\tau)\,\mathrm{ch}\,r_c\tau]^{-1/2}$$
$$\times\exp\left(-i\frac{\omega\tau}{2\gamma^2}\right)$$
$$\times\exp\left(-i\frac{\omega\tau}{2}\vartheta_{0x}^2\right)\exp\left\{-\frac{\vartheta_{0y}^2}{2\sigma_c}\frac{r_c\,\mathrm{th}\,r_c\tau}{1+r_ct\,\mathrm{th}\,r_c\tau}\right\}, \tag{4.8}$$

$$\langle J^{(y)}_{\mathbf{n},\omega}\rangle=\frac{e^2\omega^2}{2\pi^2c}\operatorname{Re}\int_0^T dt\int_0^{T-t}d\tau[(1+r_ct\,\mathrm{th}\,r_c\tau)\,\mathrm{ch}\,r_c\tau]^{-3/2}$$
$$\times\left[\sigma_ct+\frac{\vartheta_{0y}^2}{(1+r_ct\,\mathrm{th}\,r_c\tau)}\right]\exp\left(-i\frac{\omega\tau}{2\gamma^2}\right)$$
$$\times\exp\left(-i\frac{\omega\tau}{2}\vartheta_{0x}^2\right)\exp\left\{-\frac{\vartheta_{0y}^2}{2\sigma_c}\frac{r_c\,\mathrm{th}\,r_c\tau}{1+r_ct\,\mathrm{th}\,r_c\tau}\right\} \tag{4.9}$$

where $r_c=(i\omega\sigma_c)^{1/2}$, and $\vartheta_{0x},\vartheta_{0y}$ are the projections of ϑ_0, the initial deflection angle, onto $\mathbf{e}_x$ and $\mathbf{e}_y$.

Equations (4.8) and (4.9) thus comprise the desired solution, describing the effects of multiple scattering on the differential spectrum and polarization characteristics of radiation emitted by relativistic charged particles in an ideal crystal.

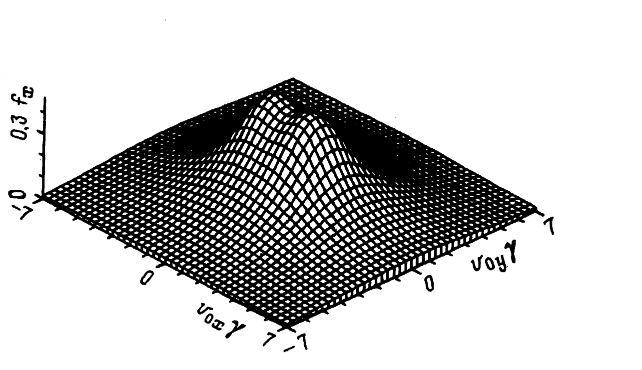

FIG. 2.

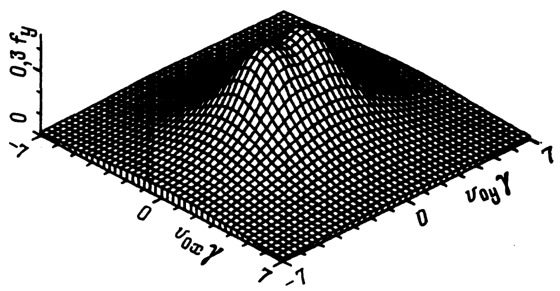

FIG. 3.

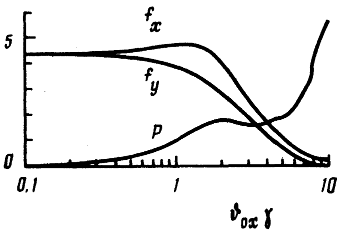

FIG. 4.

5. NUMERICAL ANALYSIS

To take an example, we analyze the angular distribution and degree of polarization of radiation emitted when an electron passes through nonoriented (amorphous target) or oriented tungsten crystals. The parameters adopted for use in Eqs. (3.9), (4.8), and (4.9) are electron energy $E = 40$ GeV, detected photon energy $\omega = 200$ MeV, and target thickness $cT = 0.01$ cm.

Amorphous target. For ease of subsequent comparison of the angular distribution of the flux density in the amorphous target with that in the crystal, we choose the polarization vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ to be arbitrarily oriented (but always in the same way) in the plane orthogonal to $\mathbf{n}$, without necessarily relating them to the physically distinguished system described in Section 3. The differential energy spectrum for radiation with polarization $i = x,y$ takes the form

$$\frac{d^2\mathscr{E}_i}{d\Omega d\omega}=\langle J_{\mathbf{n},\omega}^{(i)}\rangle=\frac{e^2}{\pi^2 c}(\omega/\sigma)^{1/2} f_i(\vartheta_{0x},\vartheta_{0y}),\qquad i=x,y, \tag{5.1}$$

where

$$f_i(\vartheta_{0x},\vartheta_{0y})=\frac{1}{2}\operatorname{Re}\int_0^h d\xi\int_0^{h-\xi} d\zeta\left\{\frac{[1+\xi\,\mathrm{th}(j\zeta)]\xi+2g(\vartheta_{0i}\gamma)^2}{\mathrm{ch}^2(j\zeta)[1+j\xi\,\mathrm{th}(j\zeta)]^3}\right\}$$
$$\times\exp\left\{-g\left[i\zeta+j\vartheta_0^2\gamma^2\frac{\mathrm{th}(j\zeta)}{1+j\xi\,\mathrm{th}(j\zeta)}\right]\right\},\qquad i=x,y, \tag{5.2}$$

$$j=i^{1/2},\quad \vartheta_0^2=\vartheta_{0x}^2+\vartheta_{0y}^2,\quad g=\frac{1}{2\gamma^2}\left(\frac{\omega}{\sigma}\right)^{1/2},\quad h=(\omega\sigma)^{1/2}T,$$

and σ is defined by Eq. (4.7).

It is clear from Figs. 2 and 3, respectively, that the functions f_x and f_y are of the same form, but with a 90° rotation between them in the $(\vartheta_{0x},\vartheta_{0y})$ plane. A cross section through f_x and f_y, and the degree of polarization

$$P(\vartheta_{0x},\ \vartheta_{0y})=|f_x-f_y|/(f_x+f_y) \tag{5.3}$$

in the plane $\vartheta_{0y} = 0$, are shown in Fig. 4. The curves in that figure depict the situation in which the physically interesting choice $\mathbf{e}_x \to \mathbf{e}_\parallel$, $\mathbf{e}_y \to \mathbf{e}_\perp$ yields the polarization basis vectors.

Crystal orientation. We rewrite Eqs. (4.8) and (4.9) in the same dimensionless notation used for amorphous medium:

$$\frac{d^2\mathscr{E}_i^c}{d\Omega d\omega}=\langle J_{\mathbf{n},\omega}^{(i)}\rangle=\frac{e^2}{\pi^2 c}(\omega/\sigma)^{1/2} f_i^c(\vartheta_{0x},\vartheta_{0y}),\quad i=x,y, \tag{5.4}$$

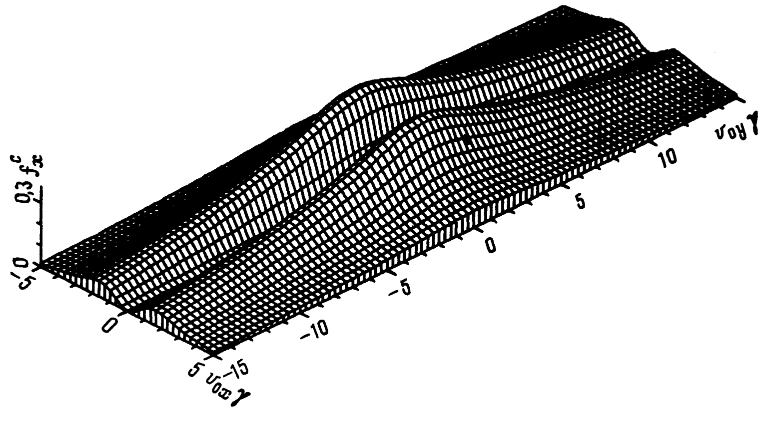

FIG. 5.

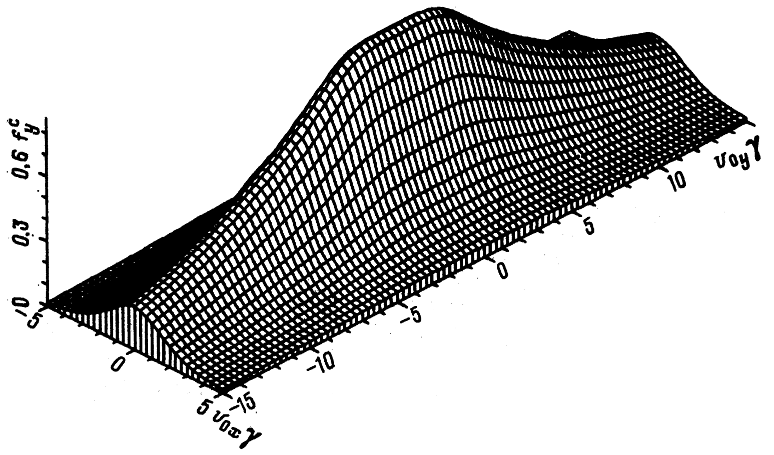

FIG. 6.

$$f_x^c(\vartheta_{0x},\vartheta_{0y})=\frac{1}{2}\left(\frac{\sigma}{\sigma_c}\right)^{1/2}\mathrm{Re}\times\int_0^h d\xi\int_0^{h-\xi}d\zeta\left\{\frac{2g(\vartheta_{0x}\gamma)^2}{\mathrm{ch}^{1/2}(j\zeta)\,[1+j\xi\,\mathrm{th}(j\zeta)]^{1/2}}\right\}\times\exp\left\{-g\left[i(1+\vartheta_{0x}^2\gamma^2)\zeta+j\vartheta_{0y}^2\gamma^2\frac{\mathrm{th}(j\zeta)}{1+j\xi\,\mathrm{th}(j\zeta)}\right]\right\},\tag{5.5}$$

$$f_y^c(\vartheta_{0x},\vartheta_{0y})=\frac{1}{2}\left(\frac{\sigma}{\sigma_c}\right)^{1/2}\mathrm{Re}\int_0^h d\xi\times\int_0^{h-\xi}d\zeta\left\{\frac{[1+j\xi\,\mathrm{th}(j\zeta)]\xi+2g(\vartheta_{0y}\gamma)^2}{\mathrm{ch}^{3/2}(j\zeta)\,[1+j\xi\,\mathrm{th}(j\zeta)]^{5/2}}\right\}\times\exp\left\{-g\left[i(1+\vartheta_{0x}^2\gamma^2)\zeta+j\vartheta_{0y}^2\gamma^2\frac{\mathrm{th}(j\zeta)}{1+j\xi\,\mathrm{th}(j\zeta)}\right]\right\},$$

$$g=\frac{1}{2\gamma^2}\left(\frac{\omega}{\sigma_c}\right)^{1/2},\quad h=(\omega\sigma_c)^{1/2}T,\tag{5.6}$$

where σ_c is given by (4.6).

Figures 5 and 6 show f_x^c and f_y^c for photons emitted by an electron moving through a tungsten crystal at an angle $\psi=2$ mrad to the ⟨100⟩ crystallographic axis. Figure 7 shows cross sections of f_x^c and f_y^c, as well as the degree of polarization P^c given by (5.3), at $\vartheta_{0y}=0$. The cross section of f_y^c at $\vartheta_{0y}=0$ is shown in Fig. 8, in which $f_x^c=0$ and $P^c=1$.

Discussion. These results demonstrate that the radiation from a relativistic charged particle moving through an oriented crystal and undergoing multiple scattering possesses a number of features that distinguish it from the radiation from a nonoriented crystal (an amorphous target). These features reflect the differences inherent in multiple scattering in the two kinds of media. Specifically, if the angle of incidence ψ is such that $\psi_c \ll \psi \ll R/a$ (ψ_c is the critical angle for channelization), then the objects responsible for multiple scattering in a crystal are chains of atoms, which produce a potential appropriate to a filament, while in an amorphous medium they are spherically symmetric scatterers. This, along with the fact that the rms scattering angle σ_c is greater in a crystal than σ, its value in a amorphous medium, makes the angular distribution of the differently polarized components of the radiation differ markedly both from each other and from the polarization components in the amorphous medium. Apart from the strong anisotropy, the radiation coming from a crystal will also show pronounced polarization and much higher total intensity.

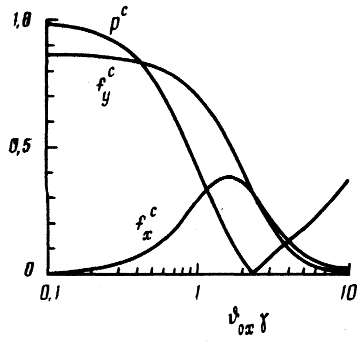

FIG. 7.

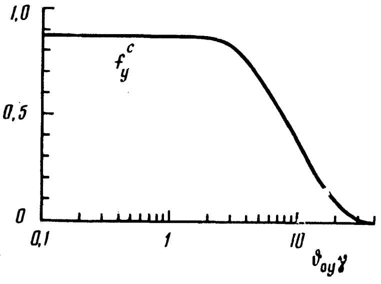

FIG. 8.

These features of the angular distribution of the radiation emanating from a crystal make it possible to investigate experimentally the appropriateness of the proposed approach for describing the effects of multiple scattering on the radiation from charged relativistic particles in matter.

APPENDIX

Let us calculate the multiple integral (3.6), using the standard procedure for computing Gaussian integrals over paths.[10] We first express $Q^{(i)}(\mu)$ in the form

$$Q(\mu)=\lim_{\substack{N\to\infty\\L\to\infty}}\int\ldots\int\frac{d\vartheta_1\ldots d\vartheta_N d\vartheta_{N+1}\ldots d\vartheta_{N+L}}{(2\pi\sigma\Delta)^{N/2}(2\pi\sigma\Delta')^{L/2}}$$

$$\times\exp\Big\{-\sum_{n=0}^{N}\frac{(\vartheta_{n+1}-\vartheta_n)^2}{2\sigma\Delta}-\sum_{n=N}^{N+L-1}\frac{(\vartheta_{n+1}-\vartheta_n)^2}{2\sigma\Delta'}$$

$$+\mu\vartheta_N\vartheta_{N+L}-i\frac{\omega}{2}\Delta'\vartheta_N^2-i\frac{\omega}{2}\Delta'\sum_{n=N+1}^{N+L}\vartheta_n^2\Big\},$$

$$\Delta\equiv t/N,\quad \Delta'\equiv\tau/L,$$

where for typographical expediency we have omitted vector indices on Q, ϑ, and σ. Carrying out the integration over $\vartheta_1,\ldots,\vartheta_{N-1}$ and transforming to variables $y_k=\vartheta_{N+k}(2\sigma\Delta')^{-1/2}$, $k=1,\ldots,L$, we may write $Q(\mu)$ in the form

$$Q(\mu)=\lim_{\substack{N\to\infty\\L\to\infty}}\int\frac{d\vartheta_N}{(2\pi\sigma\Delta N)^{1/2}}$$

$$\times\exp\Big\{-\frac{(\vartheta_N-\vartheta_0)^2}{2\sigma\Delta N}-i\frac{\omega}{2}\Delta'\vartheta_N^2-\frac{\vartheta_N^2}{2\sigma\Delta'}\Big\}$$

$$\times\int dy_1\ldots\int dy_L\exp\Big\{-\sum_{n,m=1}^{L}A_{nm}y_ny_m+\sum_{n=1}^{L}B_ny_n\Big\},\tag{A1}$$

where

$$B_1=(2/\sigma\Delta')^{1/2}\vartheta_N;\ B_n=0,\ n=2,\ldots,L-1;\ B_L=\mu(2\sigma\Delta')^{1/2}\vartheta_N,$$

and the nonzero elements of the matrix A are

$$A_{nn}=2+i\omega\sigma\Delta'^2,\ n=1,\ldots,L-1,$$
$$A_{LL}=1+i\omega\sigma\Delta'^2,\quad A_{n\,n+1}=A_{n+1\,n}=-1.$$

The integral in (A.1) can also be transformed:

$$I=\lim_{L\to\infty}\int dy_1\ldots\int dy_L\exp\Big\{-\sum_{n,m=1}^{L}A_{nm}y_ny_m+\sum_{n=1}^{L}B_ny_n\Big\}$$

$$=\lim_{L\to\infty}D_1^{(L)^{-1/2}}\exp\Big\{\frac{1}{4}\sum_{n=1}^{L}(D_n^{(L)}D_{n+1}^{(L)})^{-1}\Big[\sum_{k=n}^{L}B_kD_{k+1}^{(L)}\Big]^2\Big\}$$

$$=\lim_{L\to\infty}D_1^{(L)^{-1/2}}\exp\Big\{\vartheta_N^2\Big[\frac{D_2^{(L)}-D_1^{(L)}}{2\sigma\Delta' D_1^{(L)}}$$

$$+\frac{\mu}{D_1^{(L)}}+\frac{\sigma\mu^2\Delta'}{2}\sum_{n=1}^{L}\frac{1}{D_n^{(L)}D_{n+1}^{(L)}}\Big]\Big\},\tag{A2}$$

where $D_n^{(L)}$ is the $(L-n+1)$th order principal minor of the determinant of matrix A, located in the lower right-hand corner, and $D_{L+1}^{(L)}=1$. In the limit $L\to\infty$, $D_n^{(L)}$ tends to the value of the continuous function $D(\tau')$ satisfying the differential equation

$$\frac{d^2D(\tau')}{d\tau'^2}=i\omega\sigma D(\tau'),\quad D(\tau)=1,\quad D'(\tau)=0,$$

when the function is evaluated at the point $n\Delta'$; the solution takes the form

$$D(\tau')=\operatorname{ch} r(\tau-\tau'),\ r\equiv(i\omega\sigma)^{1/2}.$$

With this in mind, we can continue the calculation in (A2):

$$I=(D(0))^{-1/2}\exp\Big\{\vartheta_N^2\Big[\frac{D'(0)}{2\sigma D(0)}+\frac{\mu}{D(0)}+\frac{\sigma\mu^2}{2}\int_0^\tau\frac{d\tau'}{(D(\tau'))^2}\Big]\Big\}$$

$$=(\operatorname{ch} r\tau)^{-1/2}\exp\Big\{\vartheta_N^2\Big[\frac{r}{2\sigma}\operatorname{th} r\tau+\frac{\mu}{D(0)}+\frac{\sigma\mu^2}{2r}\operatorname{th} r\tau\Big]\Big\}.\tag{A3}$$

Substituting (A.3) into (A.1), we carry out the integration over ϑ_N, and in the limit $N\to\infty$ we obtain

$$Q^{(i)}(\mu)=\Big(\operatorname{ch} r_i\tau\Big(1+r_it\operatorname{th} r_i\tau-t\frac{2\sigma_i\mu}{\operatorname{ch} r_i\tau}-t\frac{\sigma_i\mu^2}{r_i}\operatorname{th} r_i\tau\Big)\Big)^{-1/2}$$

$$\times\exp\Big\{-\frac{\vartheta_{0i}^2}{2\sigma_i}\frac{r_i\operatorname{th} r_i\tau-2\sigma_i\mu/\operatorname{ch} r_i\tau-\sigma_i\mu^2\operatorname{th} r_i\tau/r_i}{1+r_it\operatorname{th} r_i\tau-2\sigma_it\mu/\operatorname{ch} r_i\tau-\sigma_it\mu^2\operatorname{th} r_i\tau/r_i}\Big\},$$

$$r_i=(i\omega\sigma_i)^{1/2}.\tag{A4}$$

Here we have reinserted the vector indices $i=1,2$.

Taking $\mu=0$ in (A.4), we obtain Eq. (3.7); differentiating (A.4) with respect to μ, and then putting $\mu=0$, we obtain Eq. (3.8).

[1] We shall not consider here such factors as thermal vibration, impurities, defects, and the like, which also lead to essentially random motion.

[2] The applicability of the Gaussian approximation to a crystal is examined in Section 4.

[3] In the remainder of this section we omit subscripts $i=1,2$ on ϑ_i.

[1] J. F. Bak, J. A. Ellison, E. Uggerhof, *et al.*, Nucl. Phys. **B302**, 525 (1988).

[2] N. V. Laskin, A. S. Mazmanishvili, N. N. Nasonov, and N. F. Shul'ga, Zh. Eksp. Teor. Fiz. **88**, 763 (1985) [*sic*].

[3] V. E. Pafomov, Zh. Eksp. Teor. Fiz. **49**, 1222 (1965) [Sov. Phys. JETP **22**, 848 (1966)].

[4] A. B. Migdal, Dokl. Akad, Nauk SSSR **96**, 49 (1954); Zh. Éksp. Teor. Fiz. **32**, 633 (1957) [Sov. Phys. JETP **5**, 527 (1957)].

[5] A. I. Akhiezer and N. F. Shul'ga, Usp. Fiz. Nauk **151**, 385 (1987) [Sov. Phys. Usp. **30**, 197 (1987)].

[6] M. L. Ter-Mikaélyan, *Effects of the Medium on High-Energy Electromagnetic Processes* [in Russian], Armenian Academy of Sciences Press, Erevan (1969).

[7] A. N. Kolmogorov, *Foundations of the Theory of Probability*, Chelsea, New York (1956).

[8] I. I. Gikhman and A. V. Skorokhod, *The Theory of Stochastic Processes*, Springer, New York (1979).

[9] M. Kac, *Some Stochastic Problems in Physics and Mathematics, Magnolia Petroleum*, Dallas (1956), Ch. IV.

[10] I. M. Gel'fand and A. M. Yaglom, Usp. Mat. Nauk **11**, 17 (1956).

[11] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Pergamon, New York (1971).

Translated by Marc Damashek