Electrophysical properties of a percolation layer of finite thickness

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A scaling theory is constructed for the electrophysical properties of a percolation layer of finite thickness in order to investigate the transition from three-dimensional to two-dimensional behavior that is observed near the percolation threshold. This transition is manifest in differences in the percolation threshold, the critical exponents for the conductivity and dielectric permittivity, and the frequency dependence of the dielectric response, compared to a three-dimensional percolation system. Using the method of percolation renormalization-group transformations, we obtain the dependence of the percolation threshold of the layer on its thickness, and also scaling asymptotic forms for the conductivity and dielectric permittivity in the longitudinal and transverse directions at different frequencies. The metal–insulator system is investigated in detail. It is shown that the layer exhibits a sharpening of the peak in the dielectric permittivity at the percolation threshold compared to that of a three-dimensional system. Possible applications of the theory developed here are noted.

A large number of papers[1-13] have been devoted to calculations of the electrophysical and optical properties of percolation systems, and to the application of percolation models to qualitative and quantitative descriptions of various disordered dispersive systems: composite materials, HTSC, metalloceramics, porous bodies, emulsions, and other colloidal systems. In these papers, the systems that were investigated were isotropic two-dimensional or three-dimensional systems, either infinite or having the form of $L \times L$ squares or $L \times L \times L$ cubes. A detailed discussion of finite size effects in two-dimensional percolating systems can be found in Refs. 4, 5, 14, and 15.

The object of our investigation is a percolating $L \times L \times H$ layer ($H \ll L$). This layer is a useful model of disordered composite films, in particular metalloceramic layers deposited on a substrate. In the system under discussion here, the percolation threshold exhibits a transition from three-dimensional to two-dimensional behavior that is manifest in changes in the threshold for percolation, the critical exponents for the conductivity and dielectric permittivity, the frequency dependences of the dielectric response, and other characteristics compared to a three-dimensional percolating system. Apparently, the first time this phenomenon was noted was by the authors of Ref. 16, who gave a qualitative explanation for it. In recent time, descriptions have appeared of special experimental investigations of the electrophysical and optical properties of composite films in the neighborhood of the percolation threshold;[17-19] these investigations have stimulated the present work.

In this paper we use the method of percolation renormalization-group transformations[13,20-23] to obtain scaling expressions for the electrophysical characteristics of a percolation layer in the neighborhood of threshold. We investigate conductor–insulator, conductor–superconductor, "bad" conductor–"good" conductor, and metal–dielectric systems. We also discuss the region of applicability of the theory developed here.

1. PERCOLATION THRESHOLD AND CONDUCTIVITY OF CONDUCTING AND INSULATING LAYERS

Let us first discuss a percolation system in the form of a layer consisting of conducting and nonconducting elements of size a. Let p be the fraction of conductor and $1-p$ the fraction of insulator. We will assume that the longitudinal size of the layer is infinite but the transverse size is finite. For definiteness we will consider a cubic close-packed structure bounded by the planes $z=0$ and $z=H$, consisting of randomly placed conducting and nonconducting cubes (Fig. 1a). The specific nature of this model system does not destroy the generality of the results we obtain. The system is anisotropic; there is no percolation threshold in the transverse direction: for any value of p there exists a chain of contacting conductors which join the bounding layers of the plane. The threshold for percolation in the longitudinal directions $p_{c,\parallel}$ depends on the thickness of the layer H: for $H \to \infty, p_{c,\parallel} \to p_{c,3}$ where $p_{c,3}$ is the percolation threshold of a three-dimensional close-packed structure. For $H=a$, the system under study is two-dimensional and $p_{c,\parallel} = p_{c,2}$ where $p_{c,2}$ is the percolation threshold of the two-dimensional close-packed structure. In the case under discussion here we have $p_{c,3} \approx 0.3117$ (the percolation threshold for the problem of sites on a cubic lattice), and $p_{c,2} \approx 0.59275$ (the percolation threshold for the problem of sites on a square lattice).[21]

For a finite thickness $a \ll H < \infty$ the properties of a percolation layer are determined by the relation between the correlation length of the three-dimensional percolating system

$$\xi_3 = a|p-p_{c,3}|^{-\nu_3} \tag{1}$$

and the thickness H (here and in what follows, the subscript 3 denotes three-dimensionality; ν is the critical index for the correlation length, and $\nu_3 \approx 0.9$; see Ref. 21). For $\xi_3 \ll H$ the layer behaves like a three-dimensional system, and the individual characteristics of the layer are isotropic and do not depend on its thickness. In this case, we can use well-known methods for treating three-dimensional systems to calculate the electrophysical characteristics, e.g., for $\xi_3 \approx a$, the method of self-consistent field theory, and for $\xi_3 \gg a$ the method of scaling relations. In particular the conductivity of a layer σ for $p > p_{c,3}$ and $\xi_3 < H$ equals

$$\sigma_\perp = \sigma_\parallel = \sigma_1 (p-p_{c,3})^{t_3}. \tag{2}$$

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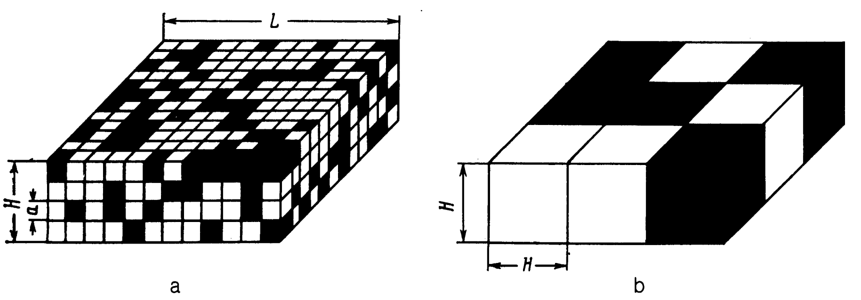

FIG. 1. a—Percolation layer. b—Equivalent two-dimensional percolation system obtained as a result of a PRGT.

Here σ_1 is the specific conductivity of the conducting elements, t is the critical exponent of the conductivity, and t_3 varies from 1.6 to 2.9 (Ref. 21).

For $\xi_3 > H$ the layer behaves as a two-dimensional system, which can be treated as if it consisted of effective blocks of size $H \times H \times H$. The characteristics of the effective blocks can be calculated by the method of percolation renormalization-group transformations (PRGT).[13,20–23] These transformations are implemented by passing from a percolation system with elements of size a to an equivalent percolation system with macroscopic properties consisting of effective elements (blocks) whose size is n times larger, including n^d elements of the original system (d is the dimensionality of the space; in the present case $d = 3$). The condition that the PRGT method can be used is the strong inequality $an \ll \xi_3$. In the transformed system the fraction of effective conductors p^* and their specific conductivity σ_1^* equal[20,21]

$$p^* = p_{c,3} + n^{1/\nu_3}(p - p_{c,3}), \tag{3}$$

$$\sigma_1^* = \sigma_1 n^{-t_3/\nu_3}. \tag{4}$$

We remind the reader that Eqs. (3) and (4) reflect the assumption that the fixed point of the PRGT is the percolation threshold $p_{c,3}$, and that the correlation length ξ defined in (1) and the effective conductivity σ defined in (2) are preserved under the PRGT.

Let us apply a PRGT to the system under discussion and set $n = H/a$. In this case we will go from a percolation layer thickness $H = na$ to a two dimensional mosaic formed by effective blocks of size $H \times H \times H$ (Fig. 1b). It properties are determined by the relations (3) and (4). Because the percolation threshold $p_{c,2}$ of the two-dimensional mosaic is larger than the percolation threshold of the corresponding three-dimensional close-packed structure, a percolation breakdown in the longitudinal direction takes place when the fraction of effective conductors p^* becomes equal to $p_{c,2}$. Consequently, the percolation threshold for a percolation layer $p_{c,\|}$ is determined from the condition

$$p_{c,2} = p_{c,3} + (H/a)^{1/\nu_3}(p_{c,\|} - p_{c,3}) \tag{5}$$

and equals

$$p_{c,\|} = p_{c,3} + (p_{c,2} - p_{c,3})(H/a)^{-1/\nu_3}. \tag{6}$$

It is interesting to note that although the PRGT method used here is justified only for sufficiently large layer thicknesses (i.e., $H/a \gg 1$), the resulting expression (6) gives the correct limit not only for $H \to \infty$, for which $p_{c,\|}(H) \to p_{c,3}$, but also for $H = a$, for which $p_{c,\|}(H) = p_{c,2}$.

Equation (6) for the percolation threshold of the layer is plotted as a function of layer thickness for a cubic close-packed structure in Fig. 2. This function agrees qualitatively with the experimental data of Ref. 16, obtained in the course of investigating the conductivity of layers of conducting and nonconducting spheres, and with the data of Ref. 17 obtained in studying metalloceramic Au–Al_2O_3 films; because the experimental data in these papers is presented in the form of plots, a quantitative comparison is not possible, and the best we can do is look for qualitative agreement with Eq. (6).

In the neighborhood of the percolation threshold, for $\xi_3 > H$ the transverse conductivity of the layer is determined by the effective conductivity σ^* of the conducting blocks, taking into account their fraction p^*

$$\sigma_\perp(H) = p^* \sigma_1^* \approx p_{c,3} \sigma_1 (H/a)^{-t_3/\nu_3}. \tag{7}$$

Expression (7) corresponds to the fractal conductivity law of an isotropic finite-dimensional percolation system. The longitudinal characteristics of the layer for $\xi_3 > H$ are determined by the critical exponents of a two-dimensional system; in particular, the correlation length equals

$$\xi_2 = H|p^* - p_{c,2}|^{-\nu_2} = a(H/a)^{1-\nu_2/\nu_3}|p - p_{c,\|}|^{-\nu_2}, \tag{8}$$

where the critical exponent is $\nu_2 = 4/3$ (see Ref. 4), while the longitudinal conductivity for $p > p_{c,\|}$ equals

$$\sigma_\| = \sigma_1^*(p^* - p_{c,2})^{t_2} = \sigma_1 (H/a)^{(t_2 - t_3)/\nu_3}(p - p_{c,\|})^{t_2}, \tag{9}$$

where the critical index is $t_2 \approx 1.3$.[21]

As we should expect, a crossover is observed between the two-dimensional (Eqs. 7–9) and three-dimensional asymptotic forms (see Eqs. 1, 2) for $\xi_3 \approx H$, and accordingly

$$p \approx p_{c,3} + (H/a)^{-1/\nu_3},$$

this can be verified by direct substitution into the equations under discussion.

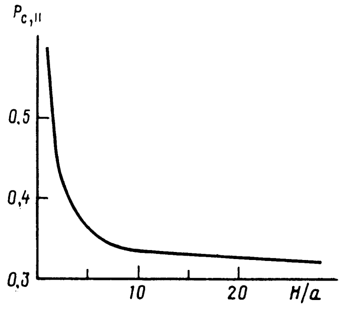

FIG. 2. Dependence of the percolation threshold in the longitudinal direction on layer thickness.

The authors of the experimental studies mentioned above[16,17] observed a transition from the three-dimensional asymptotic form (Eq. 2) to the two-dimensional (Eq. 9) in their electrical conductivity measurements. The critical exponents t_3 and t_2 determined from their experimental data agree with the theoretical values: according to the data of Ref. 16, $t_3 \approx 1.5 \pm 0.1$, $t_2 \approx 1.28 \pm 0.1$ for layers of spheres.

In calculating the conductivity in the immediate vicinity of the percolation threshold it is necessary to take into account the finite longitudinal size L of the layer. For $\xi_2 > L$ we observe the well-known fractal dependence for a two-dimensional sample of finite size; namely, with probability

$$p^{**}=p+(L/H)^{1/\nu_2}(p^{*}-p_{c,2})$$

the sample is an effective conductor with specific conductivity

$$\sigma_{\|}=\sigma_1^{*}\left(\frac{H}{L}\right)^{t_2/\nu_2}=\sigma_1\left(\frac{a}{H}\right)^{t_3/\nu_3}\left(\frac{H}{L}\right)^{t_2/\nu_2}. \qquad (10)$$

It should be noted that relations analogous to Eqs. (6) and (9) for the percolation threshold and the layer conductivity were obtained in Ref. 16 by using another method. These relations differed from ours by constant factors. In our opinion, the PRGT method we have used here is easier to justify, allows us to increase the accuracy of our results, and most importantly allows us to investigate more complex problems in a unified fashion.

2. CONDUCTOR-SUPERCONDUCTOR SYSTEM

A problem of particular relevance to the task of predicting the electrophysical properties of ceramic high-temperature superconducting films[24] is that of the conductivity of a percolation layer made up of superconducting and nonsuperconducting particles. Let p be the fraction of superconducting particles and $1-p$ the fraction of conducting particles with specific conductivity σ_2. In such a system, a transition is observed in the neighborhood of the percolation threshold from three-dimensional behavior to two-dimensional. For $\xi_3(p) < H$ and $p < p_{c,3}$ the system is isotropic and the effective longitudinal conductivity exhibits the usual scaling dependence:[1]

$$\sigma_{\|}=\sigma_2(p_{c,3}-p)^{-s_3}. \qquad (11)$$

Here, s is the critical exponent of superconductivity $s_3 \approx 0.7$ (see Ref. 21). For such a system, the conductivity across a layer of finite thickness is always infinite.

For $\xi_3 > H$ this system is equivalent to a two-dimensional mosaic. Let us use the PRGT method to calculate its characteristics. Applying a PRGT with a scaling change coefficient n leads to an equivalent percolating system consisting of blocks of size $na \times na \times na$ in which the fraction of effective superconducting blocks p^* is determined by Eq. (3) while the effective conductivity σ_2^* of nonsuperconducting blocks equals[13]

$$\sigma_2^{*}=\sigma_2 n^{s_3/\nu_3}. \qquad (12)$$

Expression (12) follows from the condition of conservation of effective conductivity of the system under PRGT. Applying a PRGT with $n = H/a$ to the percolation layer, we obtain an equivalent two-dimensional mosaic made of superconducting and nonsuperconducting blocks of size $H \times H \times H$ which becomes superconducting in the longitudinal direction for $p = p_{c,\|}$ [see Eq. (6)]. For $\xi_3 > H$ and $p < p_{c,\|}$, the effective longitudinal conductivity $\sigma_{\|}$ is determined by the two-dimensional critical exponent s_2, where $s_2 = t_2 \approx 1.3$ (see Ref. 21):

$$\sigma_{\|}=\sigma_2^{*}(p_{c,2}-p^{*})^{-s_2}=\sigma_2(H/a)^{-(t_2-s_3)/\nu_3}(p_{c,\|}-p)^{-t_2}. \qquad (13)$$

A crossover between the two-dimensional [see Eq. (13)] and three-dimensional [see Eq. (11)] asymptotic forms is observed for $\xi_3 \approx H$ and, correspondingly, $p \approx p_c - (H/a)^{-1/\nu_3}$.

3. THE "GOOD" CONDUCTOR–"BAD" CONDUCTOR SYSTEM

The problems investigated above are limiting cases of the general problem of calculating the conductivity of a percolation layer formed by "good" conductors with specific conductivity σ_1 (with fraction p) and "bad" conductors with specific conductivity σ_2 (with fraction $1-p$). We assume $\gamma = \sigma_2/\sigma_1 \ll 1$. As shown in Refs. 13 and 25, in a percolating system made up of conductors that differ widely in conductivity, in addition to the correlation length ξ_3 there is still another characteristic scale: the mixed conductivity scale $l_{p,3}$, which equals

$$l_{p,3}=a\gamma^{-\nu_3/(s_3+t_3)}. \qquad (14)$$

The mixed conductivity scale $l_{p,3}$ possesses the following property: when a PRGT is applied that increases the microscale by a factor of $l_{p,3}/a$, the transformed system consists of blocks of sizes $l_{p,3} \times l_{p,3} \times l_{p,3}$ that are equivalent in their properties, whose effective conductivity equals the conductivity of a three-dimensional "bad" conductor–"good" conductor system in the neighborhood of the percolation threshold, Ref. 1:

$$\sigma_{c,3}=\sigma_1^{s_3/(s_3+t_3)}\,\sigma_2^{t_3/(s_3+t_3)}=\sigma_1(l_{p,3}/a)^{-t_3/\nu_3}=\sigma_2(l_{p,3}/a)^{s_3/\nu_3}. \qquad (15)$$

If the PRGT parameter $n < (l_{p,3}/a)$, then the transformed system consists of blocks with effective conductivities σ_1^* and σ_2^*, which are determined by relations (4) and (12) (Ref. 13). In this case $\sigma_1^* > \sigma_2^*$, and the fraction p^* of "good" conductors (blocks with conductivity σ_1^*) is determined by Eq. (3).

From this we see the conductivity of the percolation layer made up of "bad" and "good" conductors is determined by the ratio of three scales: the correlation length ξ_3 [see Eq. (1)], the mixed conductivity scale $l_{p,3}$ [see Eq. (14)], and the layer thickness H. In this case it is important to determine whether or not the thickness of the layer H exceeds the scale of the mixed conductivity.

The three-dimensional asymptotic forms apply when the correlation length ξ_3 is the largest of these scales:

$$\sigma_{\perp}=\sigma_{\|}=\sigma_1(p-p_{c,3})^{t_3}=\sigma_1(\xi_3/a)^{-t_3/\nu_3}$$
$$\text{for } \xi_3<\min(l_{p,3}, H), \quad p>p_{c,3}, \qquad (16a)$$

$$\sigma_{\perp}=\sigma_{\|}=\sigma_2(p_{c,3}-p)^{-s_3}=\sigma_2(\xi_3/a)^{s_3/\nu_3}$$
$$\text{for } \xi_3<\min(l_{p,3}, H), \quad p<p_{c,3}. \qquad (16b)$$

In the case $l_{p,3} < H$, the properties of the percolation layer do not differ from the properties of an unbounded three-dimensional system and do not depend on the thickness of the layer H. The crossover between asymptotic forms (16a) and

(16b) takes place for $\xi_3 = l_{p,3}$, in the neighborhood of the percolation threshold, the effective conductivity is determined by Eq. (15):

$$\sigma_\perp = \sigma_\parallel = \sigma_{c,3} \quad \text{for} \quad \xi_3 > l_{p,3}, \ l_{p,3} < H. \tag{16c}$$

Eq. (16c) follows from the condition that it is possible to apply a PRGT with transformation parameter $n = l_{p,3}/a$ when $\xi_3 > l_{p,3}$.

For the case $l_{p,3} > H$, a transition is observed from three-dimensional to two-dimensional behavior. For $\xi_3 < H$ the scaling asymptotic forms (16a) and (16b) apply. For $\xi_3 > H$ a PRGT with transformation parameter $n = H/a$ converts the percolation layer to a monolayer (i.e., a two-dimensional mosaic) of elements of size $H \times H \times H$ whose "good" conductors have an effective conductivity σ_1^* [see Eq. (4)] which is $p^*/(1 - p^*)$ times smaller than the "bad" conductors with effective conductivity σ_2^* [see Eq. (12)]. The ratio of the conductivities

$$\gamma^* = \frac{\sigma_2^*}{\sigma_1^*} = \gamma\left(\frac{H}{a}\right)^{(s_3+t_3)/\nu_3} \tag{17}$$

is a small parameter. The transverse conductivity $\sigma_\perp$ of such a layer of conductors connected in parallel equals, Ref. 13

$$\sigma_\perp(H) = (1-p^*)\sigma_2^* + p^*\sigma_1^* = (1-p_c)\sigma_2\left(\frac{H}{a}\right)^{s_3/\nu_3} + p_c\sigma_1\left(\frac{H}{a}\right)^{-t_3/\nu_3} + \left(\frac{H}{a}\right)^{-1/\nu_3}(p-p_c)\left[\sigma_1\left(\frac{H}{a}\right)^{-t_3/\nu_3} + \sigma_2\left(\frac{H}{a}\right)^{s_3/\nu_3}\right]. \tag{18}$$

The crossover between relations (18) and the asymptotic forms (16a) and (16b) takes place at $\xi_3 \approx H$, i.e., accordingly for

$$p = p_+ = p_{c,3} + (H/a)^{-1/\nu_3}(1 - p_{c,3})$$

and

$$p = p_- = p_{c,3} - (H/a)^{-1/\nu_3} p_{c,3}.$$

The longitudinal conductivity $\sigma_\parallel$ of the layer is the conductivity of a two-dimensional percolating system of "bad" and "good" conductors. Its properties, in analogy with the properties discussed above of three-dimensional systems, are determined by three characteristic scales: the correlation length ξ_2 [see Eq. (8)], a new "two-dimensional" mixed conductivity scale $l_{p,2}$, which equals

$$l_{p,2} = H(\gamma^*)^{-\nu_2/(s_2+t_2)} = a\gamma^{-\nu_2/2t_2}(H/a)^{1-\nu_2(s_3+t_3)/2\nu_3 t_2}, \tag{19}$$

and the longitudinal size of the layer L. We have used the relation $s_2 = t_2$ in this argument. When the correlation length ξ_2 is the largest of these scales, the two-dimensional scaling asymptotic forms Eqs. (9) and (13) apply, i.e.,

$$\sigma_\parallel = \sigma_1^*(\xi_2/H)^{-t_2/\nu_2} \quad \text{for} \quad \xi_2 < \min(l_{p,2}, L), \ p > p_{c,\parallel}, \tag{20a}$$

$$\sigma_\parallel = \sigma_2^*(\xi_2/H)^{t_2/\nu_2} \quad \text{for} \quad \xi_2 < \min(l_{p,2}, L), \ p < p_{c,\parallel}. \tag{20b}$$

The two-dimensional asymptotic forms (20a) and (20b) merge with the three-dimensional asymptotic forms (16a) and (16b) for $\xi_2 \approx \xi_3 \approx H$. For the case $l_{p,2} < L$, $\xi_2 > l_{p,2}$, it is possible to apply a PRGT to this two-dimensional system with transformation parameter $n = l_{p,2}/H$. As a result of this transformation, we obtain a two-dimensional mosaic of elements of size $l_{p,2} \times l_{p,2}$ with effective conductivity equal to the conductivity $\sigma_{c,2}$ of a two-dimensional percolation system of "bad" and "good" conductors at the percolation threshold:

$$\sigma_\parallel(H/a) = \sigma_{c,2}^* = (\sigma_1^*)^{s_2/(s_2+t_2)}(\sigma_2^*)^{t_2/(s_2+t_2)} = (\sigma_1^*\sigma_2^*)^{1/2} = (\sigma_1\sigma_2)^{1/2}(H/a)^{-(t_3-s_3)/2\nu_3} = \sigma_{c,3}(l_{p,3}/H)^{(t_3-s_3)/2\nu_3}. \tag{20c}$$

Relation (20c) shows that the crossover between the asymptotic forms (20a) and (20b) for $l_{p,2} < L$ takes place for $\xi_2 = l_{p,2}$ where both these asymptotic forms lead to the same value $\sigma_{c,2}$ [see Eq. (20c)]. Equations (20c) and (16c) join smoothly at $l_{p,3} = H$, which can easily be verified by direct substitution.

For $l_{p,3} > L$ the finiteness of the longitudinal size of the layer L appears directly in the percolation threshold. The scaling laws (20a) and (20b) are found to be valid up to $\xi_2 \approx L$, while for $\xi_2 > L$, the effective longitudinal conductivity depends both on the transverse and on the longitudinal layer dimensions. Applying a two-dimensional PRGT with transformation parameter $n = L/H$, we find that with probability

$$p^{**} = p_{c,2} + (L/H)^{1/\nu_2}(p^* - p_{c,2}) \tag{21}$$

the sample as a whole is a "good" conductor with a longitudinal conductivity

$$\sigma_1^{**} = \sigma_1^*\left(\frac{L}{H}\right)^{-t_2/\nu_2} = \sigma_1\left(\frac{H}{a}\right)^{-t_3/\nu_3}\left(\frac{L}{H}\right)^{-t_2/\nu_2}, \tag{22}$$

while with probability $1 - p^{**}$ it is a "bad" conductor with effective longitudinal conductivity

$$\sigma_2^{**} = \sigma_2^*\left(\frac{L}{H}\right)^{t_2/\nu_2} = \sigma_2\left(\frac{H}{a}\right)^{s_3/\nu_3}\left(\frac{L}{H}\right)^{t_2/\nu_2}. \tag{23}$$

On the average the effective longitudinal conductivity of the layer at the percolation threshold equals

$$\sigma_\parallel(H, L) = p^{**}\sigma_1^{**} + (1 - p^{**})\sigma_2^{**} \quad \text{for} \quad \xi_2 > L, \quad l_{p,2} > L. \tag{24}$$

This expression is the two-dimensional analogue of Eq. (18) for the transverse conductivity. The crossover between Eq. (24) and the asymptotic forms (20a) and (20b) takes place for $\xi_2 = L$.

Summarizing the analysis presented above, we can identify three characteristic regions for the parameters of the system under study:

$$1) \quad \frac{\sigma_2}{\sigma_1} > \left(\frac{a}{H}\right)^{(s_3+t_3)/\nu_3}. \tag{25a}$$

For a system with characteristically three-dimensional behavior, Eq. (16) is valid.

$$2) \quad \left(\frac{H}{L}\right)^{2t_2/\nu_2}\left(\frac{a}{H}\right)^{(s_3+t_3)/\nu_3} < \frac{\sigma_2}{\sigma_1} < \left(\frac{a}{H}\right)^{(s_3+t_3)/\nu_3}. \tag{25b}$$

The system undergoes a transition from "three-dimensional" behavior [Eq. (16)] to "two-dimensional" behavior [see Eq. (20)] for $\xi_3 \approx \xi_2 \approx H$.

$$3) \quad \frac{\sigma_2}{\sigma_1} < \left(\frac{H}{L}\right)^{2t_2/\nu_2}\left(\frac{a}{H}\right)^{(s_3+t_3)/\nu_3}. \tag{25c}$$

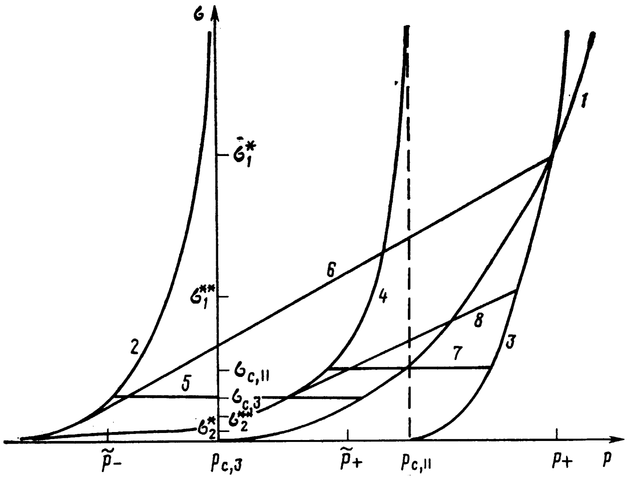

FIG. 3. Schematic representation of the asymptotic forms of the conductivity of a percolation layer in the neighborhood of the percolation threshold: *1*—three-dimensional asymptotic form (2) for $\sigma_\perp$ and $\sigma_\parallel$ with $\xi_3 < \min(l_{p,3}, H), p > p_{c,3}$; *2*—three-dimensional asymptotic form (11) for $\sigma_\perp$ and $\sigma_\parallel$ with $\xi_3 < \min(l_{p,3}, H)$ $p < p_{c,3}$; *3*—two-dimensional asymptotic form (9) for $\sigma_\parallel$ with $\xi_2 < \min(l_{p,2}, L)$, $l_{p,3} < H$, $p > p_{c,3}$; *4*—two-dimensional asymptotic form (13) for $\sigma_\parallel$ with $\xi_2 < \min(l_{p,2}, L)$, $l_{p,3} > H$, $p < p_{c,3}$; *5*—crossover (15) of the three-dimensional asymptotic forms (2) and (11) for $\sigma_\perp$ and $\sigma_\parallel$ with $l_{p,3} < \min(\xi_3, H)$; *6*—crossover (18) of the three-dimensional asymptotic forms (2) and (11) for $\sigma_\perp$ with $H < \min(\xi_3, l_{p,3})$; *7*—crossover (20c) of the two-dimensional asymptotic forms (9) and (13) for $\sigma_\parallel$ with $l_{p,2} < \min(\xi_2, L)$; *8*—crossover (24) of the two-dimensional asymptotic forms (9) and (13) for $\sigma_\parallel$ with $L < \min(\xi_2, l_{p,2})$; $\tilde{p}_\pm = p_{c,3} \pm (l_{p,3}/a)^{-1/\nu_3}$, $p_+ = p_{c,3} + (H/a)^{-1/\nu_3}(1 - p_{c,3})$.

After the transition to the two-dimensional state, the system exhibits fractal properties in the longitudinal direction in the neighborhood of the percolation threshold.

Figure 3 illustrates all the asymptotic forms we have obtained for the conductivity for a percolation layer, along with the crossovers between them.

4. DIELECTRIC PROPERTIES OF A PERCOLATION LAYER

In calculating the dielectric properties of a percolation layer, we will use the method of Efros and Shklovskii.[1] Let us first extend the relations obtained above to the case of conductivities with frequency-dependent complex values, i.e., $\sigma_1(\omega)$ and $\sigma_2(\omega)$ for the individual phases are functions of the frequency ω. Once we determine the effective complex conductivity $\sigma(\omega)$ of the system, we then calculate the complex dielectric permittivity $\varepsilon(\omega)$ according to the well-known equation

$$\varepsilon(\omega) = 1 + i4\pi\sigma(\omega)/\omega. \tag{26}$$

Let us put the specific phase conductivities $\sigma_1(\omega)$ and $\sigma_2(\omega)$ in a form which is inverse to Eq. (26):

$$\sigma_j(\omega) = \operatorname{Re}\sigma_j - i\omega \operatorname{Re}\chi_j, \tag{27}$$

where $\chi_j(\omega) = [\varepsilon_j(\omega) - 1]/4\pi$ is the specific dielectric susceptibility of the jth phase, and $j = 1,2$. We will use the same relations for the effective complex conductivity of the system

$$\sigma(\omega) = \operatorname{Re}\sigma - i\omega \operatorname{Re}\chi, \tag{28}$$

where $\chi(\omega) = [\varepsilon(\omega) - 1]/4\pi$ is the effective dielectric susceptibility.

The sole limitation of the theory we will develop below is the condition that the absolute value of the ratio of the phase conductivities $\gamma = \sigma_2(\omega)/\sigma_1(\omega)$ be sufficiently small. For the case $\gamma \ll 1$, in order to calculate the effective complex conductivity of a percolation system of unbounded size we will start with the scaling asymptotic forms presented in Ref. 1, which we will write in a form that is convenient for further analysis:

$$\begin{aligned}\sigma(\omega) &= \sigma_1(\omega)(p - p_{c,3})^{t_3} + A_3\sigma_2(\omega)(p - p_{c,3})^{-s_3}\\ &= \sigma_1(\omega)(\xi_3/a)^{-t_3/\nu_3} + A_3\sigma_2(\xi_3/a)^{s_3/\nu_3} \quad \text{for} \quad p > p_{c,3},\end{aligned} \tag{29}$$

$$\begin{aligned}\sigma(\omega) &= \sigma_2(\omega)(p_{c,3} - p)^{-s_3} - B_3\frac{[\sigma_2(\omega)]^2}{\sigma_1(\omega)}(p_{c,3} - p)^{-2s_3 - t_3}\\ &= \sigma_2(\omega)\left(\frac{\xi_3}{a}\right)^{s_3/\nu_3} - B_3\frac{[\sigma_2(\omega)]^2}{\sigma_1(\omega)}\left(\frac{\xi_3}{a}\right)^{(2s_3 + t_3)/\nu_3}\\ &\text{for} \quad p < p_{c,3},\end{aligned} \tag{30}$$

here A_3 and B_3 are constants of order unity. These relations contain the first two terms in an expansion of $\sigma(\omega,p)$ in the small parameter $\gamma(\omega)$.

As in the case considered above, i.e., zero frequency in a two-phase system with complex phase conductivities, we introduce a mixed conductivity scale $l_{p,3}(\omega)$ equal to

$$l_{p,3}(\omega) = a\gamma(\omega)^{-\nu_3/(s_3 + t_3)}. \tag{31}$$

This scale corresponds to the correlation length at which all the terms in the right sides of the scaling Eqs. (29) and (30) are of order unity, i.e., for $\xi_3 = l_{p,3}(\omega)$ a crossover occurs between the asymptotic forms (29) and (30). In this case, in the vicinity of the percolation threshold where the correlation length ξ_3 exceeds the mixed conductivity scale $l_{p,3}(\omega)$, the effective complex conductivity $\sigma_{c,3}(\omega)$ equals, in order of magnitude,[1]

$$\sigma_{c,3}(\omega) = [\sigma_1(\omega)]^{s_3/(s_3 + t_3)}[\sigma_2(\omega)]^{t_3/(s_3 + t_3)}. \tag{32}$$

Under a PRGT, the effective phase conductivities in the transformed system are determined by the relations

$$\sigma_1^*(\omega) = \sigma_1(\omega)n^{-t/\nu}, \quad \sigma_2^*(\omega) = \sigma_2(\omega)n^{s/\nu}, \tag{33}$$

which follow from the condition of invariance of the effective conductivity of the system, i.e., Eqs. (29) and (30), under a PRGT. Relations (32) and (33) are extensions of Eqs. (15), (4) and (12) to nonzero frequency. If the scale of the PRGT is chosen to equal the mixed conductivity scale and if we set $n = l_p(\omega)/a$, then the effective phase conductivities in the transformed system are found to be of the same order of magnitude. This implies that at scales larger than the mixed conductivity scale the system under discussion ceases to be strongly disordered. Consequently, if the scale of mixed conductivity is smaller than the thickness of the percolation layer $[l_{p,3}(\omega) < H]$ then the effective characteristics of the layer do not differ from the characteristics of a three-dimensional percolation system of unbounded size and are determined by Eqs. (29), (30) and (32). In the opposite case $[H < l_{p,3}(\omega)]$, the three-dimensional asymptotic forms (29) and (30) are obtained only for $\xi_3 < H$.

For $\xi_3 > H$, by applying a PRGT with transformation parameter $n = H/a$ we obtain an equivalent two-dimensional percolation system with effective phase conductivities

$\sigma_1^*(\omega)$ and $\sigma_2^*(\omega)$ [see Eq. (33)]. The condition $l_{p,3}(\omega) > H$ requires, in addition, the condition

$$\gamma^*(\omega) = |\sigma_2^*(\omega)/\sigma_1^*(\omega)| < 1.$$

In this case, the effective transverse complex conductivity $\sigma_\perp(\omega)$ of the layer is determined by Eq. (18), in which the complex phase conductivities $\sigma_1(\omega)$, $\sigma_2(\omega)$, $\sigma_1^*(\omega)$, and $\sigma_2^*(\omega)$ appear. The effective longitudinal complex conductivity of the layer $\sigma_\parallel(\omega)$ is determined by two-dimensional asymptotic forms analogous to (29), i.e.,

$$\sigma_\parallel(\omega) = \sigma_1^*(\omega)(p^* - p_{c,2})^{t_2} + A_2\sigma_2^*(\omega)(p^* - p_{c,2})^{-s_2}$$
$$= \sigma_1^*(\omega)\left(\frac{\xi_2}{H}\right)^{-t_2/\nu_2} + A_2\sigma_2^*\left(\frac{\xi_2}{H}\right)^{t_2/\nu_2} \quad \text{for} \quad p^* > p_{c,2}, \tag{34a}$$

$$\sigma_\parallel(\omega) = \sigma_2^*(\omega)(p_{c,2} - p^*)^{-s_2} - B_2\frac{[\sigma_2^*(\omega)]^2}{\sigma_1^*(\omega)}(p_{c,2} - p^*)^{-2s_2 - t_2}$$
$$= \sigma^*(\omega)\left(\frac{\xi_2}{H}\right)^{t_2/\nu_2} - B_2\frac{[\sigma_2^*(\omega)]^2}{\sigma_1^*(\omega)}\xi_2^{3t_2/\nu_2} \quad \text{for} \quad p^* < p_{c,2}. \tag{34b}$$

The two-dimensional mixed conductivity scale $l_{p,2}(\omega)$, which [by analogy with Eq. (19)] equals

$$l_{p,2}(\omega) = H[\gamma^*(\omega)]^{-\nu_2/(s_2+t_2)} = H(H/a)^{(s_3+t_3)/\nu_3}[\gamma(\omega)]^{-\nu_2/2t_2}, \tag{35}$$

determines the longitudinal correlation length $\xi_2 = l_{p,2}$ at which all terms on the right sides of (34a) and (34b) are of the same order, corresponding to crossover of these asymptotic forms. In the immediate vicinity of the percolation threshold, for $\xi_2 > l_{p,2}$, the effective longitudinal complex conductivity $\sigma_{c,2}(\omega)$ equals

$$\sigma_\parallel(H, \omega) = \sigma_{c,2}^*(\omega) = [\sigma_1^*(\omega)\sigma_2^*(\omega)]^{1/2}$$
$$= [\sigma_1(\omega)\sigma_2(\omega)]^{1/2}(H/a)^{-(t_3-s_3)/2\nu_3} \tag{34c}$$

Equation (34c) is valid whenever the longitudinal scale of mixed conductivity $l_{p,2}(\omega)$ is smaller than the longitudinal size of the layer L. In the opposite case, i.e., $l_{p,2}(\omega) > L$, the average longitudinal conductivity for $\xi_2 < L$ is determined by the scaling relations (34a) and (34b), while for $\xi_2 > L$ it is determined by Eq. (24) in which the complex phase conductivities $\sigma_1(\omega)$ and $\sigma_2(\omega)$ appear.

Equations (29), (30), (32) and (34) imply analogous relations for the complex dielectric susceptibility $\chi(\omega) = -i\sigma(\omega)/\omega$ and the complex dielectric permittivity $\varepsilon(\omega)$ [see Eq. (26)].

5. THE METAL–DIELECTRIC SYSTEM

Composite films, one of whose components is metal and the other dielectric (e.g., ceramics or polymers), are important objects for practical application of the theory presented here. The real conditions for synthesis of metalloceramics and metal-filled polymers lead to random distributions of metal and dielectric particles. Therefore, the percolation model for such systems finds very wide application and experimental confirmation, Refs. 6, 9, and 17.

For the complex metal and dielectric conductivities, we use the following standard expressions:

$$\sigma_1(\omega) = \sigma_m^0 + i\omega\sigma_m^0\tau_m, \tag{36}$$

$$\sigma_d(\omega) = \sigma_d^0 - i\omega\varepsilon_d^0/4\pi. \tag{37}$$

Here $\sigma_m^0(\omega)$ and $\sigma_d^0(\omega)$ are the specific conductivities of the metal and dielectric at zero frequency, ε_d^0 is the dielectric permittivity of the dielectric at zero frequency, and τ_m is the relaxation time in the metal. Equation (36) is the consequence of the Drude model for frequencies that are not too high, i.e., $\omega \ll \gamma_m^{-1}$. For these frequencies, the condition that the absolute value of the ratio $\sigma_2(\omega)/\sigma_1(\omega)$ be small, which ensures the validity of the general expressions obtained in the previous section, is certainly fulfilled. The features specific to a metal–dielectric system include the fact that, in contrast to a dielectric in metal, the dielectric susceptibility is negative and $\operatorname{sgn}(\operatorname{Im}\sigma_1) = -\operatorname{sgn}(\operatorname{Im}\sigma_2)$.

It does not take much work to obtain general expressions for the conductivity and dielectric permittivity of a percolation layer by substituting Eqs. (36) and (37) into Eqs. (29)–(34). However, the most interesting results are obtained for two regions of parameters—the low frequency region ($\omega \ll 4\pi\sigma_d^0/\varepsilon_d^0$) and the high frequency region ($4\pi\sigma_d^0/\varepsilon_d^0 \ll \omega \ll \tau_m^{-1}$), to which there correspond two characteristic mixed conductivity scales:

$$l_p^0 = a\left(\frac{\sigma_d^0}{\sigma_m^0}\right)^{-\nu_3/(s_3+t_3)}, \tag{38a}$$

$$l_p^\omega = a\left(\frac{\omega\varepsilon_d}{4\pi\sigma_m^0}\right)^{-\nu_3/(s_3+t_3)}. \tag{38b}$$

Extremely low frequencies

In the frequency region $\omega \ll 4\pi\sigma_d^0/\varepsilon_d^0$, the real part of the effective conductivity of a percolation layer does not depend on frequency and can be calculated approximately from the relations obtained in Section 3 for $\sigma_1 = \sigma_m^0$ and $\sigma_2 = \sigma_d^0$. The effective dielectric permittivity of the layer ε^0 is expressed through the imaginary part of the effective complex permittivity $\sigma(\omega)$. In this case, the leading asymptotic order is determined by the relation between the mixed conductivity scale $l_{p,3}^0$, the correlation length $\xi_3(p)$, and the thickness of the layer H. For $l_{p,3}^0 < H$ the three-dimensional asymptotic forms are then valid, and Eqs. (29), (30), and (32) lead to the scaling equations obtained previously in Ref. 1 for the real part of the effective dielectric permittivity of a percolation metal–dielectric system of unbounded size:

$$\varepsilon_\perp^0 = \varepsilon_\parallel^0 = A_1\varepsilon_d^0(p - p_{c,3})^{-s_3} - 4\pi\sigma_m^0\tau_m(p - p_{c,3})^{t_3}$$
$$= A_1\varepsilon_d^0(\xi_3/a)^{s_3/\nu_3} - 4\pi\sigma_m^0\tau_m(\xi_3/a)^{-t_3/\nu_3} \tag{39a}$$

$$\text{for} \quad \xi_3 < l_{p,3}^0, \quad l_{p,3}^0 < H, \quad p > p_{c,3};$$

$$\varepsilon_\perp^0 = \varepsilon_\parallel^0 = \varepsilon_d^0(p_{c,3} - p)^{-s_3} = \varepsilon_d^0(\xi_3/a)^{s_3/\nu_3} \tag{39b}$$

$$\text{for} \quad \xi_3 < l_{p,3}^0, \quad l_{p,3} < H, \quad p < p_{c,3};$$

$$\varepsilon_\perp^0 = \varepsilon_\parallel^0 = \varepsilon_{c,3}^0 = \varepsilon_d^0(\sigma_m^0/\sigma_d^0)^{s_3/(s_3+t_3)} = \varepsilon_d^0(l_p^0/a)^{s_3/\nu_3} \tag{39c}$$

$$\text{for} \quad \xi_3 > l_{p,3}^0, \quad l_{p,3}^0 < H.$$

The second term in Eq. (39a) becomes important for sufficiently large departures from the percolation threshold, where the correlation length is commensurate (or smaller) with the metallization scale

$$l_p{}^m = (\varepsilon_d{}^0/4\pi\sigma_m{}^0\tau_m)^{-\nu_3/(s_3+t_3)}. \tag{40}$$

For $\xi_3 > l_p^m$ the dielectric susceptibility becomes negative under conditions for which a finite cluster of the metallic phase exists. The asymptotic forms (39a) and (39c) attest to the extremal character of the dielectric permittivity in the neighborhood of the percolation transition point. The characteristic form of the function $\varepsilon(\omega)$ is shown in Fig. 4. The curve of dielectric permittivity is symmetric about the extremum because the corresponding critical exponents up to and beyond the percolation threshold are the same. The maximum is determined by Eq. (39c).

The transition to two-dimensional behavior is observed for layers of thickness $H < l_{p,3}^0$. In this case, Eqs. (39a) and (39b) remain valid only for $\xi_3 < H$, while for $\xi_3 > H$ the following two-dimensional asymptotic forms are obtained from Eqs. (18) and (34)

$$\varepsilon_\perp{}^0 = \overset{0}{\varepsilon}_{c,\perp} = \varepsilon_d{}^0 (H/a)^{s_3/\nu_3}; \tag{41}$$

$$\varepsilon_\parallel{}^0 = A_2 \varepsilon_d{}^0 \left(\frac{H}{a}\right)^{s_3/\nu_3} \left(\frac{\xi_2}{H}\right)^{t_2/\nu_2} = A_2 \varepsilon_d{}^0 \left(\frac{H}{a}\right)^{-(t_2-s_3)/\nu_3} (p-p_{c,\parallel})^{-t_2} \tag{42a}$$

$$\text{for} \quad \xi_2 < \overset{0}{l}_{p,2}, \quad p^* > p_{c,2};$$

$$\varepsilon_\parallel{}^0 = \varepsilon_d{}^0 \left(\frac{H}{a}\right)^{s_3/\nu_3} \left(\frac{\xi_2}{H}\right)^{t_2/\nu_2} = \varepsilon_d{}^0 \left(\frac{H}{a}\right)^{-(t_2-s_3)/\nu_3} (p_{c,\parallel}-p)^{-t_2} \tag{42b}$$

$$\text{for} \quad \xi_2 < l_{p,2}, \quad p^* < p_{c,2};$$

$$\varepsilon_\parallel{}^0(H) = \overset{0}{\varepsilon}_{c,\parallel} = \varepsilon_d{}^0 \left(\frac{\sigma_m{}^0}{\sigma_d{}^0}\right)^{1/2} \left(\frac{H}{a}\right)^{-(t_3-s_3)/2\nu_3}$$

$$\text{for} \quad \xi_2 > l_{p,2}. \tag{42c}$$

Here we have introduced the longitudinal mixed conductivity scale

$$\overset{0}{l}_{p,2} = H \left(\frac{\sigma_d{}^0}{\sigma_m{}^0}\right)^{-\nu_2/2t_2} \left(\frac{H}{a}\right)^{-(s_3+t_3)\nu_2/2t_2\nu_3}, \tag{43}$$

which determines the upper boundary (with respect to ξ_2) of the scaling asymptotic forms (42a) and (42b). Relations (41) and (42) show that in composite films an increase occurs in the anisotropy of the dielectric permittivity with decreasing thickness H. At the percolation threshold, the maximum of the dielectric permittivity in the transverse direction smooths out, while in the longitudinal direction it grows sharper in comparison with a three-dimensional system of unbounded size:

$$\frac{\overset{0}{\varepsilon}_{c,\perp}}{\varepsilon_{c,3}} = \left(\frac{\overset{0}{l}_{p,3}}{H}\right)^{-s_3/\nu_3},$$

$$\frac{\overset{0}{\varepsilon}_{c,\parallel}}{\varepsilon_{c,3}} = \left(\frac{\overset{0}{l}_{p,3}}{H}\right)^{(t_3-s_3)/2\nu_3} \tag{44}$$

Apparently, previous investigators have not noticed this property.

The critical exponent of the longitudinal dielectric permittivity for a layer of thickness $H < l_{p,3}^0$ in the immediate vicinity of the percolation threshold $p_{c,\parallel}$ changes from the three-dimensional value s_3 to the two-dimensional value t_2.

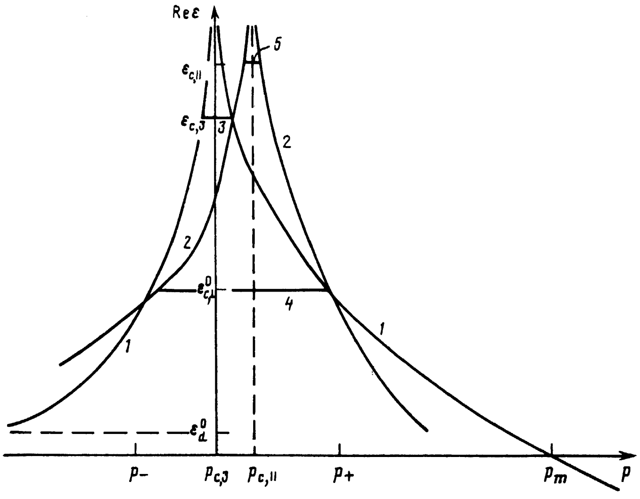

FIG. 4. Schematic representation of the asymptotic forms of the real part of the dielectric permittivity of a metal–dielectric percolation layer in the neighborhood of the percolation threshold: *1*—three-dimensional asymptotic form (39) for $\varepsilon_\perp$ and $\varepsilon_\parallel$ with $\xi_3 < \min\,(l_{p,3}(\omega), H)$; *2*—two-dimensional asymptotic form (42) for $\varepsilon_\parallel$ with $\xi_2 < \min[l_{p,2}(\omega), L\,]$, $l_{p,3}(\omega) > H$; *3*—crossover of the three-dimensional asymptotic forms (39) for $\varepsilon_\perp$ and $\varepsilon_\parallel$ with $l_{p,3}(\omega) < H$; *4*—crossover of the three-dimensional asymptotic forms (39) for $\varepsilon_\perp$ with $l_{p,3}(\omega) > H$; *5*—crossover of the two-dimensional asymptotic forms (42) for $\varepsilon_\parallel$ with $l_{p,3}(\omega) > H$, $l_{p,2}(\omega) < L$. For $\omega < 4\pi\sigma_d^0$, ε_d^0 $l_{p,3}(\omega) = l_{p,3}^0$ (38a), $\varepsilon_{c,\parallel} = \varepsilon_{c,\parallel}^0$ (42b), $\varepsilon_{c,\perp} = \varepsilon_{c,\perp}^0$ (41). For $\omega > 4\pi\sigma_d^0/\varepsilon_d^0$, $l_{p,3}(\omega) = l_{p,3}^\omega$ (38b), $\varepsilon_{c,\parallel} = \varepsilon_{c,\parallel}^\omega$ (53), $\varepsilon_{c,3} = \varepsilon_{c,3}^\omega$ (48). Here, $p_+ = p_{c,3} + (H/a)^{-1/\nu_3}(1-p_{c,3})$, $p_- = p_{c,3} - (H/a)^{-1/\nu_3} p_{c,3}$, $p_m = p_{c,3} + (l_p^m/a)^{-1/\nu_3}$.

When the difference in the conductivities of the metal and dielectric is large enough that the longitudinal mixed conductivity scale $l_{p,2}^0$ exceeds the longitudinal size of the layer L, the scaling asymptotic forms (42a) and (42c) are limited by the conditions $\xi_2 < L$, while for $\xi_2 > L$ the longitudinal dielectric permittivity attains its highest value $\varepsilon_\parallel^0$, which equals

$$\varepsilon_\parallel{}^0(H, L) \approx \varepsilon_d{}^0 \left(\frac{H}{a}\right)^{s_3/\nu_3} \left(\frac{L}{H}\right)^{t_2/\nu_2}. \tag{45}$$

From this it follows that the longitudinal dielectric permittivity of a composite film can exceed the transverse permittivity at the percolation threshold by a factor of $(L/H)^{t_2/\nu_2}$. The characteristic form of the dielectric permittivity of a composite layer as a function of concentration of the metallic phase for various parameters is shown in Fig. 4.

Frequency dependence of the conductivity and dielectric permittivity (for $4\pi\sigma_d^0/\varepsilon_d^0 \ll \omega \ll \tau_m^{-1}$)

For $\omega \ll 4\pi\sigma_d^0/\varepsilon_d^0$ Eqs. (29) contain the principal asymptotic forms presented above for the conductivity Re $\sigma(\omega)$ that apply in the region of three-dimensional behavior of the percolation layer, i.e., for $\xi_3 < \min(l_{p,3}^\omega, H)$:

$$\mathrm{Re}\,\sigma_\perp(\omega) = \mathrm{Re}\,\sigma_\parallel(\omega) = \sigma_m{}^0 (p-p_{c,3})^{t_3} = \sigma_m{}^0 (\xi_3/a)^{-t_3/\nu_3}$$

$$\text{for} \quad p > p_{c,3}, \tag{46a}$$

$$\operatorname{Re}\sigma_\perp(\omega)=\operatorname{Re}\sigma_\parallel(\omega)$$

$$=B_3\frac{(\omega\varepsilon_d^0)^2}{(4\pi)^2\sigma_m}(p_{c,3}-p)^{-(2s_3+t_3)}+\sigma_d^0(p_{c,3}-p)^{-s_3}$$

$$=B_3\frac{\omega^2\varepsilon_d^2}{(4\pi)^2\sigma_m}\left(\frac{\xi_3}{a}\right)^{(2s_3+t_3)/\nu_3}+\sigma_d^0\left(\frac{\xi_3}{a}\right)^{s_3/\nu_3}$$

$$\text{for}\quad p<p_{c,3}. \tag{46b}$$

The second term in (46b) is important only for sufficiently large departures from the percolation threshold when $\xi_3<(l_p^\omega)^2/l_p^0$. The corresponding asymptotic forms for the dielectric permittivity Re $\sigma(\omega)$ do not differ from the asymptotic form (39) obtained for extremely low frequencies (they differ only ion the boundaries of their regions of applicability). The merging of these asymptotic forms takes place for $\xi_3=l_{p,3}^\omega$ [Eq. (38b)]. If the mixed conductivity scale $l_{p,3}^\omega$ does not exceed the layer width H, then the crossover to three-dimensional asymptotic behavior may also occur for rather thick layers, independent of H. The conductivity and dielectric permittivity at the percolation threshold are power-law functions of frequency:

$$\operatorname{Re}\sigma_\perp(\omega)=\operatorname{Re}\sigma_\parallel(\omega)=\sigma_{c,3}^\omega=\sigma_m(\omega\varepsilon_d^0/4\pi\sigma_m^0)^{t_3/(s_3+t_3)}\propto\omega^{t_3/(s_3+t_3)}$$

$$\text{for}\quad \xi_3>l_{p,3}^\omega,\quad l_{p,3}^\omega<H, \tag{47}$$

$$\operatorname{Re}\varepsilon_\perp(\omega)=\operatorname{Re}\varepsilon_\parallel(\omega)=\varepsilon_{c,3}^\omega=\varepsilon_d^0(\omega\varepsilon_d^0/4\pi\sigma_m^0)^{-s_3/(s_3+t_3)}\propto\omega^{s_3/(s_3+t_3)}$$

$$\text{for}\quad \xi_3>l_{p,3}^\omega,\quad l_{p,3}^\omega<H. \tag{48}$$

As ω decreases to its limiting value $4\pi\sigma_d^0/\varepsilon_d^0$, the quantity Re $\sigma_{c,3}^\omega(\omega)$ decreases with an exponent $t_3/(s_3+t_3)$ to its minimum value (15), while Re $\varepsilon_{c,3}^\omega(\omega)$ grows with an exponent $-s_3/(s_3+t_3)$ to its maximum value (39c).

A transition to two-dimensional behavior is observed for layers of thickness $H<l_{p,3}^\omega$. In this case, Eqs. (46a), (46b), (39a), and (39b) remain valid only for $\xi_3<H$, while for $\xi_3>H$ the following asymptotic forms are obtained from Eqs. (18) and (34):

$$\begin{aligned}\operatorname{Re}\sigma_\perp(\omega)&=\sigma_{c,\perp}^\omega=\sigma_m^0(H/a)^{-t_3/\nu_3},\\ \operatorname{Re}\varepsilon_\perp(\omega)&=\varepsilon_{c,\perp}^0=\varepsilon_d^0(H/a)^{s_3/\nu_3};\end{aligned} \tag{49}$$

$$\begin{aligned}\operatorname{Re}\sigma_\parallel(\omega)&=\sigma_m^0=\sigma_m^0\left(\frac{H}{a}\right)^{-t_3/\nu_3}\left(\frac{\xi_2}{H}\right)^{-t_2/\nu_3},\\ \operatorname{Re}\varepsilon_\parallel(\omega)&=A_2\varepsilon_d^0\left(\frac{H}{a}\right)^{s_3/\nu_3}\left(\frac{\xi_2}{H}\right)^{t_2/\nu_2}\end{aligned} \tag{50a}$$

$$\text{for}\quad \xi_2<\min(l_{p,2}^\omega,L),\qquad p>p_{c,\parallel};$$

$$\begin{aligned}\operatorname{Re}\sigma_\parallel(\omega)&=B_2\frac{(\omega\varepsilon_d^0)^2}{(4\pi)^2\sigma_m^0}\left(\frac{H}{a}\right)^{-t_3/\nu_3}\left(\frac{\xi_2}{H}\right)^{3s_2/\nu_2},\\ \operatorname{Re}\varepsilon_\parallel(\omega)&=\varepsilon_d^0\left(\frac{H}{a}\right)^{s_3/\nu_3}\left(\frac{\xi_2}{H}\right)^{t_2/\nu_2}\end{aligned} \tag{50b}$$

$$\text{for}\quad \xi_2<\min(l_{p,2}^\omega,L),\qquad p<p_{c,\parallel}.$$

Here we have introduced a new [compared to (43)] longitudinal mixed conductivity scale

$$l_{p,2}^\omega=H\left(\frac{\omega\varepsilon_d^0}{4\pi\sigma_m^0}\right)^{-\nu_2/2t_2}\left(\frac{H}{a}\right)^{-(s_3+t_3)/2t_2\nu_3}, \tag{51}$$

When the correlation length ξ_2 becomes equal to this scale, there occurs a crossover of the asymptotic forms (50a) and (50b). In this case, directly at the percolation threshold, for $\xi_2>l_{p,2}^\omega$, $l_{p,2}^\omega<L$, $l_{p,3}^\omega>H$, we have

$$\operatorname{Re}\sigma_\parallel(\omega)=\sigma_{c,\parallel}^\omega=\sigma_m^0\left(\frac{\omega\varepsilon_d^0}{4\pi\sigma_m^0}\right)^{1/2}\left(\frac{H}{a}\right)^{-(t_3-s_3)/2\nu_3}\propto\omega^{1/2}, \tag{52}$$

$$\operatorname{Re}\varepsilon_\parallel(\omega)=\varepsilon_{c,\parallel}^\omega=\varepsilon_d^0\left(\frac{\omega\varepsilon_d^0}{4\pi\sigma_m^0}\right)^{-1/2}\left(\frac{H}{a}\right)^{-(t_3-s_3)/2\nu_3}\propto\omega^{-1/2}. \tag{53}$$

These Eqs. (49)–(53) demonstrate the anisotropy of the dielectric properties of a metal–dielectric layer of thickness $H>l_{p,3}^\omega$. In other words, the percolation layer is anisotropic at frequencies

$$\omega<\left(\frac{4\pi\sigma_m}{\varepsilon_d^0}\right)\left(\frac{H}{a}\right)^{-(s_3+t_3)/\nu_3} \tag{52}$$

and the lower the frequency the larger the anisotropy. In this case, the peak in the dielectric permittivity as a function of the concentration of the metallic phase grows sharper compared to the three-dimensional system (Fig. 4). The exponent in the frequency dependence of the conductivity and dielectric permittivity at the percolation threshold also varies. A maximum in the longitudinal permittivity $\varepsilon_{c,\parallel}^\omega$ is observed at the percolation threshold of the layer. The following relations hold:

$$\frac{\sigma_{c,\parallel}^\omega}{\sigma_{c,3}^\omega}=\frac{\varepsilon_{c,\parallel}^\omega}{\varepsilon_{c,3}^\omega}=\left(\frac{\omega\varepsilon_d^0}{4\pi\sigma_m^0}\right)^{-(t_3-s_3)/2(s_3+t_3)}\left(\frac{H}{a}\right)^{-(t_3-s_3)/2\nu_3}$$

$$=\left(\frac{l_{p,3}^\omega}{H}\right)^{(t_3-s_3)/2\nu_3}, \tag{54}$$

$$\frac{\sigma_{c,\parallel}^\omega}{\sigma_{c,\perp}^\omega}=\left(\frac{\omega\varepsilon_d^0}{4\pi\sigma_m^0}\right)^{1/2}\left(\frac{H}{a}\right)^{(s_3+t_3)/\nu_3}=\left(\frac{l_{p,3}^\omega}{H}\right)^{-(t_3+s_3)/2\nu_3}, \tag{55}$$

$$\frac{\varepsilon_{c,\parallel}^\omega}{\varepsilon_{c,\perp}^\omega}=\left(\frac{\omega\varepsilon_d^0}{4\pi\sigma_m^0}\right)^{-1/2}\left(\frac{H}{a}\right)^{-(s_3+t_3)/2\nu_3}=\left(\frac{l_{p,3}^\omega}{H}\right)^{(t_3+s_3)/2\nu_3}. \tag{56}$$

6. POSSIBLE APPLICATIONS OF THE THEORY

Let us pause briefly to discuss certain possible applications of the theory presented here.

In the first place, there is the problem of calculating the electrophysical properties of dispersive layers with pairs of phase complex conductivities other than those of the metal–dielectric system discussed above. Current systems under study are ion-exchange membranes, porous electrodes partially saturated with electrolyte, current-carrying wires, diaphragms, emulsion and suspension films, and ceramic HTSC films deposited on substrates.

Second, the theory investigated here is, in fact, also a theory of viscoelasticity of composite films which, as is known from Refs. 26 and 27, can be formulated in the language of complex elastic moduli. In this case, the problem of determining the effective complex elastic modulus of a two-phase system is mathematically equivalent to the problem investigated here of finding the effective conductivity. The analogue of the conductivity is the elastic modulus while the analog of the dielectric permittivity is the viscosity.

A third problem of current interest is the investigation of the propagation of elastic and electromagnetic waves in percolation layers of finite thickness. Similar problems for two-dimensional and three-dimensional systems are discussed in Refs. 15 and 26–28.

A fourth problem, which is very current from the point of view of practical applications, is the investigation of systems consisting of percolation layers of finite thickness deposited on three-dimensional substrates with definite dielectric properties. The dielectric properties of a two-dimensional conductor–dielectric percolation system on a dielectric substrate were discussed in Ref. 6.

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