Anisotropic Heisenberg ferromagnet for a thin film

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The renormalization-group method is applied to an anisotropic Heisenberg ferromagnet in a thin film with magnetic-dipole interaction. It is shown that the renormalizations lead to an effective two-dimensional Hamiltonian with constants that depend on the thickness of the film and on the parameters of the starting Hamiltonian. The phase diagram in temperature-anisotropy variables is constructed. It is concluded that this diagram contains a tricritical point which is the boundary of three phases: uniaxial, planar, and disordered. The dependence of the critical temperature and anisotropy on the thickness of the film is determined.

1. INTRODUCTION

Thanks to the achievements of modern technology thin magnetic films are now being actively studied as promising materials for technological applications. In this connection it is important to clarify the conditions for the existence of and the type of ordered state in them. In contrast to bulky bodies, in ultrathin films the quasi-two-dimensionality of the film is strongly manifested. Thus anisotropy, which determines the existence of magnetic order, plays a special role. Magnetic films as two-dimensional systems are a special subject of study in the physics of critical phenomena. The large fluctuations that are characteristic of two-dimensional systems preclude application of the methods of the spin-wave theory to such films. The renormalization-group (RG) method makes it possible to investigate two-dimensional systems. Based on the renormalization group, in Refs. 1 and 2 the isotropic Heisenberg model was studied, in Refs. 3 and 4 the effect of a field and anisotropy was examined, and in Refs. 5–8 the effect of magnetic-dipole interaction in an isotropic two-dimensional Heisenberg system was investigated.

In this work the renormalization-group method is applied to a Heisenberg ferromagnetic model in a film of finite thickness with uniaxial anisotropy and magnetic-dipole interaction. The renormalization-group method permits obtaining an effective two-dimensional Hamiltonian with constants that depend on the thickness of the film and the starting parameters of the Hamiltonian—the temperature, the anisotropy, and the dipole-interaction constants.

Under repeated renormalizations of the two-dimensional Hamiltonian the dipole interaction induces an anisotropy of the easy-plane type in accordance with Refs. 3–5. The starting anisotropy competes with the dipole anisotropy. The result of the renormalization-group evolution depends on the relation between the starting anisotropy, the dipole constant, and the temperature. Analysis of the renormalization-group equations shows that the phase diagram in the temperature-anisotropy variables contains three regions corresponding to a disordered phase, a uniaxial or Ising phase, and a planar or XY-phase. The tricritical point is the common boundary of the three phases.

This paper is organized as follows. In Sec. 2 the model is defined and the renormalization-group equations are derived for $d = 2$ and $d = 3$. In Sec. 3 the solutions of the renormalization-group equations for $d = 2$ are analyzed. These solutions are joined with the solutions for $d = 3$, whence there follows the dependence of the critical parameters on the thickness of the film.

2. RENORMALIZATION-GROUP EQUATIONS

We shall define the classical spin $\mathbf{n}(\mathbf{x})$ as a three-component vector of unit length, located at the site x of a three-dimensional lattice, which is infinite in two directions and has a finite thickness l.

The Hamiltonian H has the form

$$H=-\sum_{\mathbf{x},\mathbf{a}} J\mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{x}+\mathbf{a})-\sum_{\mathbf{x}} Kn_3(\mathbf{x})n_3(\mathbf{x}) + \frac{1}{2}\mu_0^2\sum_{\mathbf{x},\mathbf{x}'}\frac{n_i(\mathbf{x})n_j(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}(\delta_{ij}-3\nu_i\nu_j). \quad (1)$$

The first term describes the exchange interaction for neighboring spins; here $\mathbf{a}$ is the lattice vector. The second term is the energy of uniaxial anisotropy; the axis "3" is oriented normally to the film. The third term describes the magnetic-dipole interaction, $\boldsymbol{\nu} = (\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|$, $i, j = 1, 2, 3$.

We shall redefine the constants as follows. We include the temperature T in the Hamiltonian ($H/T \to H$) and we shall transform to dimensionless quantities by making the substitutions $T \to TJ$, $K \to 1/2J\lambda a^2$, and $\mu_0^2 \to \mu^2 Ja^5$. Dropping the insignificant constant, we write the exchange term in terms of the derivative $\partial n = [n(x+a)-n(x)]/a$ and the sum $\Sigma a^3(\ldots)$ in terms of the integral $\int d^3x(\ldots)$. Then

$$H=\frac{1}{2Ta}\int d^3x\left\{(\partial_i\mathbf{n})^2-\lambda n_3^2+\mu^2\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|^3}n_in_j(\delta_{ij}-3\nu_i\nu_j)\right\}. \quad (2)$$

Following Refs. 1 and 2, we represent $\mathbf{n}(\mathbf{x})$ as a combination of the slow $\mathbf{n}_0(\mathbf{x})$ and fast $\phi(\mathbf{x})$ parts:

$$\mathbf{n}(\mathbf{x})=\mathbf{n}_0(\mathbf{x})(1-\phi\phi)^{1/2}+\phi_a\mathbf{e}_a,$$

$$\mathbf{n}_0\mathbf{e}_a=0,\quad \mathbf{e}_a\mathbf{e}_b=\delta_{ab},\quad a, b=1, 2.$$

The fast field $\phi_a \propto \exp(\mathbf{k}\cdot\mathbf{x})$, where $\mathbf{k}$ lies near the boundary of the Brillouin zone. We shall take into account only terms that are quadratic in ϕ. Then averaging over ϕ gives the equations

$$\langle(\partial_i\mathbf{n})^2\rangle=(\partial_i\mathbf{n}_0)^2(1-{}^1/_2\langle\phi^2\rangle),\quad \langle n_3^2\rangle=n_{03}^2(1-{}^3/_2\langle\phi^2\rangle), \quad (3)$$

$$\langle n_i(x)n_j(x')\rangle=n_{0i}(x)n_{0j}(x')(1-\langle\phi^2\rangle) + \langle\phi_a(x)\phi_b(x')\rangle e_a^i(x)e_b^j(x'). \quad (4)$$

The equation (3) determines the renormalization of the ex-

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change term and the anisotropy and Eq. (4) determines the renormalization of the dipole interaction and the additional contribution made to the anisotropy by the dipole coupling. In the three-dimensional case this contribution is zero in the approximation under study, but this is not the case for $d = 2$. Indeed, since the correlation function of the field ϕ is equal to

$$\langle \phi_a(x)\,\phi_b(x')\rangle = \delta_{ab} T a \int \frac{d^3k}{(2\pi)^3} \frac{\exp[i\mathbf{k}(\mathbf{x}-\mathbf{x}')]}{k^2}, \qquad (5)$$

where the integral extends over a layer of thickness $\pi(\delta a/a^2)$ along the boundary of the Brillouin zone, the dipole contribution to the anisotropy is determined by the pair potential of the dipole coupling averaged over the boundary of the Brillouin zone:

$$I_{ij}(\mathbf{k}) = a^3 \sum{}' \frac{\exp[i\mathbf{k}(\mathbf{x}-\mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|^3}(\delta_{ij}-3\nu_i\nu_j). \qquad (6)$$

Since k is large, the contribution of the boundaries of the film to I_{ij} can be neglected and it can be assumed that $e_a^i(x)e_b^j(x') \approx e_a^i(x)e_b^j(x)$. Then the average of I_{ij} is proportional to the trace I_{ii}, which for $d = 3$ is equal to zero.

We introduce the following parametrization for the scale: $a(t) = a_0 \exp(t)$. Then from Eq. (5) we find

$$\frac{1}{2}\langle\phi^2\rangle = T\frac{\delta t}{2\pi}C_1, \quad C_1 = \frac{6}{\pi}\int_0^1\int \frac{dx\,dy}{1+x^2+y^2} = 1{,}221. \qquad (7)$$

The constant C_1 appears because of the difference between a cube and a sphere. Thus the renormalization-group equations for $d = 3$ are as follows:

$$\frac{d}{dT}\left(\frac{1}{Ta}\right) = -\frac{C_1}{2\pi a}, \quad \frac{d}{dt}\left(\frac{\lambda}{Ta}\right) = -3\lambda\frac{C_1}{2\pi a},$$
$$\frac{d}{dt}\left(\frac{\mu^2}{Ta}\right) = -2\mu^2\frac{C_1}{2\pi a}. \qquad (8)$$

The constants T, λ, and μ^2 vary as functions of the scale a according to the solutions of Eqs. (8) until the scale is equal to the thickness of the film l. Here there arises an effective two-dimensional Hamiltonian:

$$H = \frac{1}{2T}\int d^2x\left\{(\partial_i\mathbf{n})^2 - \lambda n_3^2 + \mu^2\int \frac{d^2x'\,a}{|\mathbf{x}-\mathbf{x}'|^3} n_i n_j(\delta_{ij}-3\nu_i\nu_j)\right\}. \qquad (9)$$

At the scale $a = l$ its constants are related as follows with the starting parameters λ_0, μ_0^2, T_0, and a_0:

$$T_1 = \frac{a_0}{l}T_0\left[1-\frac{C_1}{2\pi}T_0\left(1-\frac{a_0}{l}\right)\right]^{-1},$$
$$\lambda_1 = \lambda_0\left(\frac{T_0 a_0}{T_1 l}\right)^2, \quad \mu_1^2 = \mu_0^2\left(\frac{T_0 a_0}{T_1 l}\right). \qquad (10)$$

The renormalization-group equations for the Hamiltonian (9) are derived analogously. For $d = 2$ it is necessary to average over the perimeter of the Brillouin square. Then

$$\langle \phi_a(\mathbf{x})\,\phi_b(\mathbf{x}')\rangle = \delta_{ab} T\int \frac{d^2k}{(2\pi)^2}\frac{\exp[i\mathbf{k}(\mathbf{x}-\mathbf{x}')]}{k^2},$$
$$^1/_2\langle\phi^2\rangle = T\delta t/2\pi. \qquad (11)$$

The average of the dipole potential is now different from zero and

$$\sum_{x,x'}{}' \langle\phi_a(\mathbf{x})\,\phi_b(\mathbf{x}')\rangle e_a^i e_b^j(\delta_{ij}-3\nu_i\nu_j)$$
$$= \left(\frac{1}{2}-\frac{3}{2}n_3n_3\right)\int \frac{d^2k a^2}{(2\pi)^2k^2}\sum_x{}' \frac{\exp(i\mathbf{kx})}{|\mathbf{x}|^3}$$
$$= \left(\frac{3}{2}n_3n_3-\frac{1}{2}\right)\frac{\delta t}{2\pi a}C_2, \quad C_2 = 1{,}209. \qquad (12)$$

Thus the renormalization-group equations for the Hamiltonian (9) have the form

$$\frac{d}{dt}\left(\frac{1}{T}\right) = -\frac{1}{2\pi}, \quad \frac{d}{dt}\left(\frac{\lambda}{T}\right) = \frac{3\lambda}{2\pi} - \frac{3}{2}C_2\mu^2\frac{l}{2\pi a},$$
$$\frac{d}{dt}\left(\frac{\mu^2}{T}\right) = -\frac{2\mu^2}{2\pi}, \qquad (13)$$

where $a(t) = le^t$, and Eqs. (10) are the initial conditions at $t = 0$.

3. TEMPERATURE–ANISOTROPY PHASE DIAGRAM

The solution of Eqs. (13) has the form

$$T(t) = \left(\frac{1}{T_1}-\frac{t}{2\pi}\right)^{-1}, \quad \mu^2(t) = \mu_1^2\frac{T_1}{T(t)},$$
$$\lambda(t) = \left(1-\frac{t}{2\pi}T_1\right)^2\left\{\lambda_1 - \lambda_1^*\frac{T_1}{2\pi}\int_0^t dt\,e^{-t}\left(1-\frac{t}{2\pi}T_1\right)^{-2}\right\}, \qquad (14)$$
$$\lambda_1^* = {}^3/_2C_2\mu_1^2.$$

As follows from Eq. (14), for sufficiently high initial temperature T_i on some scale t_c the effective temperature becomes infinite, and λ and μ approach zero. This means that the system is in a disordered state.[1] The scale on which $T \to \infty$, determines the correlation length $\xi \approx l\exp(2\pi/T_i)$.

If, however, the initial temperature T_i is sufficiently low, then as the scale increases the effective anisotropy per spin can become of the order of the exchange energy, i.e., $|\lambda(t)|a^2(t) \sim 1$. In this case the dipole energy can also become large. To determine the phase that the system is in we shall study the effective Hamiltonian in the long-wavelength approximation. Let $\mathbf{n}(\mathbf{x}) = \mathbf{n}_0(1-\phi\phi)^{1/2} + \phi_a\mathbf{e}_a$, where $\mathbf{n}_0 = \text{const}$ and $\phi(x)$ is the variable part. Then

$$HT = \frac{V}{2}\left[(\lambda_c-\lambda)n_{03}^2 - \frac{1}{3}\lambda_c\right]$$
$$+\frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\phi_a(\mathbf{k})\,\phi_b(-\mathbf{k})\left\{\delta_{ab}\left[k^2+(\lambda-\lambda_c)n_{03}^2+\frac{1}{3}\lambda_c\right]\right.$$
$$\left.+\mu^2 I_{ij}(k)e_a^i e_b^j\right\}, \quad \lambda_c = \frac{3}{2}\sum{}'\frac{a^3}{|x|^3} = 4\pi C_0\mu^2, \quad C_0 = 1{,}078. \qquad (15)$$

In Eq. (15) V is the area and $I_{ij}(\mathbf{k})$ is defined in Eq. (6). Minimizing the uniform part of the Hamiltonian in Eq. (15) we find that a uniaxial state is realized for $\lambda > \lambda_c$ and a two-dimensional state is realized for $\lambda < \lambda_c$. In Appendix 1 the stability of the uniform state with respect to long-wavelength perturbations $\phi(x)$ is analyzed. The instability arises only in a narrow neighborhood of $\lambda \approx \lambda_c$. Thus the condition for the existence of an ordered state is that on a scale $t < t_c$ the equality

$$|\lambda-\lambda_c|a^2 = 1 \qquad (16)$$

must be satisfied. In this case, if $\lambda < \lambda_c$, then the magnet

becomes effectively a two-component magnet and over large scales the spin lies in a plane.[3-5] For a planar (XY) magnet the effective temperature does not increase, and the long-range order is determined by the dipole interaction.[5-8]

For $\lambda > \lambda_c$ on scales where $(\lambda - \lambda_c)a^2 \sim 1$ there arises an Ising one-component magnet in the ordered state. The temperature does not increase, since the transverse fluctuations are suppressed by the anisotropy.

Thus, depending on the initial conditions, the system is in the disordered or in the two-dimensional phase or in the uniaxial phase. The boundaries of the phases are determined by the fact that on them the condition $|\lambda - \lambda_c|a^2 \sim 1$ no longer holds. We introduce the notation $t_c = 2\pi/T_i$. Evaluating the integral in Eq. (14) we obtain

$$\lambda(t) = \left(1 - \frac{t}{t_c}\right)^2 \left(\lambda_1 - \frac{\lambda_1^*}{t_c}\right).$$

The scale t at which renormalization of the temperature terminates is determined by the equation

$$\left|\left(1 - \frac{t}{t_c}\right)^2 \left(\lambda_1 - \frac{\lambda_1^*}{t_c}\right) - \lambda_{c1}\left(1 - \frac{t}{t_c}\right)\right| l^2 = e^{-2t},$$

$$\lambda_{c1} = 4\pi C_0 \mu_1^2. \qquad (17)$$

The parameters corresponding to the boundary of the phases can be obtained from the condition that the curves described by the left and right sides of Eq. (17) intersect. The behavior of the roots of Eq. (17) is analyzed in detail in Appendix 2.

The boundary of the uniaxial phase is determined by the equation

$$\lambda_1 = \lambda_1^*/t_c + t_c\lambda_{c1} \operatorname{sh}[\theta(t_c)], \qquad (18)$$

where $\theta(t_c)$ is the solution of Eq. (A2.3) (see Appendix 2). At low temperatures we have from Eqs. (A2.4) and (18) the linear section

$$\lambda_1 = (T_c/2\pi)\lambda_1^* + \lambda_{c1}. \qquad (19)$$

At higher temperatures there follows from the same expressions a relation between the critical temperature and the anisotropy constant in the form

$$\lambda_1 = \frac{T_c}{2\pi}\lambda_1^* + \frac{2\pi}{T_c}\lambda_{c1} \operatorname{sh}\left[-\frac{4\pi}{T_c} + 2 + \ln\left(\frac{4\pi}{T_c\lambda_{c1}l^2}\right)\right]. \qquad (20)$$

In Eq. (20) the values of T_c are such that the argument of sinh is greater than zero.

The boundary of the two-dimensional phase is determined by Eq. (18), but here $\theta(t_c)$ must be the solution of Eq. (A2.5). As shown in Appendix 2 the boundaries of the two-dimensional and uniaxial phases intersect at a point determined by Eqs. (A2.6). At the boundary of the disordered phase the renormalized constants λ and μ vanish, so that here the average spin $\langle n \rangle$ also vanishes. Therefore this boundary is the line of second-order phase transitions. The common point of the boundaries is the tricritical point, since it lies at the intersection of the lines of the second-order phase transitions. The coordinates of the tricritical point follow from Eq. (A2.6):

$$\lambda_{1(3c)} = \lambda_1^*/t_3 + t_3\lambda_{c1} \operatorname{sh}\theta_{3c}. \qquad (21)$$

For $\lambda^* l^2 = 10^{-4}$ we find from Eq. (A2.6) $t_3 = 5.5799$, i.e., $T_3 = 2\pi \cdot 0.1792$ and $\lambda_{i(3c)} = 70.71\lambda^*$. In the region where the anisotropy is less than the tricritical anisotropy but greater than λ_{ci} the boundary of the uniaxial phase is determined approximately by the expression (21), where t_3 is replaced by the variable t_c.

Figure 1 shows the $T_c(\lambda_i)$ dependence for $\lambda^* l^2 = 10^{-4}$. The quantities λ_i/λ^* and $T_c/2\pi$ are plotted along the axes. The regions I, II, and III in Fig. 1 correspond to the XY phase, the Ising phase, and the disordered phase.

The solutions of the renormalization-group equations are inapplicable in the region of large fluctuations, i.e., near the boundaries of the phases. The region of applicability of the renormalization-group equations is determined by the smallness of the renormalized temperature compared with unity. This makes it possible to draw a conclusion about the existence of the XY and Ising phases, about the position of the boundary between them (I–II), and about their boundaries with the phase III (I–III) and (II–III). In the phase III the renormalization-group equations are applicable only on scales less than ξ (see above), but there is no basis for expecting here an ordered state.

Using Eqs. (10) and (18) we find the dependence of the critical physical parameters of the Hamiltonian (2) on the film thickness l:

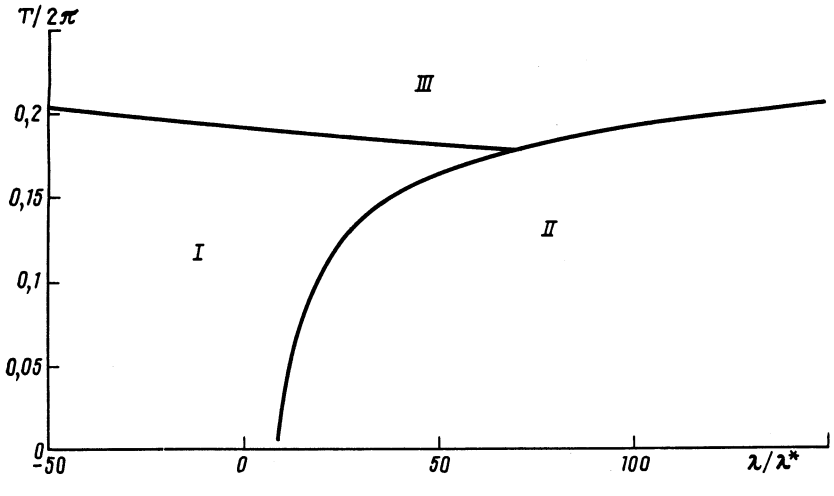

FIG. 1. The phase diagram for the two-dimensional system: I—the XY-phase, II—the Ising phase, and III—the disordered phase.

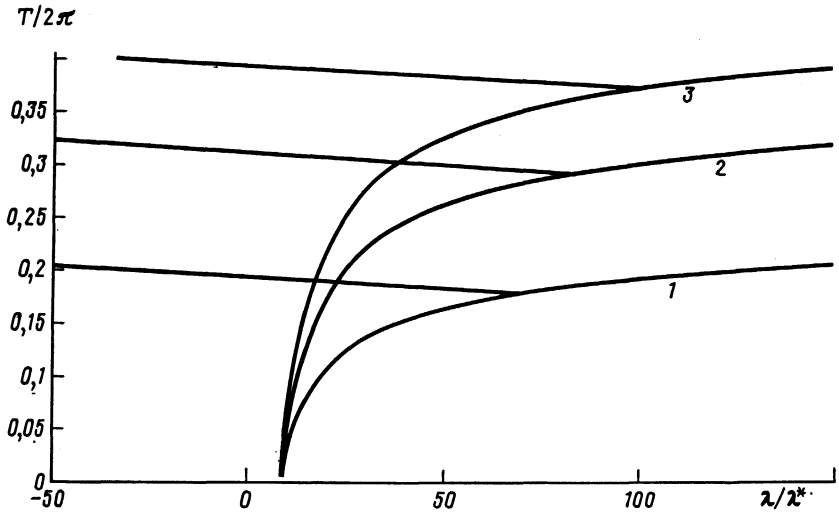

FIG. 2. The phase diagram for several layers. The numbers on the curves correspond to thicknesses equal to 1, 2, and 3 layers of atoms.

$$\lambda_0=\frac{T_0}{2\pi}\frac{a_0}{l}\lambda_0^*\left[1-C_1\frac{T_0}{2\pi}\left(1-\frac{a_0}{l}\right)\right]^{-2}+\frac{2\pi}{T_0}\frac{l}{a_0}\lambda_c\,\mathrm{sh}\,\theta, \quad (22)$$

where θ is the solution of Eq. (A2.3) or (A2.5), respectively, for the boundary of the uniaxial or two-dimensional phases with

$$t_c=\frac{2\pi l}{T_0 a_0}\left[1-C_1\frac{T_0}{2\pi}\left(1-\frac{a_0}{l}\right)\right]. \quad (23)$$

The coordinates of the tricritical point as a function of the thickness will be

$$T_3(l)=\frac{2\pi l}{t_3 a_0}\left[1+\frac{C_1}{t_3}\left(\frac{l}{a_0}-1\right)\right]^{-1},$$
$$\lambda_3(l)=\lambda_0^*\left(\frac{1}{t_3}+t_3\frac{8\pi C_0}{3C_2}\,\mathrm{sh}\,\theta_{3c}\right)\left[1-\frac{C_1}{t_3}\frac{l/a_0-1}{1+(l/a_0-1)C_1/t_3}\right]^{-1}. \quad (24)$$

Figure 2 shows the functions $T_{0c}(\lambda_0)$ for $l/a_0 = 1$, 2, and 3 for $\lambda_0^* a_0^2 = 10^{-4}$. For sufficiently large anisotropy $\lambda > \lambda^*$ we obtain

$$T_{0c}(\lambda_0, l)=2\pi\left[1+\frac{a_0}{l}\ln\left(\frac{1}{a_0^2\lambda_0^{1/2}}\right)\right]^{-1}. \quad (25)$$

For $l/a_0 \gg 1$ the expression (25) changes into a dependence, corresponding to the scaling estimates of Ref. 9, of the shift in T_c on the thickness of the film. We note that the formula (24) is also applicable for $l/a_0 \sim 1$.

In conclusion it should be noted that in this work we neglected the domain structure, whose scale for sufficiently low temperatures is comparable to the scale of the fluctuations (see Appendix 1). This requires a separate analysis.

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APPENDIX 1

It follows from Eq. (15) that instabilities of the uniform state correspond to negative eigenvalues of the matrix

$$M_{ab}=\delta_{ab}[k^2+(\lambda-\lambda_c)n_{03}^2+{}^1/_3\lambda_c]+\mu^2 I_{ij}(k)e_a^i e_b^j. \quad (A1.1)$$

We shall find the quantity $I_{ij}(k)$, defined in Eq. (6). Let

$$I_{33}(k)=\sum{}'\frac{a^3}{|\mathbf{x}|^3}\exp(i\mathbf{kx})=D(k),$$
$$I_{ab}(k)=a^3\sum{}'\frac{\exp(i\mathbf{kx})}{|\mathbf{x}|^3}(\delta_{ab}-3v_a v_b) \quad (A1.2)$$
$$=\delta_{ab}A(k)+\frac{k_a k_b}{k^2}B(k), \quad a,b=1,2.$$

Since $I_{ii}=0$, $2A(k)+B(k)=-D(k)$. For $k=0$ we have $B(k)=0$, and therefore $2A(0)=-D(0)$. For $D(0)$ we find

$$D(0)=\sum{}'\frac{a^3}{|\mathbf{x}|^3}=\frac{8\pi}{3}C_0, \quad C_0=1{,}078. \quad (A1.3)$$

For small k the contribution linear in $|\mathbf{k}|$ to the sum (A1.2) is determined by an integral which can be easily calculated. Thus we find

$$D(k)={}^8/_3\pi C_0-2\pi|\mathbf{k}|a, \quad A(k)=-{}^4/_3\pi C_0, \quad B(k)=2\pi|\mathbf{k}|a. \quad (A1.4)$$

We shall study perturbations of the uniaxial state. It follows from Eqs. (A1.1) and (A1.4) that the eigenvalues of the matrix M for the longitudinal and transverse vector $\mathbf{k}$ of the waves have the form

$$k^2+\lambda-\lambda_c+2\pi k a\mu^2 \quad \text{for} \quad \mathbf{e}\|\mathbf{k},$$
$$k^2+\lambda-\lambda_c \quad \text{for} \quad \mathbf{e}\perp\mathbf{k}.$$

For this reason, for $\lambda > \lambda_c$ the uniaxial state is stable.

For the two-dimensional state $\mathbf{n}_0$ lies in the 1–2 plane. We orient $\mathbf{e}_2$ along the "3" axis and we put $\mathbf{e}_1$ in the 1–2 plane. Then the wave ϕ_i corresponds to the eigenvalue

$$k^2+2\pi\mu^2 k a\sin^2\alpha>0,$$

where α is determined by the relation $\cos\alpha = \mathbf{kn}_0/|\mathbf{k}|$. For ϕ_2 we find

$$k^2+\lambda_c-\lambda-2\pi\mu^2 k a. \quad (A1.5)$$

It follows from Eq. (A1.5) that for $\lambda_2 < \lambda < \lambda_c$, where

$$\lambda_2=\lambda_c[1-\lambda_c a^2/(16C_0^2)], \quad (A1.6)$$

the two-dimensional state is unstable for perturbations with

$k \approx \lambda_c a/(4C_0)$. For this reason for the indicated values of λ a nonuniform state with the scale $L \approx 1/(\lambda_c a)$ is realized. For $\lambda < \lambda_c$ the two-dimensional state is stable.

APPENDIX 2

To find the limiting roots of Eq. (17) we introduce the notation $y = 1 - t/t_c$ and $x = (\lambda_1 - \lambda_1^*/t_c)/t_c$. From the condition $0 < t < t_c$ it follows that $0 < y < 1$. Then Eq. (17) can be rewritten in the form

$$|y^2 x t_c - y| = (\lambda_{c1} l^2)^{-1} \exp[-2t_c(1-y)]. \quad \text{(A2.1)}$$

Differentiating and taking (A2.1) into account, we find the equation for the tangent:

$$2yxt_c - 1 = 2t_c(y^2 x t_c - y). \quad \text{(A2.2)}$$

It follows from Eq. (A2.2) that

$$y = \frac{1}{2t_c}\left[\frac{1}{x} + 1 \pm (1+x^{-2})^{1/2}\right].$$

We set $x = \sinh\theta$. Then for $\lambda - \lambda_c > 0$ we obtain from Eq. (A2.1)

$$\theta = -2t_c + 1 + \ln\left(\frac{2t_c}{\lambda_{c1} l^2}\right) + \operatorname{cth}\frac{\theta}{2}. \quad \text{(A2.3)}$$

Since $0 < y < 1$, $\theta > 0$. The root of Eq. (A2.3) determines the boundary of the uniaxial phase. For low temperatures t_c is large and $\theta \to 0$. In this case

$$\frac{2}{\theta} = 2t_c - 1 - \ln\left(\frac{2t_c}{\lambda_{c1} l^2}\right). \quad \text{(A2.4)}$$

For $\lambda - \lambda_c < 0$ it follows from Eq. (A2.1) that

$$-\theta = -2t_c + 1 + \ln\left(\frac{2t_c}{\lambda_{c1} l^2}\right) - \operatorname{th}\frac{\theta}{2}. \quad \text{(A2.5)}$$

The root of Eq. (A2.5) determines the boundary of the two-dimensional phase. Here θ is arbitrary, but it is necessary to take into account the fact that there exist trajectories in the renormalization-group space which as t increases at first fall in the region where the uniaxial phase stabilizes, i.e., Eq. (17) is satisfied. Then, as the trajectories are continued, they fall into the region where the two-dimensional phase is stable. For such trajectories the corresponding solutions of Eq. (A2.5) must be dropped. Since for $\theta < 0$ this does not happen, in Eq. (A2.5) $\theta < \theta_{3c}$ necessarily, where θ_{3c} is the simultaneous solution of Eqs. (A2.3) and (A2.5). This solution corresponds to the intersection of the boundaries of the uniaxial and two-dimensional phases and determines the tricritical point. For θ_{3c} and t_{3c} we have from Eqs. (A2.3) and (A2.5) the equations

$$\begin{aligned} \theta_{3c} &= {}^1/_2[\operatorname{th}({}^1/_2\theta_{3c}) + \operatorname{cth}({}^1/_2\theta_{3c})], \\ 2t_{3c} - 1 - \ln\left(\frac{2t_{3c}}{\lambda_{c1} l^2}\right) &= \theta_{3c} - \operatorname{th}\frac{\theta_{3c}}{2}. \end{aligned} \quad \text{(A2.6)}$$

From the first equation we find $\theta_{3c} = 1.1997$.

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