Fermi-liquid dynamics in metals and the electron-phonon interaction

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We derive a system of dynamic equations for metals consisting of the Boltzmann kinetic equation for the conduction electrons, the equations of elasticity theory for the lattice, and the Maxwell equations. The system takes into account the electron-electron Fermi-liquid interaction, which leads to coupling between these equations. We find exact identities for the electron-phonon interaction vertices; these identities in turn determine the vertex for the interaction of the electrons with long-wavelength low-frequency phonons in terms of a given electron energy spectrum and the Landau Fermi-liquid function.

When a metal is sufficiently close to a perfect crystal, its macroscopic dynamic properties at low temperatures are not described by local equations of hydrodynamic type, which correspond to the presence of a finite number of Goldstone degrees of freedom in the metal dynamics. The reason for this is the Fermi-liquid behavior of the conduction electrons, which gives rise to the appearance of an infinite number of "Fermi-like Goldstone-ons", i.e., gapless Fermi-like excitations which are characterized by an infinitely long mean free path at zero temperature in an ideal crystal.

The equations that describe the macroscopic dynamics of metals make up a coupled system, consisting of equations from the theory of elasticity, the Boltzmann kinetic equation for the conduction electrons, and the Maxwell equations for the electromagnetic field (see Ref. 1 of Kontorovich and the literature cited therein). Puchkavor and one of the present authors[2] identified a system of nonlinear equations for the dynamics of a metal, and showed that this system could be uniquely derived from general principles of Galilean invariance, symmetries and conservation laws, as is the case for equations of hydrodynamic type used to describe the dynamics of other macroscopic bodies. However, the authors of Ref. 2 did not include the electron-electron Fermi-liquid interaction in their analysis.

The basic goal of this paper is to derive macroscopic dynamic equations for metals that include the Fermi-liquid electron-electron interaction. The equations obtained here also apply to Fermi-like quantum crystals, for example, spin-polarized crystals of ^{3}He with no vacancies (see Ref. 3), in which a quantum delocalization of the particles occurs (for this special case the electric charge of a quasiparticle equals zero and the Maxwell equations are not needed). Dzyaloshinskii, Kondratenko, and Levchekov[4] investigated the system of linear equations for a quantum crystal; the difference between these equations and the linearized version of the equations we derive here will be discussed below.

The equations obtained in this paper for an ideal crystal at zero temperature are exact in the sense that the only requirement for their applicability is a large number of spatial and temporal scales of the motion. As we will show below, one consequence of these equations is a series of exact identities for the electron-phonon interaction vertices. We emphasize that our discussion will center around the exact vertex; the introduction of any sort of "bare" electron-phonon vertex (e.g., in a Hamiltonian of Froehlich type) must necessarily be model-dependent character. Such a vertex is absent in the true Hamiltonian of electrons and nuclei which make up the metal.

1. A COMPLETE SYSTEM OF DYNAMIC EQUATIONS

Following the paper by Pushkarov and one of the authors,[2] we will describe the motion of the metal lattice in the general nonlinear case using the Lagrangian coordinates $N^\alpha = N^\alpha(\mathbf{r},t)$, which are functions of the usual Eulerian coordinates $\mathbf{r}$ and time. The local values of the vectors $\mathbf{a}^\alpha(\mathbf{r},t)$ $(\alpha = 1,2,3)$ of the reciprocal lattice and the lattice velocity $\mathbf{v}_L(\mathbf{r},t)$ are determined by the spatial and time derivatives of N^α, respectively:

$$\mathbf{a}^\alpha = \nabla N^\alpha, \quad \mathbf{v}_L = -\mathbf{a}_\alpha \dot{N}^\alpha, \tag{1}$$

where $\boldsymbol{\alpha}_\alpha$ is an elementary translation vector of the direct lattice connected with $\mathbf{a}^\alpha$ by the relations

$$\mathbf{a}_\alpha \mathbf{a}^\beta = \delta_\alpha{}^\beta, \quad a_{\alpha i} a_k{}^\alpha = \delta_{ik}. \tag{2}$$

The density of the lattice is $\rho_L = Mg^{-1/2}$ where M is the mass of the ions in a unit cell, and g is the determinant of the metric "tensor" $g_{\alpha\beta} = \mathbf{a}_\alpha \mathbf{a}_\beta$. The "tensor" $g^{\alpha\beta}$ inverse to it equals $\mathbf{a}^\alpha \mathbf{a}^\beta$. By virtue of Eqs. (1) and (2), the derivatives of the vectors $\mathbf{a}^\alpha$ and $\mathbf{a}_\alpha$ with respect to time can be expressed in terms of the spatial derivatives:

$$\dot{\mathbf{a}}_\alpha = -(\mathbf{v}_L \nabla)\mathbf{a}_\alpha + (\mathbf{a}_\alpha \nabla)\mathbf{v}_L, \tag{3}$$

$$\dot{a}_i{}^\alpha = -(\mathbf{v}_L \nabla) a_i{}^\alpha - a_k{}^\alpha \partial v_{Lk}/\partial x_i.$$

Equations (3) and (1) along with the relation $dg = -gg_{\alpha\beta} dg^{\alpha\beta}$ imply that the equation of continuity is identically satisfied for the lattice:

$$\dot{\rho}_L + \operatorname{div} \mathbf{v}_L \rho_L = 0. \tag{4}$$

Let

$$\mathscr{E}_0 = \mathscr{E}_0\{n(s_\alpha), g^{\alpha\beta}\}$$

be the energy per unit volume of the stationary metal $(\mathbf{v}_L = 0)$, which is a function of the invariant metric tensor and a functional of the electron distribution function $n(s_\alpha)$, where s_α is the component of the invariant quasimomentum that is canonically conjugate to the coordinate N^α (see Ref. 2). The physical quantities must be periodic functions of s_α with period $2\pi\hbar$. We write the variation $\delta\mathscr{E}_0$ in the form

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$$\delta\mathscr{E}_0=g^{-1/2}\int \varepsilon(s_\alpha)\,\delta n(s_\alpha)\,d\tau_s+\sigma_{\alpha\beta}\delta g^{\alpha\beta}. \tag{5}$$

Here $d\tau_s = 2d^3s/(2\pi\hbar)^3$, and $\varepsilon(s_\alpha)$ and $\sigma_{\alpha\beta}$ are functionals of $n(s_\alpha)$; $\varepsilon(s_\alpha)$ is in some sense the energy of a quasiparticle with invariant quasimomentum s_α. The appearance of the factor $g^{-1/2}$ arises because the density of states in the variables s_α, $\mathbf{r}$ equals $2g^{-1/2}d^3sd^3r/(2\pi\hbar)^3$. As is well known, the quantity $\varepsilon(s_\alpha)$ in a stationary crystal also plays the role of a Hamiltonian function for the quasiparticles.

In order to find the quasimomentum $\mathbf{p}$ that is canonically conjugate to the usual coordinate $\mathbf{r}$ and Hamilton's function $H(\mathbf{r},\mathbf{p},t)$ for a crystal moving with velocity $\mathbf{v}_L$, we will start with the formula for Galilean transformations. Since the transformation law of the Ψ-operators for interacting electrons under a Galilean transformation is the same as that for free electrons, the transformation law for $\mathbf{p}$ and H coincides with that found in Ref. 2:

$$\begin{aligned}&\mathbf{p}=\mathbf{a}^\alpha s_\alpha+m\mathbf{v}_L,\\ &H=\varepsilon+\mathbf{p}\mathbf{v}_L-m\mathbf{v}_L^2/2.\end{aligned} \tag{6}$$

Here m is the mass of a free electron. (For a Fermi-like quantum crystal we should understand by m the mass of the isolated atoms which make up the crystal.)

The electron distribution function $n(\mathbf{r},\mathbf{p},t)$ satisfies the kinetic equation which includes the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields

$$\frac{\partial n}{\partial t}+\frac{\partial n}{\partial \mathbf{r}}\frac{\partial H}{\partial \mathbf{p}}-\frac{\partial n}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{r}}-\frac{\partial n}{\partial \mathbf{p}}\left\{e\mathbf{E}+\frac{e}{c}\left[\frac{\partial H}{\partial \mathbf{p}}\mathbf{B}\right]\right\}=In, \tag{7}$$

where I is the collision operator. Here, in contrast to the equation for noninteracting electrons, the energy ε and the Hamiltonian function are functionals of the distribution function.

We transform Eq. (5) for the variation of the energy $\mathscr{E}_0$ to a system in which the lattice is stationary by introducing the usual quasimomentum $\mathbf{p}$ in place of s_α. The variation $(\delta n)_s$ at constant s_α is related to the variation $(\delta n)_\mathbf{p}$ at constant $\mathbf{p}$ by the relation

$$(\delta n)_s=(\delta n)_\mathbf{p}+(\partial n/\partial \mathbf{p})\,\delta\mathbf{p},$$

where

$$\delta\mathbf{p}=s_\alpha\delta\mathbf{a}^\alpha+m\delta\mathbf{v}_L$$

is the variation of $\mathbf{p}$ for constant s_α. Expressing $\delta g^{\alpha\beta}=\delta(\mathbf{a}^\alpha\mathbf{a}^\beta)$ in terms of $\delta\mathbf{a}^\alpha$ as well, and taking into account the equation $d^3s=g^{-1/2}d^3p$, we obtain

$$\delta\mathscr{E}_0=\int \varepsilon\delta n(\mathbf{p})\,d\tau_\mathbf{p}-\mathbf{P}_0\delta\mathbf{v}_L+\mathbf{L}_\alpha\delta\mathbf{a}^\alpha, \tag{8}$$

where $d\tau_\mathbf{p}=2d^3p/(2\pi\hbar)^3$, and

$$\mathbf{P}_0=-m\int d\tau_\mathbf{p}\,\varepsilon\frac{\partial n}{\partial \mathbf{p}}=m\int d\tau_\mathbf{p}\frac{\partial\varepsilon}{\partial\mathbf{p}}n, \tag{9}$$

$$L_{\alpha i}=2\sigma_{\alpha\beta}a_i^\beta+a_{\alpha k}\left(v_{Lk}P_{0i}+\int d\tau_\mathbf{p}\varepsilon p_k\frac{\partial n}{\partial p_i}\right). \tag{10}$$

Note that the vector $\mathbf{P}_0$, according to Eq. (9), is the electron momentum per unit volume of metal in the system of the stationary lattice.

The equations of motion of the lattice are contained in the law of conservation of momentum

$$\dot P_i+\partial\Pi_{ik}/\partial x_k=0, \tag{11}$$

where $\mathbf{P}$ is the total momentum per unit metallic volume and Π_{ik} is the symmetric momentum flux tensor, which must be determined from the condition for conservation of energy. As a sequence of Eqs. (7) and (11) and the Maxwell equations, this condition requires that

$$\operatorname{div}\mathbf{B}=0,\ \operatorname{rot}\mathbf{B}=(4\pi/c)\mathbf{j},\ \operatorname{rot}\mathbf{E}=(-1/c)\dot{\mathbf{B}}, \tag{12}$$

where $\mathbf{j}=-(\mathbf{e}/\mathbf{m})P_0$, be an equation of the form

$$\dot{\mathscr{E}}+\operatorname{div}\mathbf{Q}=0, \tag{13}$$

where $\mathscr{E}$ is the total energy per unit volume of the metal and $\mathbf{Q}$ is the required definition of the energy flux vector.

The momentum $\mathbf{P}$ and energy $\mathscr{E}$ are determined from the formula for a Galilean transformation

$$\mathbf{P}=\rho\mathbf{v}_L+\mathbf{P}_0, \tag{14}$$

$$\mathscr{E}=\frac{\rho\mathbf{v}_L^2}{2}+\mathbf{v}_L\mathbf{P}_0+\mathscr{E}_0+\frac{\mathbf{B}^2}{8\pi}. \tag{15}$$

Here ρ is the total density of the metal and equals

$$\rho=\rho_L+m\int d\tau_\mathbf{p}n. \tag{16}$$

In the paper by Dzyaloshinskii *et al.*[4] no distinction was made between the energy and the Hamiltonian function of the electrons, although the first of these must, and the second need not, be a periodic function of the quasimomentum. This latter fact is connected with the circumstance that only nonperiodic Hamiltonian functions can ensure consistency of the Boltzmann kinetic equation with the condition that the distribution function be periodic (with variable period). Furthermore, in Ref. 4 the function $\mathbf{P}_0(\mathbf{p})$ was treated as indeterminate, i.e., Eq. (9) was not included. In connection with this note that Eq. (9) is valid both for the case of metals, where the momentum in the system of the stationary lattice is a pure electronic momentum (the nuclei are localized at the lattice sites), and for the case of quantum crystals, which consist of particles of a single kind. In both cases the momentum and particle flux operators in the system of the stationary lattice are proportional to one another.

By differentiating Eq. (15) with respect to time and including Eqs. (3), (8), and (12) we obtain

$$\dot{\mathscr{E}}=\frac{\mathbf{v}_L^2}{2}\dot\rho+\rho\mathbf{v}_L\dot{\mathbf{v}}_L+\dot{\mathbf{P}}_0\mathbf{v}_L+\int d\tau_\mathbf{p}\,\varepsilon\dot n+\mathbf{L}_\alpha\dot{\mathbf{a}}^\alpha-\operatorname{div}\mathbf{S}-\mathbf{E}\mathbf{j}, \tag{17}$$

where $\mathbf{S}$ is the Poynting vector. Here and below we will follow the rule formulated in Ref. 2, according to which we may omit integrals that appear in intermediate equations over the boundaries of the Brillouin zone. Such integrals cancel out in the final expressions.

From Eqs. (4), (6), (7), (11), (14), and (16) there follow the equations

$$\dot\rho=-\operatorname{div}\left(\rho_L\mathbf{v}_L+m\int d\tau_\mathbf{p}\frac{\partial H}{\partial\mathbf{p}}n\right), \tag{18}$$

$$\rho\dot v_{Li}+\dot P_{0i}=-\frac{\partial\Pi_{ik}}{\partial x_k}+v_{Li}\operatorname{div}\left(\rho_L\mathbf{v}_L+m\int d\tau_\mathbf{p}\frac{\partial H}{\partial\mathbf{p}}n\right), \tag{19}$$

$$\int d\tau_\mathbf{p}\,\varepsilon\dot n=\int d\tau_\mathbf{p}\,\varepsilon\dot{f}n+\mathbf{j}\left(\mathbf{E}+\frac{1}{c}[\mathbf{v}_L\mathbf{B}]\right)-\operatorname{div}\left(\int d\tau_\mathbf{p}\,\varepsilon\frac{\partial\varepsilon}{\partial\mathbf{p}}n\right)$$

$$-\mathbf{v}_L \int d\tau_p \varepsilon \frac{\partial n}{\partial \mathbf{r}} - \frac{\partial v_{Li}}{\partial x_k} \int d\tau_p n \left\{ p_i \frac{\partial \varepsilon}{\partial p_k} - m v_{Li} \frac{\partial \varepsilon}{\partial p_k} + \delta_{ik} \varepsilon \right\}. \tag{20}$$

With the help of identity (3) we find

$$\mathbf{L}_\alpha \dot{\mathbf{a}}^\alpha = -\mathbf{L}_\alpha (\mathbf{v}_L \nabla) \mathbf{a}^\alpha - \mathbf{L}_{\alpha i} \mathbf{a}_k^\alpha \frac{\partial v_{Lk}}{\partial x_i}. \tag{21}$$

The first term in the right side of Eq. (21) is conveniently transformed by using the equation

$$(\mathbf{v}_L \nabla) \mathscr{E}_0 = \mathbf{v}_L \int d\tau_p \varepsilon \frac{\partial n}{\partial \mathbf{r}} - m v_{Lk} \frac{\partial v_{Li}}{\partial x_k} \int d\tau_p \frac{\partial \varepsilon}{\partial p_i} n + \mathbf{L}_\alpha (\mathbf{v}_L \nabla) \mathbf{a}^\alpha,$$

which is a direct consequence of identity (8). We then have

$$\begin{aligned} \mathbf{L}_\alpha \dot{\mathbf{a}}^\alpha = & -(\mathbf{v}_L \nabla) \mathscr{E}_0 + \mathbf{v}_L \int d\tau_p \varepsilon \frac{\partial n}{\partial \mathbf{r}} \\ & - \frac{\partial v_{Li}}{\partial x_k} \left\{ 2\sigma_{\alpha\beta} a_i^\alpha a_k^\beta + P_{0i} v_{Lk} + v_{Li} P_{0k} \right. \\ & \left. - \int d\tau_p n \left(p_i \frac{\partial \varepsilon}{\partial p_k} + \varepsilon \delta_{ik} \right) \right\}. \end{aligned} \tag{22}$$

Finally, substituting Eqs. (18)–(20) and (22) into Eq. (17) we obtain

$$\begin{aligned} & \dot{\mathscr{E}} + \operatorname{div} \left\{ \mathbf{v}_L \mathscr{E}_0 + \int d\tau_p \varepsilon \frac{\partial \varepsilon}{\partial \mathbf{p}} n - \frac{\mathbf{v}_L^2}{2} \mathbf{P} + v_{Lk} (\Pi_{ik} + t_{ik}) + \mathbf{S} \right\} \\ & = \int d\tau_p \varepsilon \dot{f} n + \frac{\partial v_{Li}}{\partial x_k} \{ \Pi_{ik} + t_{ik} - \rho v_{Li} v_{Lk} - v_{Li} P_{0k} - v_{Lk} P_{0i} \\ & - 2\sigma_{\alpha\beta} a_i^\alpha a_k^\beta + \mathscr{E}_0 \delta_{ik} \}, \end{aligned} \tag{23}$$

where t_{ik} is the Maxwell tensor for the magnetic field intensity.

By comparing Eqs. (23) and (13) we find the currents we are looking for:

$$\Pi_{ik} = -t_{ik} + \rho v_{Li} v_{Lk} + v_{Li} P_{0k} + v_{Lk} P_{0i} + 2\sigma_{\alpha\beta} a_i^\alpha a_k^\beta - \mathscr{E}_0 \delta_{ik}, \tag{24}$$

$$\mathbf{Q} = \mathbf{v}_L \mathscr{E}_0 + \int d\tau_p \varepsilon \frac{\partial \varepsilon}{\partial \mathbf{p}} n - \frac{\mathbf{v}_L^2}{2} \mathbf{P} + v_{Lk} (\Pi_{ik} + t_{ik}) + \mathbf{S}. \tag{25}$$

For noninteracting electrons, expressions (24) and (25) reduce to the corresponding expressions from Ref. 2. In fact, from identity (5) we obtain an equation for the second mixed derivatives:

$$\frac{\delta \sigma_{\alpha\beta}}{\delta n(\mathbf{s})} = \frac{\partial}{\partial g^{\alpha\beta}} (g^{-1/2} \varepsilon) = g^{-1/2} \frac{\partial \varepsilon}{\partial g^{\alpha\beta}} + \frac{1}{2} g_{\alpha\beta} \frac{\delta \mathscr{E}_0}{\delta n(\mathbf{s})},$$

from which it is clear that for noninteracting electrons we have

$$\begin{aligned} & \sigma_{\alpha\beta} a_i^\alpha a_k^\beta - \frac{1}{2} \mathscr{E}_0 \delta_{ik} \\ & = \left\{ \sigma_{L\alpha\beta} + \int d\tau_s g^{-1/2} \left(\frac{\partial \varepsilon}{\partial g^{\alpha\beta}} \right)_s n \right\} a_i^\alpha a_k^\beta - \frac{1}{2} \mathscr{E}_L \delta_{ik} \\ & = \left\{ \sigma_{L\alpha\beta} + \int d\tau_p \left(\frac{\partial \varepsilon}{\partial g^{\alpha\beta}} \right)_s n \right\} a_i^\alpha a_k^\beta - \frac{1}{2} \mathscr{E}_L \delta_{ik}, \end{aligned} \tag{26}$$

where $\sigma_{L\alpha B}$ and $\mathscr{E}_L$ do not depend on the electron distribution function and have the meaning of an invariant stress tensor for the lattice and an elastic energy for the stationary lattice respectively. After condition (26) is substituted into Eq. (24), the latter agrees identically with the expression from Ref. 2. Equation (25) for the energy flux reduces to the corresponding expression of Ref. 2 upon substitution of the expression

$$\mathscr{E}_0 = \int d\tau_p \varepsilon n + \mathscr{E}_L$$

for the energy of a metal with noninteracting electrons.

In formulating the complete system of equations it is necessary to use yet another condition: the electric quasineutrality of the system. This condition, as in Ref. 2, has the form

$$\int d\tau_s n = \text{const}. \tag{27}$$

Equations (7) and (11) together with the Maxwell Eqs. (12) and the condition of quasineutrality (27) constitute a complete system of dynamic equations for the metal. The unknown functions, in addition to the electromagnetic fields, are $N^\alpha(\mathbf{r},t)$ and $n(\mathbf{r},\mathbf{p},t)$.

For a quantum crystal consisting of electrically neutral particles, we must first set $e = 0$. In this case the complete system consists of Eqs. (7) and (11), and condition (27) need not hold.

2. ELECTRON-PHONON INTERACTION VERTEX

In this section a linearized system of dynamic equations will be derived microscopically. A comparison of the results of the two approaches allows us to establish a number of exact identities for the electron-phonon interaction vertex.

In order to compare with the microscopic theory the kinetic equation (7) is conveniently rewritten as an equation for $n(s_\alpha,\mathbf{r},t)$. The exact form of this equation formally coincides with the equation presented in Ref. 2, with the sole difference that now ε is a functional of n. The linearization gives

$$\omega \delta n - \mathbf{a}_{\alpha 0} \frac{\partial \varepsilon_0}{\partial s_\alpha} \mathbf{k} \delta n + \mathbf{a}_{\alpha 0} \frac{\partial n_0}{\partial s_\alpha} \{ \mathbf{k} \delta \varepsilon - i(e\mathbf{E} + m\dot{\mathbf{v}}_L) \} = 0. \tag{28}$$

Here ω, $\mathbf{k}$ are the frequency and wave vector, δn, $\delta\varepsilon$ are the deviations of the functions $n(s_\alpha,\mathbf{r},t)$ and $\varepsilon(s_\alpha,\mathbf{r},t)$ from their values n_0, ε_0 in the original equilibrium state, and $\mathbf{a}_{\alpha_0}$ is the period of the lattice in this state. Let us consider an ideal crystal at zero temperature, corresponding to which we set $\hat{I} = 0$.

The acceleration of the lattice $\dot{\mathbf{v}}_L$ can be eliminated from Eq. (28) with the help of Eq. (11). In this case

$$m\dot{v}_L \sim (m/\rho) k \Pi_{ik} \sim (k u_{ik}) m v_s^2,$$

where u_{ik} is the strain tensor from the linear theory of elasticity and v_s is the velocity of sound. The first term in the curly brackets of Eq. (28) is of order

$$k\delta\varepsilon \sim k\varepsilon_F u_{ik} \sim (k u_{ik}) m v_F^2,$$

where ε_F, v_F are the energy and velocity of electrons at the Fermi surface. The term with $\dot{\mathbf{v}}_L$ in Eq. (28) thus can be neglected.

In the general case, the electric field is the sum $\mathbf{E} = \mathbf{E}_l + \mathbf{E}_t$ of the longitudinal (curl $\mathbf{E}_l = 0$) and transverse (div $\mathbf{E}_t = 0$) fields. When sound propagates in a metal, it is well-known[1,5] that the transverse field can be ignored if we are not considering the region of very small sound frequencies such that $(\omega/v_s)\delta(\omega) \lesssim 1$, where δ is the electro-

magnetic skin depth. In fact, in order to neglect the field $\mathbf{E}_t$ it is sufficient to have $\omega > 10^9$ sec^{-1}. The longitudinal field must be included in order to fulfill the condition of quasineutrality (27).

Setting $e\mathbf{E}_l = \nabla\psi$, replacing s_α by the quasimomentum $\mathbf{a}_0{}^\alpha s_\alpha$ everywhere in the small terms (this corresponds to an unstrained lattice), and transforming to the new unknown function ν, according to

$$\delta n(\mathbf{p}) = -(\partial n_0/\partial\varepsilon)\nu(\mathbf{p}_F) = \delta(\varepsilon-\varepsilon_F)\nu(\mathbf{p}_F),$$

we obtain in place of (28)

$$(\omega-\mathbf{k}\mathbf{v}_F)\nu(\mathbf{p}_F) = \mathbf{k}\mathbf{v}_F\left\{\int f(\mathbf{p}_F,\mathbf{p}_F')\nu(\mathbf{p}_F')\,d\sigma' + \lambda_{ik}(\mathbf{p}_F)u_{ik} + \psi\right\}. \tag{29}$$

Here $\mathbf{p}_F$, $\mathbf{p}_F'$ are quasimomenta which lie at the Fermi surface, $\mathbf{v}_F = \partial\varepsilon/\partial\mathbf{p}$ for $\mathbf{p} = \mathbf{p}_F$, $d\sigma = 2dS/v_F(2\pi\hbar)^3$, dS is an element of area of the Fermi surface, and

$$f(\mathbf{p},\mathbf{p}') = f(\mathbf{a}_0{}^\alpha s_\alpha,\ \mathbf{a}_0{}^\alpha s_\alpha') = \left[\frac{\delta\varepsilon(\mathbf{a}_0{}^\alpha s_\alpha)}{\delta n(\mathbf{a}_0{}^\alpha s_\alpha)}\right]_{g^{\alpha\beta}} \tag{30a}$$

is the Landau f-function that is symmetric with respect to its arguments;

$$\lambda_{ik}(\mathbf{p}) = \left[\frac{\partial\varepsilon(\mathbf{a}_0{}^\alpha s_\alpha)}{\partial u_{ik}}\right]_{n(\varepsilon)} = -2\left[\frac{\partial\varepsilon(\mathbf{a}_0{}^\alpha s_\alpha)}{\partial g^{\alpha\beta}}\right]_{n(\varepsilon)} \mathbf{a}_{0i}{}^\alpha \mathbf{a}_{0k}{}^\beta \tag{30b}$$

has the sense of a deformation potential.

The condition of quasineutrality (27) can be rewritten in the form

$$\int d\sigma\nu(\mathbf{p}_F) = 0. \tag{31}$$

By virtue of the linearity of the problem we have

$$\nu(\mathbf{p}_F) = N_{ik}(\mathbf{p}_F, K)u_{ik}(K),$$
$$\psi = Q_{ik}(K)u_{ik}(K), \tag{32}$$

where $K = (\omega,\mathbf{k})$ and the kernels N_{ik} and Q_{ik} by virtue of Eqs. (29) and (31) are determined by the equations

$$(\omega-\mathbf{k}\mathbf{v}_F)N_{ik}(\mathbf{p}_F,K) = \mathbf{k}\mathbf{v}_F\left\{\lambda_{ik}(\mathbf{p}_F) + Q_{ik}(K) + \int f(\mathbf{p}_F,\mathbf{p}_F')N_{ik}(\mathbf{p}_F',K)\,d\sigma'\right., \tag{33}$$

$$\int d\sigma N_{ik}(\mathbf{p}_F,K) = 0. \tag{34}$$

Passing to the microscopic description, let us discuss the electron Green's function $G(\mathbf{r},\mathbf{r}',t-t')$ of the original equilibrium state of the metal. We first represent the coordinates of an electron in the form $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$, where $\mathbf{R}$ is a discrete-valued coordinate for a given unit cell and $\boldsymbol{\rho}$ takes on values within the unit cell. Let us set

$$G(\mathbf{r},\mathbf{r}',t-t') = \int\frac{d^4P}{(2\pi)^4}G(P,\boldsymbol{\rho},\boldsymbol{\rho}')\exp[i\mathbf{p}(\mathbf{R}-\mathbf{R}') - ip_0(t-t')],$$

where $P = (p_0,\mathbf{p})$; $\mathbf{p}$ is the quasimomentum, while p_0 is the temporal component of the momentum P; the integration with respect to $\mathbf{p}$ is carried out over an elementary unit cell of the reciprocal lattice. The Green's function $G(P,\boldsymbol{\rho},\boldsymbol{\rho}')$ in the representation we are discussing here has the usual pole-like singularities near the Fermi surface of the metal. Near such a pole we have

$$G(P,\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{u_{\mathbf{p}}(\boldsymbol{\rho})u_{\mathbf{p}}^*(\boldsymbol{\rho}')}{p_0-\varepsilon(\mathbf{p})+\varepsilon_F+i0\,\mathrm{sign}\,p_0},$$

where $u_{\mathbf{p}}(\boldsymbol{\rho})$ is some function of the coordinate $\boldsymbol{\rho}$ and the quasimomentum (see Ref. 6, § 62).

We introduce the vertices $\gamma_{ik}(P,K,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}_3)$ and $\gamma(P,K,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}_3)$, which couple the electrons to the long-wavelength and low-frequency fields $u_{ik}(\mathbf{r},t)$ and $\psi(\mathbf{r},t)$ and are defined by the formula

$$\delta G(P,P+K;\boldsymbol{\rho},\boldsymbol{\rho}') = \int d\boldsymbol{\rho}_1\,d\boldsymbol{\rho}_2\,d\boldsymbol{\rho}_3\,G(P,\boldsymbol{\rho},\boldsymbol{\rho}_1)G(P+K;\boldsymbol{\rho}_2,\boldsymbol{\rho}') \times\{\gamma(P,K;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}_3)\psi(K) + \gamma_{ik}(P,K;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}_3)u_{ik}(K)\}. \tag{35}$$

Here $u_{ik}(K)$ and $\psi(K)$ are the Fourier components of the fields u_{ik} and ψ. We have also included the fact that, because these fields vary slowly in space, we can set $u_{ik}(\mathbf{r}) \simeq u_{ik}(\mathbf{R})$ and $\gamma(\mathbf{r}) \simeq \gamma(\mathbf{R})$, i.e., we assume that these functions depend only on the discrete coordinate $\mathbf{R}$.

The vertices γ_{ik} and γ have singularities in their k-dependences that are characteristic of a Fermi liquid. To identify these singularities we follow the method described by Landau in Ref. 7 (see also Ref. 6, § 17) for the four-fermion vertex. The equations that we need differ from those given by Migdal[8] for the vertices that determine the interaction of fermions with an arbitrary boson field only in that we have taken into account the inhomogeneity of the metal. In the spatially inhomogeneous case the product of two-electron Green's functions can be written for small K in the form

$$G(P;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)G(P+K,\boldsymbol{\rho}_3,\boldsymbol{\rho}_4) = 2\pi i\delta(p_0)\delta[\varepsilon(\mathbf{p})-\varepsilon_F] \times\frac{\mathbf{k}\mathbf{v}_F}{\omega-\mathbf{k}\mathbf{v}_F}u_{\mathbf{p}}(\boldsymbol{\rho}_1)u_{\mathbf{p}}(\boldsymbol{\rho}_3)u_{\mathbf{p}}^*(\boldsymbol{\rho}_2)u_{\mathbf{p}}^*(\boldsymbol{\rho}_4) + \varphi(P;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}_3,\boldsymbol{\rho}_4), \tag{36}$$

where φ is the ω-limit of the left-hand side of Eq. (36), i.e., its limit as $K\to 0$, $|\mathbf{k}|/\omega\to 0$. The equation for the vertex has the form

$$\gamma_{ik}(P,K;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \gamma_{ik}^{\omega}(P;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) + \int d\sigma'\int d\boldsymbol{\rho}_4 d\boldsymbol{\rho}_5 \times d\boldsymbol{\rho}_6 d\boldsymbol{\rho}_7\Gamma^{\omega}(P,\mathbf{p}_F';\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}_4,\boldsymbol{\rho}_5)u_{\mathbf{p}'}(\boldsymbol{\rho}_4)u_{\mathbf{p}'}^*(\boldsymbol{\rho}_5) \times u_{\mathbf{p}'}^*(\boldsymbol{\rho}_6)u_{\mathbf{p}'}^*(\boldsymbol{\rho}_7)\frac{\mathbf{k}\mathbf{v}_F}{\omega-\mathbf{k}\mathbf{v}_F}\gamma_{ik}(\mathbf{p}_F',K,\boldsymbol{\rho}_6,\boldsymbol{\rho}_7). \tag{37}$$

The same equation is valid for the scalar vertex $\gamma(P,K,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$. Here we have introduced the averaged vertices with two arguments $\boldsymbol{\rho}$

$$\gamma(P,K;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \int d\boldsymbol{\rho}_3\gamma(P,K;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}_3)$$

(and analogously for γ_{ik}), which enter everywhere into all the expressions which follow; γ_{ik}^{ω} (and analogously γ^{ω}) is the ω-limit of the vertex γ_{ik}, and the quantity Γ^{ω} is determined by the expression

$$\Gamma^{\omega}(P,Q,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}_3,\boldsymbol{\rho}_4) = \frac{1}{4}\lim_{K\to 0,|\mathbf{k}|/\omega\to 0}\Gamma_{\alpha\beta,\alpha\beta}(P,Q;P+K,Q-K;\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}_3,\boldsymbol{\rho}_4),$$

i.e., it is the ω-limit of the four-fermion vertex taking into account its spin indices. We have omitted the spin indices in the expressions for γ_{ik}, γ, and G, keeping in mind that in our case these quantities are diagonal with respect to spin.

In what follows we will need to use one of the standard identities satisfied by the four-fermion vertex Γ^{ω}. Specifically we need the one derived in Ref. 7, § 19 by using the time-dependent gauge transformation $\hat{\Psi}\to\hat{\Psi}\exp[i\chi(t)]$. A simple generalization of the usual derivation to include spatial inhomogeneity gives

$$\frac{\partial}{\partial p_0}G(P,\rho_1,\rho_2)=-\int G(P;\rho_1,\rho')G(P;\rho',\rho_2)\,d\rho'$$
$$+2i\int\frac{d^4Q}{(2\pi)^4}\int d\rho'd\rho_3d\rho_4d\rho_5d\rho_6G(P;\rho_1,\rho_3)G(P;\rho_4,\rho_2)$$
$$\times\Gamma^{\omega}(Q,P;\rho_6,\rho_5,\rho_4,\rho_3)G(Q;\rho_5,\rho')G(Q;\rho',\rho_6), \quad (38)$$

where the singular product of the G-functions must be understood in the sense of the ω-limit. Isolating the pole-like part of Eq. (38), we find the identity

$$\frac{i}{2}[1-z(\mathbf{p}_F)]=\int\frac{d^4Q}{(2\pi)^4}\int d\rho_1d\rho_2u_p^*(\rho_1)u_p(\rho_2)$$
$$\times\int d\rho_3\,d\rho_4\Gamma^{\omega}(Q,\mathbf{p}_F;\rho_4,\rho_3,\rho_2,\rho_1)\int d\rho\varphi(Q;\rho,\rho_4,\rho_3,\rho), \quad (39)$$

where we have introduced the notation

$$z(\mathbf{p})=\int d\rho\,|u_p(\rho)|^2.$$

Let us calculate the change in the electronic density $\delta N(K)$ under the action of the field $u_{ik}(K)$. We have

$$\delta N(K)=R_{ik}(K)u_{ik}(K), \quad (40)$$

where the kernel equals

$$R_{ik}(K)=\int\frac{d^4P}{(2\pi)^4}\int d\rho\,d\rho_1\,d\rho_2G(P;\rho,\rho_1)$$
$$\times G(P+K;\rho_2,\rho)\gamma_{ik}(P,K;\rho_1,\rho_2). \quad (41)$$

We substitute Eqs. (36) and (37) into the right side of Eq. (41). As a result of this, the product of the second term in Eq. (36), which equals φ, and the first term in Eq. (37), which equals γ_{ik}^{ω}, gives the ω-limit of the kernel (41), i.e., its value for the case of a spatially homogeneous perturbation u_{ik}. However, in this case the kernel equals zero by virtue of the conservation of charge, so that this particular product in Eq. (41) can be neglected. We then have

$$R_{ik}(K)=\frac{i}{2}\int d\sigma\int d\rho_1\,d\rho_2\,z(\mathbf{p})u_p(\rho_2)u_p^*(\rho_1)$$
$$\times\gamma_{ik}(\mathbf{p},K;\rho_1\rho_2)\frac{\mathbf{kv}_F}{\omega-\mathbf{kv}_F}+\int\frac{d^4P}{(2\pi)^4}\int d\sigma'$$
$$\times\int d\rho\,d\rho_1\ldots d\rho_6\Gamma^{\omega}(P,\mathbf{p}';\rho_1,\rho_2,\rho_3,\rho_4)u_{p'}(\rho_3)$$
$$\times u_{p'}^*(\rho_4)u_{p'}^*(\rho_5)u_{p'}(\rho_6)\frac{\mathbf{kv}_F}{\omega-\mathbf{kv}_F}$$
$$\times\gamma_{ik}(\mathbf{p}',K;\rho_5,\rho_6)\varphi(P;\rho,\rho_1,\rho_2,\rho). \quad (42)$$

Using the identity (39), we obtain

$$R_{ik}(K)=\frac{i}{2}\int d\sigma\int d\rho_1\,d\rho_2\,u_p^*(\rho_1)u_p(\rho_2)$$
$$\times\frac{\mathbf{kv}_F}{\omega-\mathbf{kv}_F}\gamma_{ik}(\mathbf{p},K;\rho_1,\rho_2). \quad (43)$$

In the case of a scalar perturbation $\psi(K)$ we find in an analogous fashion that

$$\delta N(K)=R(K)\psi(K), \quad (44)$$

where

$$R(K)=\frac{i}{2}\int d\sigma\int d\rho_1\,d\rho_2\,u_p^*(\rho_1)u_p(\rho_2)\frac{\mathbf{kv}_F}{\omega-\mathbf{kv}_F}\gamma(\mathbf{p},K;\rho_1,\rho_2). \quad (45)$$

Since the scalar field in the presence of the strain $u_{ik}(K)$ in fact is determined by the quasineutrality condition, for $\psi(K)$ the second of the formulas (32) is valid, while the kernel $Q_{ik}(K)$ by virtue of Eqs. (40) and (44) is determined by the equation

$$R_{ik}+RQ_{ik}=0. \quad (46)$$

A scalar field always unavoidably accompanies the strain field $u_{ik}(K)$. Therefore it is not γ_{ik} that has physical significance but rather the effective vertex $\mathcal{T}_{ik}=\gamma_{ik}+\gamma Q_{ik}$. We introduce the effective vertex $\bar{\mathcal{T}}_{ik}$ by averaging with respect to the variable ρ according to the formula

$$\bar{\mathcal{T}}_{ik}(P,K)=\int d\rho_1\,d\rho_2\,u_p^*(\rho_1)u_p(\rho_2)\mathcal{T}_{ik}(P,K;\rho_1,\rho_2).$$

By virtue of Eqs. (43) and (45), and also Eq. (37) and the formulas analogous to it for γ, the effective vertex satisfies the equation

$$\bar{\mathcal{T}}_{ik}(\mathbf{p}_F,K)=\bar{\gamma}_{ik}^{\omega}(\mathbf{p}_F)+\bar{\gamma}^{\omega}(\mathbf{p}_F)Q_{ik}(K)$$
$$+\int d\sigma'f(\mathbf{p}_F,\mathbf{p}_F')\frac{\mathbf{kv}_F'}{\omega-\mathbf{kv}_F'}\bar{\mathcal{T}}_{ik}(\mathbf{p}_F',K), \quad (47)$$

where

$$f(\mathbf{p}_F,\mathbf{p}_F')=\int u_p^*(\rho_1)u_p(\rho_2)u_p(\rho_3)u_{p'}^*(\rho_4)\Gamma^{\omega}$$
$$\times(\mathbf{p}_F,\mathbf{p}_F';\rho_1,\rho_2,\rho_3,\rho_4)d\rho_1\ldots d\rho_4$$

is a quantity which, as we will see, plays the role of the Landau f-function.

The quasineutrality condition (46) can be written in the form

$$\int d\sigma\frac{\mathbf{kv}_F}{\omega-\mathbf{kv}_F}\bar{\mathcal{T}}_{ik}(\mathbf{p}_F,K)=0. \quad (48)$$

If we set

$$N_{ik}(\mathbf{p}_F,K)=\frac{\mathbf{kv}_F}{\omega-\mathbf{kv}_F}\bar{\mathcal{T}}_{ik}(\mathbf{p}_F,K), \quad (49)$$

$$\bar{\gamma}_{ik}^{\omega}(\mathbf{p}_F)=\lambda_{ik}(\mathbf{p}_F),\qquad \bar{\gamma}^{\omega}(\mathbf{p}_F)=1 \quad (50)$$

then the system of equations (47) and (48) becomes identical to the system of phenomenological equations (33) and (34). After substituting condition (50) in Eq. (47) the latter together with Eq. (48) constitutes the required system of identities for the vertex $\bar{\mathcal{T}}_{ik}$ when K is small, which deter-

mines the vertex for a specified electronic energy spectrum and Landau f-function.

The vertex for interaction with real phonons corresponds to the condition $\omega = v_s k \ll v_F k$, i.e., it is the k-limit ($K \to 0$, $\omega/|\mathbf{k}| \to 0$) of the overall vertex $\overline{\mathcal{T}}_{ij}^{k}(\mathbf{p}_F)$. Equations (47) and (48) imply that this vertex satisfies the equation

$$\overline{\mathcal{T}}_{ij}^{k}(\mathbf{p}_F) = \Lambda_{ij}(\mathbf{p}_F) - \int d\sigma' F(\mathbf{p}_F, \mathbf{p}_F') \overline{\mathcal{T}}_{ij}^{k}(\mathbf{p}_F', K), \quad (51)$$

where

$$\Lambda_{ij}(\mathbf{p}_F) = \lambda_{ij}(\mathbf{p}_F) - \int d\sigma \lambda_{ij}(\mathbf{p}_F) \Big/ \int d\sigma,$$

$$F(\mathbf{p}_F, \mathbf{p}_F') = f(\mathbf{p}_F, \mathbf{p}_F') - \left\{ \int f(\mathbf{p}_F, \mathbf{p}_F') d\sigma' + \int f(\mathbf{p}_F, \mathbf{p}_F') d\sigma \right\} \Big/ \int d\sigma.$$

It is easy to see that $\overline{\mathcal{T}}_{ij}^{k}(\mathbf{p}_F)$ equals the derivative of the electron energy $\varepsilon(\mathbf{p})$ at $\mathbf{p} = \mathbf{p}_F$ with respect to u_{ik}, taking into account the equilibrium change of the distribution function of the electrons, the Fermi-liquid interactions, and the equilibrium change of the electronic potential that is required in order to insure quasineutrality.

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