Nonlinear strings in relativistic MHD

V. S. Semenov and L. V. Bernikov

Leningrad State University
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A special Lagrangian coordinate system is constructed in which a magnetic force tube is the focus of attention. This tube behaves as a nonlinear string with respect to tangential forces. General and particular variational principles are given. The results show that a plasma in a magnetic field can be thought of in the MHD approximation as a gas of nonlinear strings which interact through pressure forces. A method is developed for reducing multidimensional nonlinear problems of relativistic MHD with boundary layers to a sequence of two-dimensional problems for force tubes, i.e., nonlinear strings.

1. INTRODUCTION

Relativistic magnetohydrodynamics (RMHD), like nonrelativistic MHD, has the property that magnetic field lines become frozen in a plasma, so the force tubes can be tagged with plasma particles and thereby individualized. The existence of Maxwell stresses along a field line is frequently used as justification for drawing an analogy between a magnetic force tube and a rubber band or cord, i.e., an entity capable of stretching and contracting. The picture of a force tube as a spring often leads to a clear qualitative description of complex phenomena. In nonrelativistic MHD, this analogy between a force tube and a spring or, more precisely, a nonlinear string, is not only qualitative but also quantitative, as was shown in Ref. 1. The quantitative analogy is by far the more important of the two. This concept underlies a method which has been proposed for reducing several difficult MHD problems, including three-dimensional and time-varying problems, to a sequence of two-dimensional problems for nonlinear strings (force tubes). Below we generalize the technique developed in Ref. 1 to the relativistic case.

2. RELATIVISTIC FROZEN-IN FRAME OF REFERENCE

We will be discussing a plane Minkowski space with a metric tensor $h_{ik} = \mathrm{diag}(1, -1, -1, -1)$, but all the calculations can be generalized in an obvious way to the case of the general theory of relativity. The medium in RMHD can be described by the energy-momentum tensor[2–4]

$$T^{ik} = \left(p+\varepsilon+\frac{1}{4\pi}b^2 \right)u^i u^k - \left(p+\frac{1}{8\pi}b^2\right)h^{ik} - \frac{1}{4\pi}b^i b^k, \qquad (1)$$

where p is the pressure, ε is the internal energy per unit volume of the plasma, u^i is a time-like velocity 4-vector, $u_i u^i = 1$; b^i is the space-like magnetic-field 4-vector, $b^i = F^{*ik}U_k$, F^{*ik} is a dual electromagnetic field tensor, and $b^2 = -b_i b^i$. The vectors u^i and b^i are orthogonal: $b_i u^i = 0$. In the dissipation-free RMHD discussed below, Ohm's law reduces to

$$F^{ik}u_k = 0, \qquad (2)$$

which simply means that there is no electric field in the comoving frame of reference:

$$\mathbf{E} + \frac{1}{c}[\mathbf{vB}] = 0.$$

The magnetic-field 4-vector satisfies the same equation:

$$F^{ik}b_k = 0. \qquad (3)$$

The system of RMHD equations can be written

$$\nabla_i \rho u^i = 0, \qquad (4)$$

$$\nabla_i T^{ik} = 0, \qquad (5)$$

$$\nabla_i (b^i u^k - b^k u^i) = 0. \qquad (6)$$

Here ρ is the density, Eq. (4) expresses conservation of matter, Eq. (5) expresses energy-momentum conservation, and Eq. (6) is the magnetic-induction equation.

We will attempt to construct a frame of reference such that a magnetic force tube in it is the focus of attention. For this purpose, the vectors b^i or u^i (or vectors proportional to them) must become the basis vectors in the new frame. The general theory[5,6] shows that two vectors a_1^i and a_2^i can be basis vectors of some frame if and only if their Lie bracket vanishes:

$$(a_1^i \nabla_i) a_2^k - (a_2^i \nabla_i) a_1^k = 0. \qquad (7)$$

We will attempt to choose scalar functions $k_1(x)$ and $k_2(x)$ such that the Lie bracket of the vectors $k_1 u^i$ and $k_2 b^i$ vanishes.

In general, the magnetic-field 4-vector b^i is not solenoidal: $\nabla_i b^i \neq 0$. It is, on the other hand, possible to choose a scalar function $q(x)$ such that

$$\nabla_i q b^i = 0. \qquad (8)$$

The physical meaning of q will be explained below. Using (4) and (8), we can rewrite Eq. (6) (the magnetic-induction equation) as

$$\frac{b^i}{\rho}\nabla_i \frac{u^k}{q} - \frac{u^i}{q}\nabla_i \frac{b^k}{\rho} = 0. \qquad (9)$$

In other words, the Lie bracket of the vectors u^i/q and b^i/ρ is zero. There exist thus coordinates η and α for which the vectors u^i/q and b^i/ρ are basis vectors:

$$\frac{dx^i}{d\eta} = \frac{u^i}{q}, \qquad (10)$$

$$\frac{dx^i}{d\alpha} = \frac{b^i}{\rho}. \qquad (11)$$

Since $u_i u^i = 1$, the vector x^i_η satisfies

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$$q^2 x_\eta^{\,2} = 1. \tag{12}$$

We supplement η, α with two other coordinates ψ, ζ, and we rewrite Eqs. (4) and (8) in terms of the new variables η, α, ψ, ζ:

$$\nabla_i \rho u^i = \nabla_i \rho q \frac{u^i}{q} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \eta} \rho q \sqrt{-g} = 0, \tag{13}$$

$$\nabla_i q b^i = \nabla_i \rho q \frac{b^i}{\rho} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \alpha} \rho q \sqrt{-g} = 0, \tag{14}$$

where $\sqrt{-g}$ is the Jacobian of the transformation from the coordinates x^0, x^1, x^2, x^3 to η, α, ψ, ζ. From Eqs. (13) and (14) we find $\rho q \sqrt{-g} = F(\psi,\zeta)$. We will leave the introduction of the coordinates ψ, ζ somewhat arbitrary at this point; specifically, we allow the transformations $\tilde{\psi} = \tilde{\psi}(\psi,\zeta)$ and $\tilde{\zeta} = \tilde{\zeta}(\psi,\zeta)$. We can utilize this arbitrariness to satisfy the condition $F(\psi,\zeta) = 1$. We then find

$$\frac{D(x^0, x^1, x^2, x^3)}{D(\eta, \alpha, \psi, \zeta)} = \frac{1}{\rho q}. \tag{15}$$

This is the form of the continuity equation in terms of the new variables. The induction equation, (6), is now satisfied identically.

By analogy with the nonrelativistic case,[7] we call the frame of reference η, α, ψ, ζ the "frozen-in" system. The justification for this name comes from the frozen-in property in (2), (6), which also leads to (9), the necessary condition for the introduction of the new coordinates. Different sets in the space of frozen-in coordinates have a simple physical meaning.

By virtue of their construction, the η coordinate lines are inverse transforms of the trajectories of the fluid particles in physical space, while the α coordinate lines are inverse transforms of the magnetic field lines. In formal terms, the functions $x^i(\eta,\alpha,\psi,\zeta)$ with fixed α, ψ, ζ and variable η specify the trajectory of a fluid particle. If instead η, ψ, ζ are fixed while α varies, we obtain a magnetic field line: a solution of Eqs. (11).

The parameter η has the meaning of a Lagrangian time along a trajectory, and the entire frozen-in coordinate system is also Lagrangian. It is for this reason that the continuity equation takes the form in (15). The parameter η differs from the proper time τ by an amount q:

$$d\eta = q\, d\tau. \tag{16}$$

The physical meaning of q can be seen from the equation

$$\nabla_i w b^i/\rho - T b^i \nabla_i s = 0, \tag{17}$$

which follows from Eqs. (4)–(6) (Ref. 3). Here s and $w = \varepsilon + p$ are the entropy and enthalpy per unit volume, and T is the temperature. It can be shown[3,4] that the entropy remains constant along the trajectories of the fluid particles:

$$u^i \nabla_i s = 0. \tag{18}$$

Hence $s = s_0(\alpha,\psi,\zeta)$. If a thermodynamic equilibrium is also established along a magnetic force tube, $b^i \nabla_i s = 0$ or $s = s_0(\psi,\zeta)$, it follows from (17) that the role of q is played by the enthalpy. In general, we find the following relation from (17):

$$q = w \exp \int (T s_\alpha / w)\, d\alpha, \tag{19}$$

where the integration is along the force tube. Using (2) and (3), we can show that in the frozen-in frame of reference the electromagnetic field tensor F_{ik} has only a single nonvanishing component:

$$F_{\psi\zeta} = -F_{\zeta\psi} = \rho q \sqrt{-g} = 1. \tag{20}$$

This result is generally understandable: The electric field vanishes in the comoving frame of reference [see (2)], and the magnetic-field 4-vector b^i/ρ is a basis vector. It follows from (20) that the magnetic flux through a fluid loop, F_B, does not vary with the time η as it moves at a velocity u^i/q or with the time τ as it moves at a velocity u^i [see (16)]:

$$F_B = \int F_{ik}\, dx^i \wedge dx^k = \int F_{\psi\zeta}\, d\psi \wedge d\zeta = \int d\psi \wedge d\zeta. \tag{21}$$

Here $dx^i \wedge dx^k$ is an area element of the surface spanning the fluid loop. We have thus found that the area $d\psi \wedge d\zeta$ in the space of frozen-in variables is equal to the magnetic flux through the inverse image of this area in physical space.

Let us determine the physical meaning of the parameter α along a magnetic field line. The mass of plasma in the volume element $\Delta x^1 \Delta x^2 \Delta x^3$, integrated over the time Δx^0, is

$$\begin{aligned} \rho\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 &= (\rho \sqrt{-g})\, d\eta \wedge d\alpha \wedge d\psi \wedge d\zeta \\ &= d\tau \wedge d\alpha \wedge d\psi \wedge d\zeta. \end{aligned} \tag{22}$$

We thus see that the mass of plasma in a force tube with a unit magnetic flux per unit proper time is α, since (17) must be broken up into the flux $d\psi \wedge d\zeta$ and the proper time $d\tau$. The parameter α is thus the mass of plasma in a force tube with a unit magnetic flux in the proper frame of reference.

3. MAGNETIC FORCE TUBE AS A NONLINEAR STRING

Let us rewrite equation of motion (5) in terms of the frozen-in variables. This equation can be put in the form

$$\rho q \frac{u^i}{q} \nabla_i \left(\frac{Qq}{\rho} \frac{u^k}{q} \right) - \rho q \frac{b^i}{\rho} \nabla_i \left(\frac{\rho}{4\pi q} \frac{b^k}{\rho} \right) = h^{ik} \nabla_k P, \tag{23}$$

where $P = p + b^2/8\pi$ is the total pressure, and $Q \equiv p + \varepsilon + b^2/4\pi$. In terms of the new variables, two operators involved here take the following form:

$$\frac{u^i}{q} \nabla_i = \frac{\partial}{\partial \eta}, \quad \frac{b^i}{\rho} \nabla_i = \frac{\partial}{\partial \alpha}. \tag{24}$$

We can thus rewrite Eq. (23) as

$$\frac{\partial}{\partial \eta} \left(\frac{Qq}{\rho} \frac{\partial x^i}{\partial \eta} \right) - \frac{\partial}{\partial \alpha} \left(\frac{\rho}{4\pi q} \frac{\partial x^i}{\partial \alpha} \right) = \frac{D(P, x^j, x^k, x^l)}{D(\eta, \alpha, \psi, \zeta)}, \tag{25}$$

where $i, j, k, l = 0, 1, 2, 3$ constitute a cyclic permutation of indices. The Jacobian appears on the right side because of the conversion of the pressure gradient $h^{ik} \nabla_i P$ to the new variables. This is the general position for any Lagrangian approach.[8]

Equations of motion (25) follow from energy-momentum conservation, but they have a different interpretation in terms of frozen-in variables. A one-dimensional wave operator describing a magnetic force tube has appeared on the left side of (25). This circumstance is understandable, since the force tube behaves as a nonlinear string [the left side of (25)] and is subjected to a pressure exerted by the neighboring force tubes [the right side of (25)].

It is useful to rederive this conclusion from a variational principle. The equations of motion can be found by varying the action[9]

$$S=-\int\left(\varepsilon+\frac{1}{8\pi}b^2\right)dV. \qquad (26)$$

We write the action in (26) in terms of frozen-in variables, using auxiliary condition (15):

$$S=-\int\left[\frac{e+\rho x_\alpha{}^2/8\pi}{q}-\lambda\left(\frac{D(x^0,x^1,x^2,x^3)}{D(\eta,\alpha,\psi,\zeta)}-\frac{1}{\rho q}\right)\right]$$
$$\times d\eta\, d\alpha\, d\psi\, d\zeta, \qquad (27)$$

where $e=\varepsilon/\rho$ is the internal energy per unit mass, $x_\alpha^2=-h_{ik}x_\alpha^i x_\alpha^k$, and λ is a Lagrange multiplier. The quantities x^i and ρ are to be varied. The variation of q can be found from (12):

$$\delta q=-q^3 h_{ik}x_\eta{}^i\delta x_\eta{}^k.$$

Varying x^i, we find the equations of motion (25). Varying the density ρ, and making use of the thermodynamic relation $\rho^2 e_\rho=p$, we find the equation

$$\lambda=p+\frac{1}{8\pi}b^2, \qquad (28)$$

from which we see that $\lambda=P$. The force tubes interact under the condition that the mass of plasma in them is conserved [see (15)]. This interaction leads automatically to the appearance of a total pressure P: the Lagrange multiplier which was introduced in a formal way turns out to be the total pressure. By analogy with the particles of a gas, one could say that a collision of force tubes in which the plasma mass does not change gives rise to the pressure forces of the magnetic force tubes.

We thus see that in RMHD, as in nonrelativistic MHD, a magnetic force tube is completely analogous to a nonlinear string with respect to tangential forces. Pressure forces act between tubes.

Note also that the frozen-in coordinates cause a natural stratification of Minkowski space into the space of the force tube, i.e., η, α (a layer), and the rest of space (the base). This point may prove useful in the development of numerical algorithms for RMHD problems.[10]

4. BOUNDARY LAYER

The picture of a magnetic force tube as a nonlinear string may prove useful, but, unfortunately, is of little help for a quantitative description. Solving the system of MHD equations is a difficult matter in either Eulerian form, (1)–(3), or Lagrangian form (15), (25). As in nonrelativistic MHD,[1] the problem does simplify when we consider an important particular case, namely, a subsystem of Eqs. (25) which we write in the form

$$\frac{\partial}{\partial\eta}\left(\frac{Qq}{\rho}\frac{\partial x^i}{\partial\eta}\right)-\frac{\partial}{\partial\alpha}\left(\frac{\rho}{4\pi q}\frac{\partial x^i}{\partial\alpha}\right)=\frac{1}{\rho q}h^{ik}\nabla_k P(x), \qquad (29)$$

$$P(x)=p+\frac{1}{8\pi}\rho^2 x_\alpha{}^2. \qquad (30)$$

We assume that the total pressure $P(x)$ is a given function of the Eulerian coordinates in subsystem (29), (30). The equation of state can be chosen in the form $p=p(\rho,s)$. The entropy is constant on trajectories (18), so it is determined by its value at the initial time: $s=s_0(\alpha,\psi,\zeta)$. The equation of state then gives us an expression for the pressure as a function of the density:

$$p=p(\rho,s_0(\alpha,\psi,\zeta)).$$

Equation (30) determines the nonlinear dependence of the density on x^i and x_α^i. The charges q are related to x_η^i by (12). We see now that to find expressions for the Eulerian coordinates in terms of the frozen-in coordinates $x^i(\eta,\alpha,\psi,\zeta)$, at a known function $P(x)$, we need to solve a system of two-dimensional equations for a nonlinear string (ψ and ζ appear as parameters). Solving this problem is of course much simpler than solving the original system of MHD equations. Solving two-dimensional string equations poses no particular challenge to modern computers. A question arises here: Under what conditions is the total pressure known at the outset or can at worst be calculated in the first step, without consideration of the other unknowns?

As an example we can use an Alfvén wave, for which we have $P=$ const, and for which Eq. (20) becomes the usual d'Alembert's wave equation:

$$\frac{Qq}{\rho}x_{\eta\eta}{}^i-\frac{\rho}{4\pi q}x_{\alpha\alpha}{}^i=0. \qquad (31)$$

A solution of (31) is an arbitrary function of the arguments $[\rho/q(4\pi Q)^{1/2}]\,\eta\pm\alpha$. Let us assume that the wave is propagating along the x axis in the field b_{0x}. We then have $\alpha=(\rho/b_{0x})x$, and $\eta=\tau q$, and the argument is rewritten as $[b_{0x}/(4\pi Q)^{1/2}]\,\tau\pm x$. The Alfvén velocity is therefore $V_A=b_{0x}/(4\pi Q)^{1/2}$ (Ref. 4).

There is a far wider class of problems in which the total pressure can be found beforehand. We have in mind problems with boundary layers, although the solution found in such cases is admittedly asymptotic rather than exact. An important property of a boundary layer is that the total pressure remains constant in the transverse direction. The proof is essentially the same as in the theory of a Prandtl viscous boundary layer.[11] Let us examine the orders of magnitude in the equation of motion across the layer (for definiteness, along the z axis):

$$\frac{\partial}{\partial\eta}\left(\frac{Qq}{\rho}\frac{\partial z}{\partial\eta}\right)-\frac{\partial}{\partial\alpha}\left(\frac{\rho}{4\pi q}\frac{\partial z}{\partial\alpha}\right)=-\frac{1}{\rho q}\frac{\partial P}{\partial z}. \qquad (32)$$

We assume that the ratios of the length scales of the variation of the various quantities along (x^0,x^1,x^2) and across (z) the layer satisfy z/x^0, z/x^1, $z/x^2\sim\varepsilon\ll 1$. We also assume that the magnetic field lines in the boundary layer are stretched out in the longitudinal direction, so the normal component is small in comparison with the tangential components. We then have normal values b_z, u_z, $z\sim\varepsilon$ and tangential values ~ 1. The left side of (32) is thus $\sim\varepsilon$, and the right side $\sim 1/\varepsilon$. Hence $\partial P/\partial z=0$; i.e., the total pressure $P=P(x^0,x^1,x^2)$ does not change in the direction across the boundary layer in the zeroth approximation.

Outside the boundary layer the problem usually simplifies, and the total pressure can be found from a simplified (limiting) system of equations. The total pressure, being a function of only the coordinates x^0, x^1, and x^2, tangential with respect to the layer, does not vary in the direction across the boundary layer, as we have already mentioned. To determine how the tangential Eulerian coordinates depend on the frozen-in coordinates, i.e., to determine $x^i(\eta,\alpha,\psi,\zeta)$, we can thus use nonlinear-string equations (29) and (30). These equations must be solved separately for each individual force tube at fixed values of ψ and ζ. Boundary conditions are found from the condition for joining the asymptotic expansions in the boundary layer and in the external region. The final step is to find the function $z(\eta, \alpha, \psi, \zeta)$ from the first-order linear equation in (15), in which the functions $x^i(\eta, \alpha, \psi, \zeta)$, $i=0, 1, 2$, are now known. Continuity equation (15) can be integrated by the method of characteristics, which are simply $x^i(\eta, \alpha, \psi, \zeta) = \text{const}$. There is the hope that all the problems mentioned above can be solved numerically in many cases, since the original nonlinear RMHD problem (which is generally four-dimensional) splits up into a sequence of two- and one-dimensional problems. In nonrelativistic RMHD, this method has proved successful in problems concerning magnetic reconnection[7] and a magnetic barrier.[12]

A particularly simple problem is that of the behavior of a narrow isolated magnetic field tube, with a longitudinal dimension which is much larger than its transverse dimension. This condition that the tube be narrow guarantees that the total pressure will remain constant in the transverse direction.[13,14] The pressure $P(x)$ is determined by the distribution of the gas pressure in the plasma. To find the tangential components of the velocity and the magnetic field, the density, the pressure, and the shape of the axial line of the tube, we must solve a Cauchy problem for nonlinear-string equations (29) and (30). In the zeroth approximation, the behavior of a narrow isolated tube depends on the gas pressure distribution in the medium and on only this distribution. In order to find subtler characteristics—the shape of the tube and the velocity and field components normal to the axis—it is necessary to know the nature of the plasma flow around the tube; that problem is vastly more difficult. All that one can do here is estimate the transverse dimensions, by calculating the magnetic field and knowing the magnetic flux. To an extent, the situation here is similar to that in the guiding-center approximation in the theory of the motion of charged particles in a magnetic field, in which case one follows not the particle itself but its guiding center, ignoring the fine details of the Larmor revolution.

The nonlinear-string method is based on the use of frozen-in coordinates. This approach can be taken only in the model of a dissipation-free medium, so it would appear that the same restriction is imposed on the use of this method. Since the magnetic field lines cannot rupture in the absence of dissipation, we see that the number of problems which lend themselves to this approach is small: In these problems, the final field configuration is found from the initial configuration by means of a continuous deformation. In the real world, there are relatively few such situations, so the range of applicability of the method is rather limited. Some help comes from reconnection or, more precisely, the Petschek mechanism[15] and its time-varying generalizations.[7] The actual reconnection of magnetic field lines occurs in a diffusion region, because of the dissipative processes which occur there; in the absence of dissipation, there could be no reconnection.

In general, the dimensions of the diffusion region are small at the scale of the system, so in a first approximation one can assume that the force tubes reconnect not in a reconnection region but on a reconnection line. It then becomes possible to invoke the model of dissipation-free RMHD, but with a rupture of the force tubes on the reconnection line. It is thus possible to substantially expand the range of applicability of the nonlinear-string method.

It is becoming progressively clearer that thin sheets with high currents and boundary layers in general play a key role in many MHD problems. Although these sheets constitute only a small fraction of the volume of the entire system, the processes which occur in them determine the dynamics of the system almost completely.[7,15–17] It is in studies of current sheets and other regions with stretched field lines that the nonlinear-string method is appropriate, so this method may prove useful in many problems in astrophysics and plasma physics.

5. PARTICULAR VARIATIONAL PRINCIPLE

As in the nonrelativistic case,[12] string equations (29), (30) can be found from a particular variational principle with an action:

$$S_{\text{part}} = \int \frac{Q}{\rho q}\, d\alpha\, d\eta = \int \frac{\varepsilon + \rho^2 x_\alpha^2/8\pi + P}{\rho q}\, d\alpha\, d\eta. \tag{33}$$

Here $P(x)$ is assumed to be a given function of the Eulerian coordinates; the functions $x^i(\eta, \alpha, \psi, \zeta)$ are varied. The boundary layer does not have to be planar; a particular variational principle is extremely useful in writing string equations in curvilinear Eulerian coordinates.

We can work from action (33) to construct two new conservation laws. Expression (33) can be thought of as an action with a Lagrangian density

$$L_B = \int \frac{\varepsilon + \rho^2 x_\alpha^2/8\pi + P}{\rho q}\, d\alpha; \tag{34}$$

we then find that the quantity $x^i_\eta \partial L_B/\partial x^i_\eta - L_B$ is conserved:

$$H_B = \int \frac{Q}{\rho q}\, d\alpha. \tag{35}$$

The integral in (35) is conserved as a force tube moves at a velocity u^i/q. We recall that we have $Q = p + \varepsilon + b^2/4\pi$. The frozen-in property is of a dual nature, as can be seen from the symmetry of frozen-in equation (6) with respect to u^i and b^i. For this reason, the same action, (33), can be considered in a different way, with the Lagrangian density

$$L_v = \int \frac{\varepsilon + \rho^2 x_\alpha^2/8\pi + P}{\rho q}\, d\eta. \tag{36}$$

Associated with (36) is the conserved quantity

$$H_v = \int \frac{\varepsilon + p}{\rho q}\, d\eta. \tag{37}$$

The integral in (37) is conserved as a tube of trajectories is continued along magnetic field lines.

In the nonrelativistic limit, the string equations and the general and particular variational principles become their own analogs.[1,7,12] In general, the nonrelativistic technique can be generalized without any fundamental change to the case of relativistic magnetohydrodynamics.

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