Canonical transformation in a theory of weakly nonlinear waves with a nondecay dispersion law

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A critical review is given of the widely used method of canonical transformations in the Hamiltonian theory of weakly nonlinear waves with a nondecay dispersion law. It is shown in particular that a reduced evolution equation, from which nondecay three-wave and nonresonant four-wave processes are excluded, can be derived by a canonical transformation in the form of an integer-degree series up to cubic terms inclusive. An explicit form of this transformation is given and the kernel of the reduced equation, having all the required properties, is presented.

1. INTRODUCTION

The Hamiltonian formalism for nonlinear waves in continuous media, developed in the early seventies by V. E. Zakharov, is now generally accepted and has many applications.[1-3] In this Hamiltonian method the equation of motion of what is known as a complex normal variable $a(\mathbf{k})$ ($\mathbf{k}$ is the wave vector), related by Fourier-type transformations to physical variables of a problem, is written in the following final form:

$$\frac{\partial a(\mathbf{k})}{\partial t} = -i\frac{\delta H}{\delta a^*(\mathbf{k})}, \tag{1}$$

where $H = H(a, a^*)$ is the Hamiltonian of the system (real functional of a and a^*); $i = \sqrt{-1}$; the symbol δ denotes a variational derivative; an asterisk is used for the complex conjugate. Equation (1) together with its complex conjugate forms a pair of canonical Hamiltonian equations with canonically conjugate variables $a(\mathbf{k})$ and $ia^*(\mathbf{k})$ (for simplicity, we shall consider the case with one type of wave without polarization).

In the case of weakly linear waves the Hamiltonian can be expanded as an integer-degree series in a and a^*, which can be limited to a finite number of terms. Up to the quartic terms inclusive, this expansion can be written in the form

$$H=H_2+H_3+H_4, \tag{2}$$

where

$$H_2=\int \omega_0 a_0 a_0^*\, dk_0, \tag{3}$$

$$H_3=\int U^{(1)}_{0,1,2}(a_0^* a_1 a_2 + a_0 a_1^* a_2^*)\delta_{0-1-2}\,dk_{012} + \frac{1}{3}\int U^{(3)}_{0,1,2}(a_0 a_1 a_2 + a_0^* a_1^* a_2^*)\delta_{0+1+2}\,dk_{012}, \tag{4}$$

$$H_4=\int V^{(1)}_{0,1,2,3}(a_0^* a_1 a_2 a_3 + a_0 a_1^* a_2^* a_3^*)\delta_{0-1-2-3}dk_{0123} + \frac{1}{2}\int V^{(2)}_{0,1,2,3} a_0^* a_1^* a_2 a_3 \delta_{0+1-2-3} dk_{0123} + \frac{1}{4}\int V^{(4)}_{0,1,2,3}(a_0 a_1 a_2 a_3 + a_0^* a_1^* a_2^* a_3^*)\delta_{0+1+2+3}\,dk_{0123}. \tag{5}$$

A compact notation is used above in which the arguments $\mathbf{k}_j$ of the coefficients of the expansions $U^{(n)}$ and $V^{(n)}$, of the dispersion law of linear waves ω, of the normal variable a, and of the delta function δ are replaced with the indices j, where the index zero refers to $\mathbf{k}$. For example, $U^{(1)}_{0,1,2} = U^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)$, $\omega_j = \omega(\mathbf{k}_j)$, $a_j = a(\mathbf{k}_j,t)$, $\delta_{0-1-2} = \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)$, etc. The differentials are denoted by $dk_0 = d\mathbf{k}$, $dk_{012} = d\mathbf{k}d\mathbf{k}_1 d\mathbf{k}_2$, etc. and the symbols denoting integration imply integrals of the appropriate multiplicity between the limits of $-\infty$ and ∞. For simplicity, the coefficients $U^{(n)}$ and $V^{(n)}$ are regarded as real and $\omega(\mathbf{k})$ is assumed to be an even function of the wave vector $\mathbf{k}$, i.e., it is assumed that $\omega(\mathbf{k}) = \omega(-\mathbf{k})$.

It is convenient to assume that the coefficients $U^{(n)}$ and $V^{(n)}$ satisfy the "conditions of natural symmetry," which specify that the integrals in Eqs. (4) and (5) are unaffected by relabeling of the dummy integration variables. For example, the coefficient $U^{(1)}_{0,1,2}$ is symmetric under the transposition of the arguments 1 and 2, whereas $U^{(3)}_{0,1,2}$ is symmetric under all the transpositions of 0, 1, 2; similarly $V^{(1)}_{0,1,2,3}$ is symmetric under the transpositions of 1, 2, 3, whereas $V^{(4)}_{0,1,2,3}$ has the same property under the transpositions of 0, 1, 2, and 3, and $V^{(2)}_{0,1,2,3}$ remains symmetric under transpositions of the arguments within the groups (0, 1) and (2, 3). Moreover, the coefficients should satisfy the symmetry conditions which represent the real nature of the Hamiltonian. In the case of the Hamiltonian described by Eqs. (2)–(5) there is one such condition: the coefficient $V^{(2)}_{0,1,2,3}$ should be symmetric under transpositions of the argument pairs (0, 1) and (2, 3). We note that coefficients calculated from the evolution equations of a medium do not usually satisfy the natural symmetry conditions and should be made symmetric by replacing them with a sum of unsymmetrized coefficients of all the transpositions of the relevant groups of arguments, divided by the number of such transpositions. In contrast to the unsymmetrized coefficients, those which are symmetrized are unique. The symmetry, which indicates that the Hamiltonian is real, should be obtained automatically if the equations describing a medium are written in the Hamiltonian form.

For example, the coefficient $V^{(2)}_{0,1,2,3}$ should satisfy the following symmetry conditions:

$$V^{(2)}_{0,1,2,3}=V^{(2)}_{1,0,2,3}=V^{(2)}_{0,1,3,2}=V^{(2)}_{2,3,0,1}. \tag{6}$$

An expansion of the Hamiltonian (2)–(5) should contain the minimum number of terms needed to investigate the "important" features of the evolution and instabilities of weakly nonlinear waves characterized by a nondecay dispersion law. The Hamiltonian of Eqs. (2)–(5) corresponds to the following equation of motion:

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$$\frac{\partial a_0}{\partial t}+i\omega_0 a_0$$

$$=-i\int U^{(1)}_{0,1,2}a_1a_2\delta_{0-1-2}\,dk_{12}-2i\int U^{(1)}_{2,1,0}a_1^*a_2\delta_{0+1-2}\,dk_{12}$$

$$-i\int U^{(3)}_{0,1,2}a_1^*a_2^*\delta_{0+1+2}\,dk_{12}-i\int V^{(1)}_{0,1,2,3}a_1a_2a_3\delta_{0-1-2-3}\,dk_{123}$$

$$-i\int V^{(2)}_{0,1,2,3}a_1^*a_2a_3\delta_{0+1-2-3}dk_{123}-3i\int V^{(1)}_{3,2,1,0}a_1^*a_2^*a_3\delta_{0+1+2-3}dk_{123}$$

$$-i\int V^{(4)}_{0,1,2,3}a_1^*a_2^*a_3^*\delta_{0+1+2+3}\,dk_{123}. \qquad (7)$$

In a study of the stability of waves, in the derivation of the kinetic (transport) equation for a random field of waves, and in some other tasks it is usual to replace Eq. (7) with an equation of the type

$$\frac{\partial b_0}{\partial t}+i\omega_0 b_0=-i\int \tilde{V}^{(2)}_{0,1,2,3}b_1^*b_2b_3\delta_{0+1-2-3}dk_{123} \qquad (8)$$

for a new auxiliary function $b_j=b(\mathbf{k}_j,t)$ with the coefficient $\tilde{V}^{(2)}$ (which represents the kernel of an integral equation) related in a certain manner to $U^{(n)}$ and $V^{(n)}$. Instead of Eq. (8), use is frequently made of an equation

$$\frac{\partial B_0}{\partial t}=-i\int \tilde{V}^{(2)}_{0,1,2,3}B_1^*B_2B_3 \times\exp[i(\omega_0+\omega_1-\omega_2-\omega_3)t]\delta_{0+1-2-3}dk_{123}, \qquad (9)$$

which is derived from Eq. (8) by the substitution of variables $b(\mathbf{k},t)=B(\mathbf{k},t)\exp[-i\omega(\mathbf{k})t]$. These equations (which will be called reduced) describe slow evolution, due to weak nonlinear interactions, of the wave field with a nondecay dispersion law.

Equation (9) was first derived by Zakharov[4,5] from an heuristic equation of the (7) type (with $V^{(1)}=V^{(4)}=0$) by heuristic reasoning (similar to the Van der Pol method), formalized later somewhat to an equivalent method of two-scale expansions in time.[6,7] In both methods it is assumed that a slow evolution of $B(\mathbf{k},t)$ is due to the interaction only of quartets of waves, which satisfy approximately

$$\omega(\mathbf{k})+\omega(\mathbf{k}_1)=\omega(\mathbf{k}_2)+\omega(\mathbf{k}_3) \qquad (10)$$

if

$$\mathbf{k}+\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3 \qquad (11)$$

(we note that these "resonance conditions" may be obeyed for any dispersion law). These methods yield a reduced equation whose kernel does not satisfy all the symmetry conditions of type (6) [this equation is symmetric only under transpositions of the arguments (2) and (3)] and, therefore, we can readily show that it does not conserve energy, i.e., it is not a Hamiltonian equation. In particular, in the problem of gravitational waves on the surface of an ideal liquid, this aspect is still causing confusion[7–9] because of the Hamiltonian nature of the original theory, although such reduced equations remain quite popular. It has also been suggested (and "supported" by numerical calculations) that this energy nonconservation is related to inclusion of just the cubic nonlinearity in an equation of the (9) type, whereas inclusion of quartic and higher nonlinear orders apparently results in energy conservation with increasing precision.[9]

However, in fact this energy nonconservation is unrelated to inclusion of just the cubic nonlinearity in the reduced equation (9), but is due to a defect of the methods of the Van der Pol type. This defect can be removed by the classical method of canonical transformations, generalized to the case of a continuum, in which the replacement of the variable $a(\mathbf{k})$ with $b(\mathbf{k})$ is a canonical transformation.[1] The transformation from $a(\mathbf{k})$ to $b(\mathbf{k})$ is canonical if the equation of motion for $b(\mathbf{k})$ retains the form of Eq. (1):

$$\frac{\partial b(\mathbf{k})}{\partial t}=-i\frac{\delta\tilde{H}}{\delta b^*(\mathbf{k})}, \qquad (12)$$

where $\tilde{H}=\tilde{H}(b,b^*)$ is the Hamiltonian $H=H(a,a^*)$ when the transformation $a=a(b,b^*)$ is substituted in it. The transformation $a=a(b,b^*)$ should be in the form of a finite number of terms in an integer-degree series of the type

$$a(\mathbf{k})=b(\mathbf{k})+I^{(2)}(\mathbf{k})+I^{(3)}(\mathbf{k})+\dots, \qquad (13)$$

where

$$I^{(2)}(\mathbf{k})=I_0^{(2)}=\int A^{(1)}_{0,1,2}b_1b_2\delta_{0-1-2}\,dk_{12}+\int A^{(2)}_{0,1,2}b_1^*b_2\delta_{0+1-2}\,dk_{12} + \int A^{(3)}_{0,1,2}b_1^*b_2^*\delta_{0+1+2}\,dk_{12}, \qquad (14)$$

$$I^{(3)}(\mathbf{k})=I_0^{(3)}=\int B^{(1)}_{0,1,2,3}b_1b_2b_3\delta_{0-1-2-3}\,dk_{123} +\int B^{(2)}_{0,1,2,3}b_1^*b_2b_3\delta_{0+1-2-3}\,dk_{123} +\int B^{(3)}_{0,1,2,3}b_1^*b_2^*b_3\delta_{0+1+2-3}\,dk_{123} +\int B^{(4)}_{0,1,2,3}b_1^*b_2^*b_3^*\delta_{0+1+2+3}\,dk_{123}, \qquad (15)$$

and so on. We shall assume the coefficients $A^{(n)}$ and $B^{(n)}$ to be real and satisfying the necessary natural symmetry conditions. The transformation is canonical if certain conditions are satisfied and these establish specific relationships between the coefficients $A^{(n)}$ and $B^{(n)}$ discussed in the next section.

The Hamiltonian $\tilde{H}$ is obtained by substituting Eq. (13) in Eqs. (2)–(5):

$$\tilde{H}=\int\omega_0 b_0 b_0^*\,dk_0+\tilde{H}_3+\tilde{H}_4, \qquad (16)$$

where $\tilde{H}_n$ ($n=3$ or 4 is the order of linearity in terms of b) are described by expressions similar to Eqs. (4) and (5) in which H_n is replaced by $\tilde{H}_n$, a_j by b_j, and $U^{(n)}$ and $V^{(n)}$ by new coefficients $\tilde{U}^{(n)}$ and $\tilde{V}^{(n)}$ obtained as a result of the above-mentioned substitution. In the case of a nondecay dispersion law the coefficients $A^{(n)}$ and $B^{(n)}$ can be selected so that the Hamiltonian $\tilde{H}$ is of the form

$$\tilde{H}=\int\omega_0 b_0 b_0^*\,dk_0+\frac{1}{2}\int\tilde{V}^{(2)}_{0,1,2,3}b_0^*b_1^*b_2b_3\delta_{0+1-2-3}\,dk_{0123}. \qquad (17)$$

This reduced Hamiltonian corresponds to the equation of motion (8). It follows from Eq. (17) that the coefficient $\tilde{V}^{(2)}$ should satisfy the symmetry conditions of the type given by Eq. (6). The Hamiltonian (17) is an obvious integral of motion, i.e., the reduced equation (8) conserves energy.

This method of deriving the reduced equation, which is natural within the Hamiltonian formalism framework, apparently has not been developed in detail. The expression for $\tilde{V}^{(2)}$ derived both by Zakharov,[1] who first proposed such a scheme, and by authors of subsequent reviews and monographs,[2,3] satisfies the symmetry conditions of the (6) type only on a resonant surface described by Eqs. (10) and (11) (which again results in energy nonconservation), whereas two of the advantages of the canonical transformation method are generalization of the symmetry conditions of the (6) type throughout the space of the vectors $\mathbf{k}$, $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ and also that the "resonance frequency difference" $\Delta\omega = \omega_0 + \omega_1 - \omega_2 - \omega_3$ can be arbitrary and need not be small (it will be recalled that, for example, modulation instabilities appear when $\Delta\omega \neq 0$). It therefore follows that in the cited sources the expression for $\widetilde{V}^{(2)}$ was not derived by a canonical transformation, but was simply based on methods of the Van der Pol type on the assumption that the results should be the same on a resonance surface (which is, in fact, correct). However, the appropriate canonical transformation method differs from that described in the cited sources in at least two respects. Firstly, a correct reduced Hamiltonian of Eq. (17) is obtained if the canonical transformation is in the form of Eqs. (13)–(15), i.e., if it is accurate up to cubic terms in b and not up to quadratic terms, as stated in the cited sources. Secondly, the contribution to $\widetilde{H}_4$ (and, consequently, to $\widetilde{V}^{(2)}$) appears not only due to H_3 and H_4 (which has been assumed implicitly earlier), but also due to H_2. Only if we allow for these two differences can we obtain the general expression for $\widetilde{V}^{(2)}$ satisfying all the necessary symmetry conditions and transforming on the resonance surface described by Eqs. (10) and (11) to an expression deduced by methods of the Van der Pol type. We shall review the Hamiltonian reduction procedure in this spirit below.

2. CONDITIONS FOR CANONICAL TRANSFORMATIONS

Different forms of conditions for canonical transformations are known in classical mechanics. They are readily generalized to the continuum case. The best known of them (but perhaps not the most economic in the volume of calculations needed for transformations in the form of an integer-degree series) are based on the Poisson brackets:

$$\int\left[\frac{\delta a(\mathbf{k})}{\delta b(\mathbf{q})}\frac{\delta a(\mathbf{k}')}{\delta b^*(\mathbf{q})}-\frac{\delta a(\mathbf{k})}{\delta b^*(\mathbf{q})}\frac{\delta a(\mathbf{k}')}{\delta b(\mathbf{q})}\right]d\mathbf{q}=0, \tag{18}$$

$$\int\left[\frac{\delta a(\mathbf{k})}{\delta b(\mathbf{q})}\frac{\delta a^*(\mathbf{k}')}{\delta b^*(\mathbf{q})}-\frac{\delta a(\mathbf{k})}{\delta b^*(\mathbf{q})}\frac{\delta a^*(\mathbf{k}')}{\delta b(\mathbf{q})}\right]d\mathbf{q}=\delta(\mathbf{k}-\mathbf{k}'). \tag{19}$$

In order to ensure that the transformation (13) is canonical up to cubic terms in b, the conditions (18) and (19) should be satisfied up to quadratic terms in b. The variational derivatives in Eqs. (18) and (19) are then quadratic in b. In calculation of the Poisson brackets it is sufficient to find $\delta a(\mathbf{k})/\delta b(\mathbf{q})$ and $\delta a(\mathbf{k})/\delta b^*(\mathbf{q})$; the other variational derivatives are obtained by complex conjugation and by replacing $\mathbf{k}$ with $\mathbf{k}'$. For example, in the case under consideration the conditions (18) and (19) assume the following structure:

$$P^{(0)}+P^{(1)}+P^{(2)}=0, \quad Q^{(0)}+Q^{(1)}+Q^{(2)}=\delta(\mathbf{k}-\mathbf{k}'),$$

where the upper indices of $P^{(i)}$ and $Q^{(i)}$ denote the degree in b. Hence, it follows that the following relationships should be obeyed:

$$P^{(0)}=P^{(1)}=P^{(2)}=Q^{(1)}=Q^{(2)}=0, \quad Q^{(0)}=\delta(\mathbf{k}-\mathbf{k}').$$

The equalities $P^{(0)} = 0$ and $Q^{(0)} = \delta(\mathbf{k} - \mathbf{k}')$ are satisfied identically. The equalities $P^{(1)} = P^{(2)} = 0$ and $Q^{(1)} = Q^{(2)} = 0$ specify the conditions which should be satisfied by the coefficients $A^{(n)}$ and $B^{(n)}$ of the transformation defined by Eqs. (13)–(15) in order that this transformation should be canonical. We can also call them the canonicity conditions.

The equality $P^{(1)} = 0$ yields two conditions:

$$A^{(2)}_{0,1,2}=A^{(2)}_{1,0,2}, \quad A^{(3)}_{0,1,2}=A^{(3)}_{1,0,2}. \tag{20}$$

It follows from the second condition in Eq. (20) and from the natural symmetry of the coefficient $A^{(3)}_{0,1,2}$ that this coefficient should be symmetric under transpositions of all three arguments 0, 1, 2. The equality $Q^{(1)} = 0$ yields two identical conditions, which are

$$A^{(2)}_{0,1,2}=-2A^{(1)}_{2,1,0}. \tag{21}$$

The conditions (20) and (21) represent a complete set of the conditions of canonicity of the transformation $a(\mathbf{k}) = b(\mathbf{k}) + I^{(2)}(\mathbf{k})$. It should be noted that the first of the two conditions in Eq. (20) follows from Eq. (21) as a result of the natural symmetry of $A^{(1)}_{0,1,2}$ in respect of 1 and 2. Therefore, out of the three conditions specified by Eqs. (20) and (21) only two are independent, namely the second one in Eq. (20) and that given by Eq. (21).

We shall now consider the equality $P^{(2)} = 0$. It represents a sum of three integrals which contain factors $b_1 b_2$, $b_1^* b_2$, $b_1^* b_2^*$ in the integrands. The kernels of the integrals with the factors $b_1 b_2$ and $b_1^* b_2^*$ should be symmetrized with respect to 1 and 2, and then the kernels of all three integrals should be equated to zero. This yields three canonicity conditions. Similarly, the equality $Q^{(2)} = 0$ yields three more conditions, two of which are however identical. We thus obtain a total of five canonicity conditions:

$$B^{(2)}_{0,1,2,3}-B^{(2)}_{1,0,2,3}=G^{(2)}_{0,1,2,3}, \tag{22}$$

$$B^{(3)}_{0,1,2,3}-B^{(3)}_{1,0,2,3}=G^{(3)}_{0,1,2,3}, \tag{23}$$

$$3B^{(4)}_{0,1,2,3}-3B^{(4)}_{1,0,2,3}=G^{(4)}_{0,1,2,3}, \tag{24}$$

$$B^{(3)}_{0,1,2,3}+3B^{(1)}_{3,2,1,0}=G^{(31)}_{0,1,2,3}, \tag{25}$$

$$B^{(2)}_{0,1,2,3}+B^{(2)}_{3,2,1,0}=G^{(22)}_{0,1,2,3}, \tag{26}$$

where

$$G^{(2)}_{0,1,2,3}=2[A^{(1)}_{1,2,1-2}A^{(1)}_{3,0,3-0}+A^{(1)}_{1,3,1-3}A^{(1)}_{2,0,2-0} -A^{(1)}_{0,2,0-2}A^{(1)}_{3,1,3-1}-A^{(1)}_{0,3,0-3}A^{(1)}_{2,1,2-1}], \tag{27}$$

$$G^{(3)}_{0,1,2,3}=2[A^{(1)}_{0,3,0-3}A^{(3)}_{1,2,-1-2}+A^{(1)}_{0+2,0,,2}A^{(1)}_{3,1,3-1} -A^{(1)}_{1,3,1-3}A^{(3)}_{0,2,-0-2}-A^{(1)}_{1+2,1,2}A^{(1)}_{3,0,3-0}], \tag{28}$$

$$G^{(4)}_{0,1,2,3}=2[A^{(1)}_{1+2,1,2}A^{(3)}_{0,3,-0-3}+A^{(1)}_{1+3,1,3}A^{(3)}_{0,2,-0-2} -A^{(1)}_{0+2,0,2}A^{(3)}_{1,3,-1-3}-A^{(1)}_{0+3,0,3}A^{(3)}_{1,2,-1-2}], \tag{29}$$

$$G^{(31)}_{0,1,2,3}=2[A^{(1)}_{3,1,3-1}A^{(1)}_{0+2,0,2}-A^{(1)}_{3,2,3-2}A^{(1)}_{0+1,0,1} -A^{(1)}_{1,3,1-3}A^{(3)}_{0,2,-0-2}-A^{(1)}_{2,3,2-3}A^{(3)}_{0,1,-0-1}], \tag{30}$$

$$G^{(22)}_{0,1,2,3}=2[A^{(3)}_{0,1,-0-1}A^{(3)}_{2,3,-2-3}+A^{(1)}_{2,0,2-0}A^{(1)}_{1,3,1-3} -A^{(1)}_{0,2,0-2}A^{(1)}_{3,1,3-1}-A^{(1)}_{0+1,0,1}A^{(1)}_{2+3,2,3}]. \tag{31}$$

Here, the conditions of Eqs. (22)–(24) are obtained from the equality $P^{(2)} = 0$, whereas the conditions of Eqs. (25) and (26) are obtained from $Q^{(2)} = 0$.

A direct check shows that the functions G have the following properties:

$$G^{(2)}_{0,1,2,3}=G^{(2)}_{0,1,3,2}=-G^{(2)}_{1,0,2,3}=G^{(22)}_{0,1,2,3}-G^{(22)}_{1,0,2,3}, \tag{32}$$

$$G^{(3)}_{0,1,2,3}=-G^{(3)}_{1,0,2,3}=G^{(3)}_{2,1,0,3}-G^{(3)}_{2,0,1,3}=G^{(31)}_{0,1,2,3}-G^{(31)}_{1,0,2,3}, \tag{33}$$

$$G^{(4)}_{0,1,2,3}=G^{(4)}_{0,1,3,2}=-G^{(4)}_{1,0,2,3}=G^{(4)}_{2,1,0,3}-G^{(4)}_{2,0,1,3}, \tag{34}$$

$$G^{(31)}_{0,1,2,3}=G^{(31)}_{0,2,1,3}, \tag{35}$$

$$G^{(22)}_{0,1,2,3}=G^{(22)}_{3,2,1,0}. \tag{36}$$

Among the five canonicity conditions given by Eqs. (22)–(26) only three are independent and they are given by Eqs. (24)–(26).

Let us turn back to Eq. (26) and transpose the arguments 0 and 1, and then subtract the resultant equality from the original one given by Eq. (26). Then, using the natural symmetry condition $B^{(2)}_{3,2,1,0}=B^{(2)}_{3,2,0,1}$ and Eq. (32), we obtain Eq. (22). Therefore, the condition (22) follows from Eq. (26) and we can thus exclude Eq. (22) from the group of independent conditions among the five given by Eqs. (22)–(26). We can similarly show that the condition (23) follows from Eq. (25). It should be noted however that in practical applications it may be useful to employ all five conditions (22)–(26).

It therefore follows that the coefficients $A^{(n)}$ in front of the quadratic (in respect of b) terms of the canonical transformation are subject to two independent canonicity conditions, whereas the coefficients $B^{(n)}$ in front of the cubic terms are subject to three independent conditions. Obviously, we can show that the canonical transformation coefficients in front of terms of the nth degree in b should obey n independent canonicity conditions.

3. REDUCED HAMILTONIAN AND CANONICAL TRANSFORMATION

The reduced Hamiltonian of Eq. (17) can be derived by finding first the representation of the Hamiltonian $\widetilde{H}$ in the form of an expansion, given by Eq. (16), in powers of the variable b. Substitution of Eqs. (13)–(15) into Eqs. (2)–(5) gives expansions of the type

$$\widetilde{H}_3=\widetilde{H}_2^{(3)}+\widetilde{H}_3^{(3)},\quad \widetilde{H}_4=\widetilde{H}_2^{(4)}+\widetilde{H}_3^{(4)}+\widetilde{H}_4^{(4)},$$

where $\widetilde{H}_n^{(m)}$ is the contribution to $\widetilde{H}_m$ of the order of m in b, which follows from the term H_n of the order of n terms of a. If we make the substitution mentioned above and use at this stage only the canonicity conditions (20) and (21), we find—after appropriate symmetrization of the kernels of the integrals in $\widetilde{H}_3$ and $\widetilde{H}_4$—the following expressions for the coefficients $\widetilde{U}^{(n)}$ and $\widetilde{V}^{(n)}$:

$$\widetilde{U}^{(1)}_{0,1,2}=(\omega_0-\omega_1-\omega_2)A^{(1)}_{0,1,2}+U^{(1)}_{0,1,2}, \tag{37}$$

$$\widetilde{U}^{(3)}_{0,1,2}=(\omega_0+\omega_1+\omega_2)A^{(3)}_{0,1,2}+U^{(3)}_{0,1,2}, \tag{38}$$

$$\widetilde{V}^{(1)}_{0,1,2,3}=R^{(1)}_{0,1,2,3}+R^{(2)}_{0,1,2,3}+R^{(3)}_{0,1,2,3}+V^{(1)}_{0,1,2,3}, \tag{39}$$

$$\widetilde{V}^{(2)}_{0,1,2,3}=S^{(1)}_{0,1,2,3}+S^{(2)}_{0,1,2,3}+S^{(3)}_{0,1,2,3}+V^{(2)}_{0,1,2,3}, \tag{40}$$

$$\widetilde{V}^{(4)}_{0,1,2,3}=T^{(1)}_{0,1,2,3}+T^{(2)}_{0,1,2,3}+T^{(3)}_{0,1,2,3}+V^{(4)}_{0,1,2,3}, \tag{41}$$

where

$$\begin{aligned}R^{(1)}_{0,1,2,3}=-{}^2/_3[&\omega_{1+3}A^{(1)}_{1+3,1,3}A^{(1)}_{0,2,0-2}\\&+\omega_{1+2}A^{(1)}_{1+2,1,2}A^{(1)}_{0,3,0-3}+\omega_{2+3}A^{(1)}_{2+3,2,3}A^{(1)}_{0,1,0-1}\\&+\omega_{2+3}A^{(1)}_{1,0,1-0}A^{(3)}_{2,3,-2-3}\\&+\omega_{1+3}A^{(1)}_{2,0,2-0}A^{(3)}_{1,3,-1-3}+\omega_{1+2}A^{(1)}_{3,0,3-0}A^{(3)}_{1,2,-1-2}],\end{aligned} \tag{42}$$

$$R^{(2)}_{0,1,2,3}=\omega_0B^{(1)}_{0,1,2,3}+{}^1/_3[\omega_1B^{(3)}_{1,2,3,0}+\omega_2B^{(3)}_{2,1,3,0}+\omega_3B^{(3)}_{3,1,2,0}], \tag{43}$$

$$\begin{aligned}R^{(3)}_{0,1,2,3}={}^2/_3[&U^{(1)}_{0,1,0-1}A^{(1)}_{2+3,2,3}+U^{(1)}_{0,2,0-2}A^{(1)}_{1+3,1,3}\\&+U^{(1)}_{0,3,0-3}A^{(1)}_{1+2,1,2}-U^{(1)}_{1+2,1,2}A^{(1)}_{0,3,0-3}\\&-U^{(1)}_{1+3,1,3}A^{(1)}_{0,2,0-2}-U^{(1)}_{2+3,2,3}A^{(1)}_{0,1,0-1}\\&-U^{(3)}_{1,2,-1-2}A^{(1)}_{3,0,3-0}-U^{(3)}_{1,3,-1-3}A^{(1)}_{2,0,2-0}\\&-U^{(3)}_{2,3,-2-3}A^{(1)}_{1,0,1-0}+U^{(1)}_{1,0,1-0}A^{(3)}_{2,3,-2-3}\\&+U^{(1)}_{2,0,2-0}A^{(3)}_{1,3-1-3}+U^{(1)}_{3,0,3-0}A^{(3)}_{1,2,-1-2}],\end{aligned} \tag{44}$$

$$\begin{aligned}S^{(1)}_{0,1,2,3}=2[&\omega_{0+1}A^{(1)}_{0+1,0,1}A^{(1)}_{2+3,2,3}\\&+\omega_{3-0}A^{(1)}_{3,0,3-0}A^{(1)}_{1,2,1-2}+\omega_{3-1}A^{(1)}_{3,1,3-1}A^{(1)}_{0,2,0-2}\\&+\omega_{2-0}A^{(1)}_{2,0,2-0}A^{(1)}_{1,3,1-3}\\&+\omega_{2-1}A^{(1)}_{2,1,2-1}A^{(1)}_{0,3,0-3}+\omega_{0+1}A^{(3)}_{0,1,-0-1}A^{(3)}_{2,3,-2-3}],\end{aligned} \tag{45}$$

$$S^{(2)}_{0,1,2,3}=\omega_0B^{(2)}_{0,1,2,3}+\omega_1B^{(2)}_{1,0,2,3}+\omega_2B^{(2)}_{2,3,0,1}+\omega_3B^{(2)}_{3,2,1,0}, \tag{46}$$

$$\begin{aligned}S^{(3)}_{0,1,2,3}=2[&U^{(3)}_{0,1,-0-1}A^{(3)}_{2,3,-2-3}+U^{(3)}_{2,3,-2-3}A^{(3)}_{0,1,-0-1}\\&+U^{(1)}_{0+1,0,1}A^{(1)}_{2+3,2,3}+U^{(1)}_{2+3,2,3}A^{(1)}_{0+1,0,1}\\&-U^{(1)}_{0,2,0-2}A^{(1)}_{3,1,3-1}-U^{(1)}_{1,2,1-2}A^{(1)}_{3,0,3-0}\\&-U^{(1)}_{0,3,0-3}A^{(1)}_{2,1,2-1}-U^{(1)}_{1,3,1-3}A^{(1)}_{2,0,2-0}\\&-U^{(1)}_{2,0,2-0}A^{(1)}_{1,3,1-3}-U^{(1)}_{2,1,2-1}A^{(1)}_{0,3,0-3}\\&-U^{(1)}_{3,0,3-0}A^{(1)}_{1,2,1-2}-U^{(1)}_{3,1,3-1}A^{(1)}_{0,2,0-2}],\end{aligned} \tag{47}$$

$$\begin{aligned}T^{(1)}_{0,1,2,3}={}^2/_3[&\omega_{0+1}A^{(1)}_{0+1,0,1}A^{(3)}_{2,3,-2-3}\\&+\omega_{0+2}A^{(1)}_{0+2,0,2}A^{(3)}_{1,3,-1-3}+\omega_{0+3}A^{(1)}_{0+3,0,3}A^{(3)}_{1,2,-1-2}\\&+\omega_{1+2}A^{(1)}_{1+2,1,2}A^{(3)}_{0,3,-0-3}+\omega_{1+3}A^{(1)}_{1+3,1,3}A^{(3)}_{0,2,-0-2}\\&+\omega_{2+3}A^{(1)}_{2+3,2,3}A^{(3)}_{0,1,-0-1}],\end{aligned} \tag{48}$$

$$T^{(2)}_{0,1,2,3}=\omega_0B^{(4)}_{0,1,2,3}+\omega_1B^{(4)}_{1,0,2,3}+\omega_2B^{(4)}_{2,0,1,3}+\omega_3B^{(4)}_{3,0,1,2}, \tag{49}$$

$$\begin{aligned}T^{(3)}_{0,1,2,3}={}^2/_3[&U^{(3)}_{-0-1,0,1}A^{(1)}_{2+3,2,3}+U^{(3)}_{-0-2,0,2}A^{(1)}_{1+3,1,3}\\&+U^{(3)}_{-0-3,0,3}A^{(1)}_{1+2,1,2}+U^{(3)}_{-1-2,1,2}A^{(1)}_{0+3,0,3}\\&+U^{(3)}_{-1-3,1,3}A^{(1)}_{0+2,0,2}+U^{(3)}_{-2-3,2,3}A^{(1)}_{0+1,0,1}\\&+U^{(1)}_{0+1,0,1}A^{(3)}_{-2-3,2,3}+U^{(1)}_{0+2,0,2}A^{(3)}_{-1-3,1,3}\\&+U^{(1)}_{0+3,0,3}A^{(3)}_{-1-2,1,2}+U^{(1)}_{1+2,1,2}A^{(3)}_{-0-3,0,3}\\&+U^{(1)}_{1+3,1,3}A^{(3)}_{-0-2,0,2}+U^{(1)}_{2+3,2,3}A^{(3)}_{-0-1,0,1}].\end{aligned} \tag{50}$$

Here, $R^{(n)}$, $S^{(n)}$, and $T^{(n)}$ with $n=1$ and 2 are the kernels of $\widetilde{H}_2^{(4)}$, whereas for $n=3$ they are the kernels of $\widetilde{H}_3^{(4)}$; $V^{(1)}$, $V^{(2)}$, and $V^{(4)}$ are the kernels of $\widetilde{H}_4^{(4)}$ of Eq. (5).

We can go over from the complete Hamiltonian $\widetilde{H}$ to its reduced form by excluding from $\widetilde{H}$ the cubic terms and the nonresonant terms of the fourth degree in b. The cubic terms can be excluded by assuming that $\widetilde{U}^{(1)}=\widetilde{U}^{(3)}=0$. Then, Eqs. (37) and (38) yield directly the expressions for the coefficients in the quadratic part of the canonical transformation, which exclude the cubic terms of the Hamiltonian:

$$A^{(1)}_{0,1,2}=-\frac{U^{(1)}_{0,1,2}}{\omega_0-\omega_1-\omega_2},\quad A^{(3)}_{0,1,2}=-\frac{U^{(3)}_{0,1,2}}{\omega_0+\omega_1+\omega_2}. \tag{51}$$

The coefficient $A^{(2)}_{0,1,2}$ is found from Eq. (21).

The nonresonant fourth-degree terms correspond to the coefficients $\widetilde{V}^{(1)}$ and $\widetilde{V}^{(4)}$. Vanishing of these coefficients makes it possible to find $B^{(1)}$, $B^{(3)}$, and $B^{(4)}$. However, the calculations are not as simple as in the case of exclusion of the cubic part of the Hamiltonian and we have to consider them in greater detail.

Assuming in Eqs. (39) and (41) that $\widetilde{V}^{(1)} = \widetilde{V}^{(4)} = 0$, and using Eqs. (43) and (49), we can rewrite the resultant equations in the form

$$D^{(1)}_{0,1,2,3}+\omega_0 B^{(1)}_{0,1,2,3}+{}^1/_3[\omega_1 B^{(3)}_{1,2,3,0}+\omega_2 B^{(3)}_{2,1,3,0}+\omega_3 B^{(3)}_{3,1,2,0}]=0, \quad (52)$$

$$D^{(4)}_{0,1,2,3}+\omega_0 B^{(4)}_{0,1,2,3}+\omega_1 B^{(4)}_{1,0,2,3}+\omega_2 B^{(4)}_{2,0,1,3}+\omega_3 B^{(4)}_{3,0,1,2}=0, \quad (53)$$

where for brevity we introduced the notation $D^{(1)} = R^{(1)} + R^{(3)} + V^{(1)}$, $D^{(4)} = T^{(1)} + T^{(3)} + V^{(4)}$. We shall now turn back to Eq. (52). The definition of $B^{(1)}$ is based on the canonicity condition (25), from which we can find the coefficients $B^{(3)}$ for all the combinations of the arguments occurring in Eq. (52). The result is an equation for $B^{(1)}_{0,1,2,3}$, whose solution gives

$$B^{(1)}_{0,1,2,3}=-\frac{{}^1/_3[\omega_1 G^{(31)}_{1,3,2,0}+\omega_2 G^{(31)}_{2,3,1,0}+\omega_3 G^{(31)}_{3,2,1,0}]+D^{(1)}_{0,1,2,3}}{\omega_0-\omega_1-\omega_2-\omega_3}. \quad (54)$$

We can now find $B^{(3)}$ from Eq. (25):

$$B^{(3)}_{0,1,2,3}=-\frac{\omega_1 G^{(3)}_{1,0,2,3}+\omega_2 G^{(3)}_{2,0,1,3}+\omega_3 G^{(31)}_{0,1,2,3}+3D^{(1)}_{3,1,2,0}}{\omega_0+\omega_1+\omega_2-\omega_3}, \quad (55)$$

where after algebraic transformations we can use Eqs. (33) and (35). We can find $B^{(4)}$ from Eq. (53) using the condition (24) and then deriving the coefficients $B^{(4)}$ for all the combinations of the arguments occurring in Eq. (53). Allowing for the natural symmetry of $B^{(4)}$, we find that all these coefficients can be expressed in terms of $B^{(4)}_{0,1,2,3}$. We then obtain

$$B^{(4)}_{0,1,2,3}=-\frac{{}^1/_3[\omega_1 G^{(4)}_{1,0,2,3}+\omega_2 G^{(4)}_{2,0,1,3}+\omega_3 G^{(4)}_{3,0,1,2}]+D^{(4)}_{0,1,2,3}}{\omega_0+\omega_1+\omega_2+\omega_3}. \quad (56)$$

A direct check shows that the coefficients $B^{(1)}$, $B^{(3)}$, and $B^{(4)}$ satisfy all the necessary properties of the natural symmetry. It should be noted that the canonicity condition of Eq. (23) is not used to find the coefficient $B^{(3)}$, but it is naturally satisfied when the expression (55) found above is used. This can be demonstrated by a direct check. In fact, it is a consequence of the fact that the condition (23) follows from the condition (25).

Using the above explicit expressions for the functions $G^{(31)}$, $G^{(3)}$, and $G^{(4)}$ as well as for the functions $R^{(1)}$, $R^{(3)}$, $T^{(1)}$, $T^{(3)}$ occurring in $D^{(1)}$ and $D^{(4)}$, as well as the expressions in Eq. (51), we can simplify greatly the expressions for $B^{(1)}$, $B^{(3)}$, $B^{(4)}$ in Eqs. (54)–(56):

$$B^{(1)}_{0,1,2,3}=-\frac{1}{\omega_0-\omega_1-\omega_2-\omega_3}\Big\{\frac{2}{3}[U^{(1)}_{0,1,0-1}A^{(1)}_{2+3,2,3}+U^{(1)}_{0,2,0-2}A^{(1)}_{1+3,1,3}+U^{(1)}_{0,3,0-3}A^{(1)}_{1+2,1,2}$$
$$+U^{(1)}_{1,0,1-0}A^{(3)}_{2,3,-2-3}+U^{(1)}_{2,0,2-0}A^{(3)}_{1,3,-1-3}+U^{(1)}_{3,0,3-0}A^{(3)}_{1,2,-1-2}]+V^{(1)}_{0,1,2,3}\Big\}, \quad (57)$$

$$B^{(3)}_{0,1,2,3}=-\frac{1}{\omega_0+\omega_1+\omega_2-\omega_3}\{2[U^{(1)}_{3,0,3-0}A^{(1)}_{1+2,1,2}-U^{(1)}_{1+0,1,0}A^{(1)}_{3,2,3-2}-U^{(1)}_{2+0,2,0}A^{(1)}_{3,1,3-1}$$
$$-U^{(3)}_{1,0,-1-0}A^{(1)}_{2,3,2-3}-U^{(3)}_{2,0,-2-0}A^{(1)}_{1,3,1-3}+U^{(1)}_{0,3,0-3}A^{(3)}_{1,2,-1-2}]+3V^{(1)}_{3,1,2,0}\}, \quad (58)$$

$$B^{(4)}_{0,1,2,3}=-\frac{1}{\omega_0+\omega_1+\omega_2+\omega_3}\Big\{\frac{2}{3}[U^{(3)}_{-0-1,0,1}A^{(1)}_{2+3,2,3}+U^{(3)}_{-0-2,0,2}A^{(1)}_{1+3,1,3}$$
$$+U^{(3)}_{-0-3,0,3}A^{(1)}_{1+2,1,2}+U^{(1)}_{0+1,0,1}A^{(3)}_{-2-3,2,3}+U^{(1)}_{0+2,0,2}A^{(3)}_{-1-3,1,3}$$
$$+U^{(1)}_{0+3,0,3}A^{(3)}_{-1-2,1,2}]+V^{(4)}_{0,1,2,3}\Big\}. \quad (59)$$

We shall now find the kernel of the reduced Hamiltonian $\widetilde{V}^{(2)}$ and the canonical transformation coefficient $B^{(2)}$. Using Eq. (46), we shall rewrite Eq. (40) in the form

$$\widetilde{V}^{(2)}_{0,1,2,3}=S^{(1)}_{0,1,2,3}+S^{(3)}_{0,1,2,3}+V^{(2)}_{0,1,2,3}+\omega_0 B^{(2)}_{0,1,2,3}$$
$$+\omega_1 B^{(2)}_{1,0,2,3}+\omega_2 B^{(2)}_{2,3,0,1}+\omega_3 B^{(2)}_{3,2,1,0}. \quad (60)$$

We shall find $B^{(2)}_{1,0,2,3}$, from Eq. (22) and $B^{(2)}_{3,2,1,0}$ and $B^{(2)}_{2,3,0,1}$ from Eq. (26). Using these expressions, we can modify Eq. (60) to

$$\widetilde{V}^{(2)}_{0,1,2,3}=Z_{0,1,2,3}+(\omega_0+\omega_1-\omega_2-\omega_3)B^{(2)}_{0,1,2,3}, \quad (61)$$

where

$$Z_{0,1,2,3}=S^{(1)}_{0,1,2,3}+S^{(3)}_{0,1,2,3}+V^{(2)}_{0,1,2,3}-\omega_1 G^{(2)}_{0,1,2,3}+\omega_2 G^{(22)}_{0,1,3,2}+\omega_3 G^{(22)}_{0,1,2,3}. \quad (62)$$

Calculations similar to those leading to Eqs. (57)–(59) yield a fairly simple expression:

$$Z_{0,1,2,3}=-2[U^{(1)}_{0,2,0-2}A^{(1)}_{3,1,3-1}+U^{(1)}_{2,0,2-0}A^{(1)}_{1,3,1-3}+U^{(1)}_{0,3,0-3}A^{(1)}_{2,1,2-1}$$
$$+U^{(1)}_{3,0,3-0}A^{(1)}_{1,2,1-2}-U^{(1)}_{0+1,0,1}A^{(1)}_{2+3,2,3}-U^{(3)}_{-0-1,0,1}A^{(3)}_{-2-3,2,3}]+V^{(2)}_{0,1,2,3}. \quad (63)$$

We can determine the explicit form of $\widetilde{V}^{(2)}$ and of the canonical transformation itself if we now find the coefficient $B^{(2)}$. It should satisfy the canonicity conditions (22) and (26) (where the first follows from the second, as demonstrated above). However, these conditions are insufficient for an unambiguous determination of $B^{(2)}$. This is due to the fact that the canonical transformation admits a certain freedom (in the case of the coefficients $B^{(1)}$, $B^{(3)}$, and $B^{(4)}$ this freedom is limited by the condition of exclusion of the nonresonant terms from $\widetilde{H}_4$). We can easily show that the coefficient $B^{(2)}$ should be represented in the form

$$B^{(2)}_{0,1,2,3}=\lambda_{0,1,2,3}+\Lambda_{0,1,2,3}, \quad (64)$$

where λ is an arbitrary function satisfying the conditions

$$\lambda_{0,1,2,3}=\lambda_{1,0,2,3}=\lambda_{0,1,3,2}=-\lambda_{3,2,1,0}, \quad (65)$$

and

$$\Lambda_{0,1,2,3}={}^1/_2 G^{(22)}_{0,1,2,3}+{}^1/_4[G^{(2)}_{0,1,2,3}-G^{(2)}_{3,2,1,0}]$$
$$=A^{(3)}_{0,1,-0-1}A^{(3)}_{2,3,-2-3}+A^{(1)}_{1,2,1-2}A^{(1)}_{3,0,3-0}$$
$$+A^{(1)}_{1,3,1-3}A^{(1)}_{2,0,2-0}-A^{(1)}_{0+1,0,1}A^{(1)}_{2+3,2,3}-A^{(1)}_{0,2,0-2}A^{(1)}_{3,1,3-1}-A^{(1)}_{0,3,0-3}A^{(1)}_{2,1,2-1}. \quad (66)$$

The function Λ has the following properties:

$$\Lambda_{0,1,2,3}=\Lambda_{0,1,3,2},$$
$$\Lambda_{0,1,2,3}-\Lambda_{1,0,2,3}=G^{(2)}_{0,1,2,3}, \qquad \Lambda_{0,1,2,3}+\Lambda_{3,2,1,0}=G^{(22)}_{0,1,2,3}. \quad (67)$$

The function λ can be selected in a form which is convenient [variation of $B^{(2)}$ alters simultaneously both $\widetilde{V}^{(2)}$ and $b(\mathbf{k})$, but leaves unchanged $a(\mathbf{k})$ in the canonical transformation]. In particular, we can substitute $\lambda \equiv 0$. It then follows from Eq. (61) that

$$\tilde{V}^{(2)}_{0,1,2,3}=Z_{0,1,2,3}+(\omega_0+\omega_1-\omega_2-\omega_3)\Lambda_{0,1,2,3}. \tag{68}$$

This kernel satisfies all the necessary symmetry conditions. It should be noted that our calculations do not impose any restrictions on the smallness of the resonance frequency difference $\Delta\omega=\omega_0+\omega_1-\omega_2-\omega_3$, and this is one of the differences between our approximation and those proposed earlier.

4. DISCUSSION

If we assume that the resonance frequency difference $\Delta\omega$ is small, then instead of Eq. (68), we have the formal relationship

$$\tilde{V}^{(2)}_{0,1,2,3}=Z_{0,1,2,3}. \tag{69}$$

The kernel $Z_{0,1,2,3}$ no longer satisfies all the symmetry conditions of the (6) type: it is symmetric only in respect of transpositions of the arguments 2 and 3, but not symmetric for 0 and 1 or for transpositions of the pairs (0, 1) and (2, 3) [in Eq. (68) a symmetry of the type described by Eq. (6) is exhibited by the whole sum on the right-hand side, but not by each of the separate terms].

Some comments should be made on the kernel Z. It was first obtained by Zakharov[4,5] without recourse to the Hamiltonian formalism or the technique of canonical transformations, but Zakharov's papers contained a number of inaccuracies. These inaccuracies were corrected by Crawford *et al.*,[6] and the kernel obtained by these authors (after removal of two obvious misprints) was fully identical with our kernel of Eq. (63). In the original English-language edition of the book of Yuen and Lake[7] there are also inaccuracies, which are only partly removed in the Russian edition.

When the resonance conditions of Eqs. (10) and (11) are satisfied exactly (i.e., on the resonance surface itself), the kernel of Eq. (63) transforms to

$$\begin{aligned} Z_{0,1,2,3}=-2\Bigg[&\frac{U^{(1)}_{3,1,3-1}U^{(1)}_{0,2,0-2}}{\omega_{1-3}+\omega_1-\omega_3}+\frac{U^{(1)}_{1,3,1-3}U^{(1)}_{2,0,2-0}}{\omega_{0-2}+\omega_0-\omega_2}+\frac{U^{(1)}_{2,1,2-1}U^{(1)}_{0,3,0-3}}{\omega_{1-2}+\omega_1-\omega_2}\\ &+\frac{U^{(1)}_{1,2,1-2}U^{(1)}_{3,0,3-0}}{\omega_{0-3}+\omega_0-\omega_3}+\frac{U^{(1)}_{2+3,2,3}U^{(1)}_{0+1,0,1}}{\omega_{0+1}-\omega_0-\omega_1}\\ &+\frac{U^{(3)}_{-2-3,2,3}U^{(3)}_{-0-1,0,1}}{\omega_{0+1}+\omega_0+\omega_1}\Bigg]+V^{(2)}_{0,1,2,3}. \end{aligned} \tag{70}$$

This expression is now explicitly symmetric in respect of 0, 1 and 2, 3 and implicitly [i.e., when we allow for Eqs. (10) and (11)] symmetric under transpositions of the pairs (0, 1) and (2, 3). This was the form used by them in Refs. 1–3, where however it was incorrectly stated that it is derived by a canonical transformation which includes only the quadratic terms. Therefore, on the resonance surface itself where Eqs. (10) and (11) apply, the kernel Z has the symmetry described by Eq. (6), but such a kernel is generally incompatible with the reduced equation (9), when it is used, because the exponential function of the latter contain $\Delta\omega\neq 0$.

In addition to the Hamiltonian (energy) $\tilde{H}$, the reduced equations (8)–(9) with the kernel given by Eq. (68) contain two additional integrals of motion of the type

$$M=\int f(\mathbf{k})\,b(\mathbf{k})\,b^*(\mathbf{k})\,d\mathbf{k}\equiv\int f(\mathbf{k})B(\mathbf{k})B^*(\mathbf{k})\,d\mathbf{k}. \tag{71}$$

In fact, it follows from Eq. (8) that M evolves in accordance with the equation

$$\frac{\partial M}{\partial t}=-\frac{i}{2}\int(f_0+f_1-f_2-f_3)\tilde{V}^{(2)}_{0,1,2,3}b_0^*b_1^*b_2b_3\delta_{0+1-2-3}\,dk_{0123},$$

from which it follows that M is conserved if $f(\mathbf{k})=1$ and $f(\mathbf{k})=\mathbf{k}$.

We can derive Eq. (72) using all the symmetry properties of the type given by Eq. (6). If we replace $\tilde{V}^{(2)}$ in Eq. (8) with Z given by Eq. (63), then an equation of the (72) type can no longer be obtained and the above laws of conservation are not satisfied. However, if we confine ourselves to the interactions of just four waves, satisfying the resonance conditions of Eqs. (10) and (11) sufficiently accurately [so that, for example, the kernel of Eq. (70) is well approximated by Eq. (68)], then the laws of conservation should be satisfied approximately. In this case there should be one further approximate law of conservation characterized by $f(\mathbf{k})=\omega(\mathbf{k})$, which can be deduced from Eq. (72). However, it is not clear *a priori* to what extent are the laws of conservation satisfied. This can be estimated by numerical solution of the reduced equations using the exact kernel of (68) and the approximate expressions (70) or (63). It should be pointed out that numerical solution of the reduced equations with the exact kernel is no more complex than when the approximate expressions are used; therefore, there is no advantage in using the approximate kernels.

Finally, we shall consider one statistical aspect associated with the canonical transformation (13). The spectrum $N(\mathbf{k})$ of a physically real random wave field is given by $\langle a(\mathbf{k})a^*(\mathbf{k}')\rangle=N(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')$. Its evolution under the influence of nonlinear resonant four-wave interactions is described by a kinetic equation which is usually derived from the reduced equation (8) for the "spectrum" $n(\mathbf{k})$, defined by a similar expression $\langle b(\mathbf{k})b^*(\mathbf{k}')\rangle=n(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')$ (see, for example, Ref. 1). The spectrum $n(\mathbf{k})$ includes also all the consequences of the kinetic equation. In particular, the concepts of what are known as the Kolmogorov power-law spectra of weak turbulence[10] apply to $n(\mathbf{k})$. Usually the difference between the spectra $N(\mathbf{k})$ and $n(\mathbf{k})$ is either ignored and these spectra are simply identified, or else is not mentioned. In practical applications we need specifically the physical spectrum $N(\mathbf{k})$, so that we have to consider its relationship to $n(\mathbf{k})$.

We can find this relationship using the canonical transformation (13) and a statistical hypothesis similar to that employed in the derivation of the kinetic equation. Using Eq. (13), we have to calculate the correlation function $\langle a(\mathbf{k})a^*(\mathbf{k}')\rangle$, and to apply the Gaussian hypothesis (equivalent to the random phase approximation) to the correlation functions of higher orders in b, which appear on the right-hand side of the equation; according to this hypothesis all the odd moments vanish, whereas the even moments can be expressed in terms of the second moments (among the even moments the only nonzero moments are those which contain the same numbers of the factors b and b^*). This calculation procedure yields

$$\begin{aligned} N_0=n_0+2\int\{&[A^{(1)}_{0,1,0-1}]^2n_1n_{0-1}+2[A^{(1)}_{0+1,0,1}]^2n_1n_{0+1}\\ &+[A^{(3)}_{0,1,-0-1}]^2n_1n_{-0-1}\}\,dk_1+\ldots, \end{aligned} \tag{73}$$

where the dots at the end of the equation represent terms which are cubic in n (not written down here) and which originate from terms cubic in b in the canonical transformation.

In practical applications Eq. (73) calls for numerical calculations. However, two simple conclusions can be drawn even using its general structure. Firstly, the power-law spectra $n(\mathbf{k})$ (for example, the Kolmogorov spectra) when transformed to the physical spectra $N(\mathbf{k})$, are no longer of the power-law type, at least because of the complex dependence of $A^{(m)}$ on $\mathbf{k}$. Secondly, in the case of the spectra $n(\mathbf{k})$ which are narrow in the $\mathbf{k}$ space and concentrated, for example, in the vicinity of the wave vector $\mathbf{k}_0$, the spectra exhibit additional "secondary" peaks at $\mathbf{k} \pm 2\mathbf{k}_0$ and $\mathbf{k} \pm 3\mathbf{k}_0$ [the latter are due to the terms cubic in n which are not included in Eq. (73)]. This is easily seen in the limiting case of a monochromatic wave, when $n(\mathbf{k}) = c\delta(\mathbf{k} - \mathbf{k}_0)$, $c = \text{const}$, and Eq. (73) yields

$$N(\mathbf{k}) = c\delta(\mathbf{k}-\mathbf{k}_0) + 2c^2\{[A^{(1)}(2\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0)]^2\delta(\mathbf{k}-2\mathbf{k}_0) + [A^{(3)}(-2\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0)]^2\delta(\mathbf{k}+2\mathbf{k}_0)\} + \dots .$$

Such secondary peaks are frequently observed in the experimental spectra of the wind waves induced in the ocean (in this case $\mathbf{k}_0$ is the wave vector of the main maximum in the spectrum).

[1] V. E. Zakharov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **17**, 431 (1974).
[2] V. E. Zakharov and E. A. Kuznetsov, Sov. Sci. Rev. Sect. C **4**, 167 (1984).
[3] V. S. L'vov, *Nonlinear Spin Waves* [in Russian], Nauka, Moscow (1987).
[4] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **51**, 1107 (1966) [Sov. Phys. JETP **24**, 740 (1967)].
[5] V. E. Zakharov, Zh. Prikl. Mekh. Tekh. Fiz. No. 2, 86 (1968).
[6] D. R. Crawford, P. G. Saffman, and H. C. Yuen, Wave Motion **2**, 1 (1980).
[7] H. C. Yuen and B. M. Lake, *Nonlinear Dynamics of Gravitational Waves in Deep Water* [Russian translation], Mir, Moscow (1987).
[8] E. A. Caponi, P. G. Saffman, and H. C. Yuen, Phys. Fluids **25**, 2159 (1982).
[9] M. Stiassnie and L. Shemer, J. Fluid Mech. **174**, 299 (1987).
[10] V. E. Zakharov, *Fundamentals of Plasma Physics* [in Russian], Vol. 2, Énergomashizdat, Moscow (1984), p. 48.

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