# Vortices in a lattice model of a two-dimensional nematic

Ya. I. Kogan, S. K. Nechaev, and D. V. Khveshchenko

*Institute of Theoretical and Experimental Physics, Moscow;*
*N. N. Semenov Institute of Chemical Physics, Academy of Sciences of the USSR, Moscow;*
*L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR, Chernogolovka*
(Submitted 25 May 1990)
Zh. Eksp. Teor. Fiz. **98**, 1847–1856 (November 1990)

The mean field approximation is used in an investigation of a phase transition in a two-dimensional lattice nematic described by the $O(3)$ $\sigma$ model on a projective sphere $RP^2$. The tight-binding approximation is used to show that a phase transition takes place at a certain temperature to a vortex free phase relative to a gauge field which determines the projective sphere topology; there is no nematic order in the system on either side of the transition point.

## 1. INTRODUCTION. FORMULATION OF THE TWO-DIMENSIONAL $\sigma$ MODEL ON A PROJECTIVE SPHERE $RP^2$

By analogy with collinear two-dimensional ($2d$) magnetic materials described by the $\sigma$ model on a sphere $S^2$ [or $S^3$–$SU(2)$], a natural description of a planar nematic is given by the $\sigma$ model on a projective sphere $RP^2 = S^2/Z^2$. The projective sphere $RP^2$ appears because the order parameter of a nematic (i.e., its director $\mathbf{d}$) specifies only a line in space but not its direction. Therefore, the action of the $RP^2$ model can be represented in the form

$$S=\frac{1}{2g^2}\int \partial_\mu \mathbf{d}\,\partial_\mu \mathbf{d}\,d^2r=\frac{1}{2g^2}\sum(\mathbf{d}_i\mathbf{d}_j)+\text{const},\quad \mathbf{d}\in RP^2, \tag{1.1}$$

or using the Berezinskii–Villain (BV) formalism[1,2]

$$S=\frac{1}{2g^2}\sum_{i,j}(\mathbf{n}_i-v_{ij}\mathbf{n}_j)^2=-\frac{1}{g^2}\sum_{i,j}\mathbf{n}_i\mathbf{n}_jv_{ij}+\text{const},$$
$$\mathbf{n}_i\in S^2,\quad v_{ij}=\pm1, \tag{1.2}$$

where $v_{ij}$ is a gauge field ensuring a transition of the vector $\mathbf{n}$ to the director $\mathbf{d}$. The partition function of the model is

$$\begin{aligned}Z&=\sum_v Z_v,\\ Z_v&=\int D\mathbf{n}\exp\Big(\sum_{i,j}\frac{1}{g^2}\mathbf{n}_i\mathbf{n}_jv_{ij}\Big)\\ &=\int\prod D\mathbf{n}\prod_{i,j}\exp\Big(\frac{1}{g^2}\mathbf{n}_i\mathbf{n}_jv_{ij}\Big).\end{aligned} \tag{1.3}$$

The action (1.2) for $v_{ij}\equiv 1$ is identical with the lattice action of the $O(3)$ $\sigma$ model, whose continuous limit is

$$S_{O(3)}=\frac{1}{2g^2}\int d^2r\,\partial_\mu\mathbf{n}\,\partial_\mu\mathbf{n},\quad \mathbf{n}\in S^2. \tag{1.4}$$

We cannot obtain the continuous limit of the action (1.2) for an arbitrary distribution of $v_{ij}$ because the field $v_{ij}$ assumes the discrete values $\pm 1$ and it cannot be described in the continuous limit, which is the general property of all discrete-valued fields. Therefore, in the continuous limit we have to use directly the action (1.1).

Nontrivial configurations of the field $v_{ij}$ are described by classical vortex configurations of the field of $\mathbf{d}$. Bearing in mind that $\mathbf{d}\in RP^2 = S^2/Z^2$, in the case of single-vortex solutions we can select the following parametrization [$\mathbf{n}=(n_1,n_2,n_3)$]:

$$n_1=\cos(\varphi/2),\qquad n_2=\sin(\varphi/2),\qquad n_3=0, \tag{1.5}$$

where $(r,\varphi)$ is the coordinate in a plane. We can see that if we go around the point $r=0$ along a closed contour, the angle $\varphi$ changes by $2\pi$, so that we have $\mathbf{n}_i\to-\mathbf{n}_i$. On an ordinary sphere $S^2$ this would not have been a topologically nontrivial solution, but on a projective sphere $RP^2$ obtained by identification of diametrically opposite points ($\mathbf{n}\equiv-\mathbf{n}$) such a vortex solution is valid. As usual (see Ref. 3), it is characterized by a logarithmically large action

$$S=\frac{1}{2g^2}\int r\,dr\,d\varphi\frac{1}{r^2}(\partial_\varphi\mathbf{n})^2=\frac{1}{16g^2}\ln\Big(\frac{A}{a^2}\Big), \tag{1.6}$$

where $A$ is the volume (total area) of the system and $a$ is the ultraviolet cutoff parameter, which in the case of the lattice model is identical with the size of a packet.

Using the standard arguments of the Berezinskii–Kosterlitz–Thouless (BKT) theory[1-3] and comparing Eq. (1.6) with the vortex entropy, we find that the BKT transition should occur for $g^2=1/16$, but this is incorrect. In fact, the theory of Eq. (1.4) is asymptotically free and the coupling constant (charge) $g$ depends on the gauge of $\rho$. In the one-loop approximation,[4] we obtain

$$g^{-2}(\rho)\approx b\ln(\rho_c/\rho),$$

where $\rho_c$ is the correlation radius governing the mass gauge of the theory. Including the quantum corrections, we can modify Eq. (1.6) to

$$S_{1-loop}=\frac{1}{16g^2(\rho)}\ln\Big(\frac{\rho}{a^2}\Big)=\frac{b}{16}\ln\Big(\frac{\rho_c}{\rho}\Big)\ln\Big(\frac{\rho}{a^2}\Big). \tag{1.7}$$

Comparing Eq. (1.7) with the entropy $\ln(\rho/a^2)$ of a pair of vortices of size $\rho$, we can see that if $\rho\approx\rho_c$, the entropy is always higher, i.e., irrespective of the unrenormalized (bare) value of $g$, the quantum corrections always make $g_{\text{eff}}=g(\rho)$ sufficiently large, so that the vortices always "survive" in one plasma phase where the excitations have a gap. We shall confine our treatment to just the tight-binding approximation.

The gauge field $v_{ij}$ introduced above and governing the topology of the projective sphere $RP^2$ can be regarded as an order parameter which is an indicator of the presence of vortices in the system.

We shall show in Sec. 2 that in the mean field approximation at some critical temperature $T_c$ a phase transition takes place to a state (which is stable below $T_c$) correspond-

 0038-5646/90/111038-05$03.00  

ing to the absence of vortices in the system. However, such a transition does not create a nematic order.

It therefore follows that in the adopted approximation the exponential function of the action of Eq. (1.2) can be represented by the first two terms of the expansion in powers of $1/g^2$:

$$\exp\Bigl(\sum_{i,j}\frac{1}{g^2}\mathbf{n}_i\mathbf{n}_j v_{ij}\Bigr)=\prod_{i,j}\Bigl(1+\frac{1}{g^2}\mathbf{n}_i\mathbf{n}_j v_{ij}\Bigr).$$

Then, the partition function $Z_v$ becomes

$$Z_v=\int D\mathbf{n}\prod_{i,j}\Bigl(1+\frac{1}{\sigma^2}\mathbf{n}_i\mathbf{n}_j v_{ij}\Bigr). \qquad (1.8)$$

The integration measure is $D\mathbf{n}=\Pi d\mathbf{n}_i$, so that

$$\int d\mathbf{n}_i 1=1, \qquad (1.8a)$$

$$\int d\mathbf{n}_i\, n_i^a=0, \qquad (1.8b)$$

$$\int d\mathbf{n}_i\, n_i^a n_i^b={}^1/_3\delta^{ab}, \qquad (1.8c)$$

and so on.

Calculating the integrals with respect to $\mathbf{n}_i$ in the system (1.8), we can easily show that the partition function $Z_v$ can be reduced to a sum over closed contours [open contours contain the odd number of $\mathbf{n}_i$, and on the basis of the condition (1.8b), make zero contribution to the expansion]:[5]

$$Z_{\langle v\rangle}=\sum_{N=0}^{\infty}3^N\sum_{\{C_1\ldots C_N\}}\Bigl(\frac{1}{3g^2}\Bigr)^{L(C_1)+\ldots+L(C_N)}\prod_{i=1}^{N}v(C_i), \qquad (1.9)$$

where $C_1,\ldots C_N$ are the close contours on a plane (generally speaking with self-intersections), whereas

$$v(C)=\prod_{i,j\in C}v_{ij}$$

is a phase factor along a closed contour, which appears as a product of the values of the gauge field specified at the bonds. We can show that $v_{ij}$ is simply $Z^2$ and it is an analog of the Wilson operator $\exp(\oint A_\mu dx^\mu)$.

We shall now sum Eq. (1.9) over all the closed contours $N$, but one naturally should reduce this summation to some function of one contour. It is natural to detect the geometric predictions in the form of an exponential function of Eq. (1.9), i.e.,

$$Z_{\langle v\rangle}=\exp\Bigl\{\sum_{\{C\}}3\exp[-L(C)\ln(3g^2)]\exp[i\Phi(C,\{v\})]\Bigr\}, \qquad (1.10)$$

where $\exp[i\Phi(C,\{v\})]=V(C,\{v\})$ is the phase factor for a contour $C$ corresponding to a given configuration of the gauge (vortex) field $\{v\}$. In fact, Eq. (1.10) is not quite an exact analog of Eq. (1.9), because integration of the exponential function of some contours is allowed for using an incorrect weight and, in particular, there are difficulties with self-intersecting contours. This problem can be avoided either by going over to a hexagonal lattice right from the beginning, because there are no such contours for this lattice, or retaining a square lattice and recalling that Eq. (1.8) is only the high-temperature approximation in which nevertheless the contours with double passes along one line are ignored; consequently, the sum over all the configurations of the contours $\{C\}$ in Eq. (1.10) denote summation only for contours that do not self-intersect.[6]

It therefore follows that in the tight-binding limit the problem of calculation of the free energy of one or another vortex configuration reduces to a study of the sum of close contours in an "external field" allowed for by a phase factor $V(C,\{v\})=\exp i\Phi(C,\{v\})]$, where the flux across the contour $C$ is

$$\Phi(C,\{v\})=\pi m,$$

whereas $m$ is the number of vortices inside the contour. The average flux (i.e., the flux per one placket) does not exceed $\pi$ (see Sec. 2). It should be pointed out that if we consider self-intersecting contours, then

$$\Phi(C,\{v\})=\sum_{i=1}^{m}\pi n_i,$$

where $n_i$ is the algebraic number of passes made by each contour around the $i$th vortex.

Using the $RP^2$ model and the high-temperature approximation we can write the complete partition function as follows:

$$\begin{aligned}Z_{RP^2}&=2^S\sum_v Z_v=2^S\sum_v\exp[-F\{v\}]\\&=2^S\exp\Bigl\{3\sum_{\{C\}}\exp[-L(C)\ln(3g^2)]\Bigr\}\\&\quad\times\sum_v\exp\Bigl\{3\sum_{\{C\}}\exp[-L(C)\ln(3g^2)][V(C,\{v\})-1]\Bigr\}\\&=Z_{S^2}\cdot 2^S\sum_v\exp\Bigl\{3\sum_{\{C\}}\exp[-L(C)\ln(3g^2)]\\&\qquad\times[V(C,\{v\})-1]\Bigr\}=Z_{S^2}2^S Z_v.\end{aligned} \qquad (1.11)$$

Here, $Z_{s^2}$ acts as a normalization factor, typical of all the field configurations, and equal simply to the sum over all the contours, whereas $Z_v$ is the partition function of the vortices in the form of a sum of the contributions made by 0, 1, 2, ... vortices:

$$Z_v=\sum_{m=0}^{\infty}Z^{(m)},$$

where

$$Z^{(0)}=1,$$

$$Z^{(1)}=\int\frac{d^2a}{\varepsilon^2}\exp\Bigl\{-6\sum_{C_-(a)}\exp[-L(C_-(a))\ln(3g^2)]\Bigr\}, \qquad (1.12)$$

$$Z^{(2)}=\frac{1}{2}\int\frac{d^2a}{\varepsilon^2}\frac{d^2b}{\varepsilon^2}\times\exp\Bigl(-6\sum_{C_-(a)}\exp[-L(C_-(a,b))\ln(3g^2)]\Bigr),\ldots$$

Here, $S=\int d^2a/\varepsilon^2$ is the number of possible positions of a vortex on a plane; $C_-(a)$ are the contours containing a vortex at a point $a$ (in general, there is an odd number of passes of contours with self-intersection); $C_-(a,b)$ are the contours with round vortices located at points $a$ and $b$, where the number of the terms in the sum is odd; we can then proceed similarly dealing with the other cases. The fact that $2^s$, which we shall omit in future, is the volume of the calibration group $Z^2$; in the case of an arbitrary group $G$, it is replaced in a natural manner by $(\dim G)^s$.

## 2. PHASE TRANSITION IN A SYSTEM OF VORTICES IN THE FRAMEWORK OF THE MEAN FIELD THEORY

By way of illustration we shall first consider the limit when there is a vortex at each placket, i.e., a flux $\Phi = \pi$ crosses each placket in the lattice. Using Eqs. (1.10) and (1.11), we can easily show that in this situation the total number of contours of length $L$ which enclose in "an odd manner" a given number of vortices (see above) is equal to the number of contours of length $L$ including the whole fixed area $S = \pi(2k+1)$, where $k = 0, \pm 1, \pm 2, \ldots$ (in this case the number $m = 2k+1$ represents the number of plackets of vortices within a contour).

In the case of a grating not quite filled with vortices, we shall use the mean field approximation which involves a distribution of the fluxes

$$\Phi(C, \{v\}) = \begin{cases} \pi & \text{for a placket with a vortex,} \\ 0 & \text{for other plackets,} \end{cases}$$

which can be replaced by the mean value

$$\Phi = \pi m/N, \tag{2.1}$$

where $N$ is the total number of cells in the system (total area) and $m$ is the number of vortices.

Since the flux $\Phi$ and the area $S$ surrounded by a contour are conjugate quantities, then in this approximation the partition function $Z_v$ [see Eq. (1.10)], identical with the number of all the contours $Z(\Phi)$ characterized by a fixed flux $\Phi$, can be represented by a path integral in the continuous limit [$L = \mathcal{N}\varepsilon = \text{const}, \mathcal{N} \to \infty, \varepsilon \to 0$, where $\mathcal{N}$ is the number of steps along a path, whereas $\varepsilon$ is the size of a lattice cell (length of a step)]:

$$Z_v \approx Z(\Phi) = \frac{1}{2\pi} \int dS \exp(i\Phi S) Z(S), \tag{2.2}$$

where

$$Z(S) = \sum_{\text{even } L} \exp[-L \ln(3g^2)] Z(S, L), \tag{2.2a}$$

and $Z(S, L)$ is described by the expression

$$Z(S,L) = \int \ldots \int D\{r\} \exp\Big\{ -\frac{1}{\varepsilon} \oint_c dS\, \dot{\mathbf{r}}^2(s) - \frac{\tau}{2a^2} \oint_c ds \oint_c ds'\, \delta(\mathbf{r}(s) - \mathbf{r}(s')) \Big\} \delta \times \Big( S - \frac{1}{2} \oint_c ds\, \mathbf{A}(\mathbf{r}(s)) \dot{\mathbf{r}}(s) \Big). \tag{2.3}$$

In Eq. (2.3) the second term in the exponential function allows for the fact that the path cannot be self-intersecting ($\tau > 0$), whereas the last factor with the $\delta$ function selects contours with a fixed algebraic area $S$ expressed in terms of the Green formula:

$$S = {}^1/_2 \oint_c d\mathbf{r}\, \mathbf{A}(\mathbf{r}), \qquad \mathbf{A} = [\boldsymbol{\xi}\mathbf{r}], \quad \boldsymbol{\xi} = (0, 0, 1). \tag{2.4}$$

Using Eqs. (2.3) and (2.4), we can rewrite the expression for $Z(\overline{\Phi}, L)$ in the form

$$Z(\Phi, L) = \int \ldots \int D\{\mathbf{r}\} \times \exp\Big\{ -\frac{1}{\varepsilon} \oint ds \Big( \dot{\mathbf{r}}^2(s) + \frac{1}{2} i\varepsilon\Phi[\mathbf{r}(s)\dot{\mathbf{r}}(s)] \Big) \Big\} \times \exp\Big[ -\frac{\tau}{2a^2} \oint ds \oint ds'\, \delta(\mathbf{r}(s) - \mathbf{r}(s')) \Big] \tag{2.5}$$

(for convenience, we shall from now on assume that $\varepsilon \equiv 1$).

The problem of calculation of $Z(\Phi, L)$ reduces to calculation of the partition function of $n$-component gauge theory $\Psi^4$ ($\Psi = (\psi_1, \psi_2, \ldots, \psi_n)$) in the limit $n \to 0$ (Ref. 7):

$$Z(\Phi, L) = \frac{1}{2\pi i} \oint_{\alpha - i\infty}^{\alpha + i\infty} d\lambda\, e^{-\lambda L} Z(\Phi, \lambda),$$

$$Z(\Phi, \lambda) = \lim_{n \to 0} \int \ldots \int d\lambda \prod_{\sigma=1}^{n} \delta\psi_\sigma \delta\psi_\sigma^* e^{-F[\Psi]}, \tag{2.6}$$

where

$$F[\Psi] = \int d\mathbf{r} \Big\{ \frac{1}{2} |(\nabla - i\Phi\mathbf{A})\Psi|^2 + \lambda|\Psi|^2 + \tau|\Psi|^4 \Big\}.$$

Unfortunately, the partition functions of Eq. (2.5) and, consequently, of Eq. (2.6) cannot be determined accurately, which means that an analysis of Eq. (2.5) can be started from the case when $\tau = 0$, coresponding to lifting of the forbiddenness of contour self-intersections. We shall generalize our expressions to the case of contours free of self-intersections and we shall do this by applying the scaling relationships.

For example, if $\tau = 0$, the partition function of Eq. (2.5), or, more exactly, $P_0(\Phi, L)$ for paths in continuous space can be calculated exactly:[8]

$$P_0(\Phi, L) = \frac{1}{\pi L\varepsilon} \frac{{}^1/_4 L\varepsilon\Phi}{\operatorname{sh}({}^1/_4 L\varepsilon\Phi)}. \tag{2.7}$$

In the limit $\Phi \to 0$, we have

$$\lim_{\Phi \to 0} P_0(\Phi, L) = \frac{1}{\pi L\varepsilon}, \tag{2.7a}$$

which agrees with the probability density of formation of a close contour with a fixed point on the surface and corresponds to the condition of normalization of the function $P_0(S, L)$ to the Gaussian distribution [see Eq. (2.2)]:

$$\int_{-\infty}^{\infty} dS\, P_0(S, L) = \frac{1}{\pi L\varepsilon}.$$

The partition function $Z_0(\Phi, L)$ for a lattice with the coordination number $Z$ can now be represented in the form

$$Z_0(\Phi, L) = z^{L/\varepsilon} P_0(\Phi, L). \tag{2.7b}$$

In the Appendix we shall show that calculation of the partition function of Eq. (2.5) for $\tau = 0$ can be reduced in a discrete case to calculation of the Green function of the Hofstadter problem of the dynamics of an electron in a magnetic field on a plane square lattice.[9]

Using Eqs. (1.11), (2.2), and (2.7) for $F_0\{v\}$, we obtain the following expression for the square lattice with the coordination number $z = 4$:

$$F\{v\} = F_0(\Phi) = 6 \sum_{\text{even } L} \exp[-L\ln(3g^2)]z^L \frac{1}{\pi L\varepsilon} \frac{{}^1/_4 L\varepsilon\Phi}{\text{sh}({}^1/_4 L\varepsilon\Phi)}. \quad (2.8)$$

We recall that the expression for $P_0(\Phi, L)$, in Eq. (2.8) is obtained in the continuous limit for self-intersecting contours.

In order to allow more correctly for the dependence on $g$ and $\Phi$ in the first terms of the sum over $L$ in $F_0(\Phi)$, we shall write them down explicitly and calculate accurately the number of contours of length $L = 4$ and 6 and employ a discrete analog of Eq. (2.2), whereas beginning from $L = 8$ we shall replace this sum in Eq. (2.8) with an integral with respect to $L$. In this way we obtain the following expression for $F_0(\Phi)$:

$$F_0(\Phi) = \frac{16\cos\Phi}{(3g^2)^4} + \frac{(46656/25 + 432)\cos\Phi + 48\cos 2\Phi}{(3g^2)^6} + 6\int_8^\infty d\tilde{L}\left(\frac{z}{3g^2}\right)^{\tilde{L}} \frac{1}{\pi\tilde{L}} \frac{{}^1/_4\tilde{L}\Phi}{\text{sh}({}^1/_4\tilde{L}\Phi)} + \text{const}. \quad (2.9)$$

In the mean field approximation the free energy of a vortex configuration with an average flux $\Phi$ has an entropy associated with the possibility of transposition of the vortices in the interior of the system. Two vortices at one point annihilate each other and, therefore, they satisfy the Fermi statistics and the entropy of such transpositions is

$$\bar{S} = \frac{\Phi}{\pi}\ln\frac{\Phi}{\pi} + \left(1 - \frac{\Phi}{\pi}\right)\ln\left(1 - \frac{\Phi}{\pi}\right). \quad (2.10)$$

The flux $\Phi$ varies from a state $\Phi = 0$, corresponding to the complete absence of vortices, to a state $\Phi = \pi$, which represents a system filled completely with vortices.

It should be stressed that a state of the system with $\Phi = 0$ determines uniquely (apart from the gauge transformation) the distribution of the field $\{v\}$ at bonds, so that it represents the situation when in all cases we have $v_{ij} = 1$ or $v_{ij} = -1$. Using geometric considerations, we can readily establish that any other distribution of $v_{ij}$ leads to a state with $\Phi \neq 0$ (more accurately, to a state with $\Phi > 0$).

The total free energy of the system $F_0^\Sigma$ is governed by the difference between the contributions (2.9) and (2.10). The temperature dependence of the original model is contained within the binding constant $g$ (we can assume that $g^2 \propto T$).

Figure 1 shows the relief of the function $F_0^\Sigma(g, \Phi)$. We can see that if $g < g_c^0$ ($g_c^0 = 1.31$), a phase transition takes place to a state corresponding to the mean flux $\Phi = 0$, i.e., a state with a complete absence of vortices in the system. In the light of the adopted mean field theory, this is a transition of the first order and there is a possibility that a correct allowance for fluctuations may alter the order of the transition.

Applying now the scaling relationships to allow for the circumstance that in reality the contours surrounding vortices cannot self-intersect. If $\Phi = 0$, then instead of Eq. (2.7a), we now have[10]

$$P(\Phi=0, L) = 1/\pi R^2, \quad (2.11)$$

where

$$R^2 = L^{2\nu},$$

whereas the critical exponent $\nu$ for the two-dimensional problem is exactly 3/4.

If $\Phi \neq 0$, then by analogy with Eq. (2.7) we find that the following scaling relationship applies to all closed nonself-intersecting contours:

$$P(\Phi, L) = \frac{1}{\pi R^2}\frac{{}^1/_4 R^2\Phi}{\text{sh}({}^1/_4 R^2\Phi)} = \frac{1}{\pi L^{2\nu}}\frac{{}^1/_4 L^{2\nu}\Phi}{\text{sh}({}^1/_4 L^{2\nu}\Phi)}. \quad (2.12)$$

Subtracting Eq. (2.10) from Eq. (2.12), we find that the total free energy $F^\Sigma$ is given by:

$$F^\Sigma = \frac{16\cos\Phi}{(3g^2)^4} + \frac{48\cos 2\Phi}{(3g^2)^6} + 3\int_8^\infty d\tilde{L}\left(\frac{z}{3g^2}\right)^{2\tilde{L}} \frac{1}{\pi\tilde{L}^{2\nu}} \frac{{}^1/_4\tilde{L}^{2\nu}\Phi}{\text{sh}({}^1/_4\tilde{L}^{2\nu}\Phi)} - \frac{\Phi}{\pi}\ln\frac{\Phi}{\pi} + \left(1 - \frac{\Phi}{\pi}\right)\ln\left(1 - \frac{\Phi}{\pi}\right) + \text{const}, \quad (2.13)$$

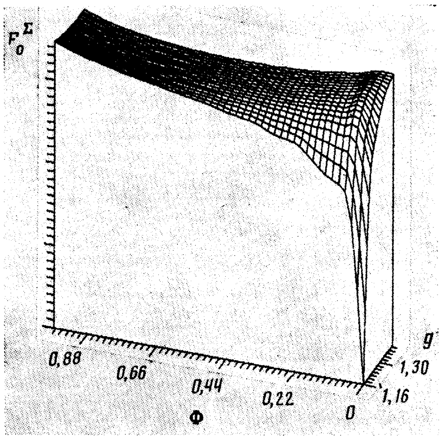


FIG. 1. Relief of the function $F_0^\Sigma(g, \Phi/\pi)$ for self-intersecting contours.

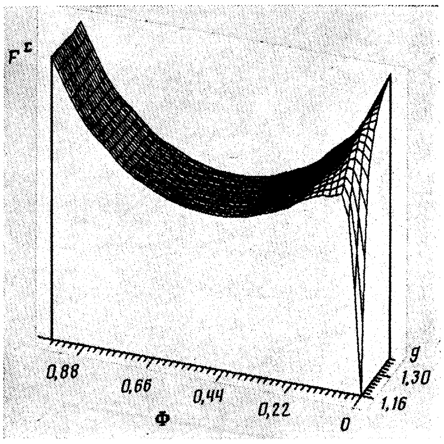


FIG. 2. Relief of the function $F^\Sigma(g, \Phi/\pi)$ for nonself-intersecting contours.

where, as in Eq. (2.9) the contributions of the contours with $L=4$ and 6 are allowed for exactly.

The corresponding relief of the function $F^{\Sigma}(g, \Phi)$ is shown in Fig. 2. The critical value $g_c = 1.21$ corresponds to a first-order phase transition (in the mean field approximation) to the phase with $\Phi = 0$, which remains thermodynamically stable when $g < g_c$.

The mean field analysis given above agrees with the qualitative ideas on the interaction between two vortices against the background of vacuum. We know that two vortices located alongside each other in a lattice repel tending to reduce the mean flux inside the area bounded by vortices.

## CONCLUSIONS

We posed for the first time the problem of the possibility of existence of a "vortex" phase transition in the $RP^2\sigma$ model. However, it should be pointed out that the mean field analysis of the stability of a vortex-free phase in the case when $g < g_c$ does not determine uniquely the nature of the transition since, as usual, an allowance for fluctuations can alter the pattern qualitatively. Formally, a rigorous analysis of this problem can be made by considering the conformal properties of the $O(n)$ model on planar lattices of different types.

The authors are grateful to V. L. Pokrovskiĭ and A. V. Chubukov for valuable comments.

## APPENDIX

We shall show that Eq. (2.5) with $\tau = 0$ in the discrete case (for a random walk on a square lattice) satisfies the Hofstadter equation describing the dynamics of an electron on a square lattice in a homogeneous magnetic field.

The recurrence equation for the partition function $Z(\mathbf{R}, \Phi, L)$ is

$$Z(\mathbf{R}, \Phi, L+1) = \int d\mathbf{R}'\, g(\mathbf{R}-\mathbf{R}') \times \exp(-{}^1/_2 i\Phi\xi[\mathbf{R}\mathbf{R}'])Z(\mathbf{R}', \Phi, L), \quad \text{(A1)}$$

where $g(\mathbf{R}-\mathbf{R}')$ is the condition of probability of a transition in one step. We must bear in mind that in the discrete case we have to make the following simple substitution in Eq. (A1):

$$\int d\mathbf{R}'\, g(\mathbf{R}-\mathbf{R}')(\ldots) \to \sum w(\mathbf{R}-\mathbf{R}')(\ldots), \quad \text{(A2)}$$

where $w(\mathbf{R}-\mathbf{R}')$ is the matrix of local jumps on a lattice in one step. Equations (A1) and (A2) yield directly the difference equation for a square lattice with a symmetric matrix $w(\mathbf{R}-\mathbf{R}')$:

$$\begin{aligned} Z(x, y, \Phi, L+1) = {}^1/_4\{&\exp({}^1/_2 i\lambda x)Z(x, y-1, \Phi, L) \\ &+\exp(-{}^1/_2 i\lambda x)Z(x, y+1, \Phi, L)+\exp(-{}^1/_2 i\lambda y) \\ &\times Z(x-1, y, \Phi, L)+\exp({}^1/_2 i\lambda y)Z(x+1, y, \Phi, L)\}, \end{aligned} \quad \text{(A3)}$$

where $\Phi$ is the magnetic flux and Eq. (A3) is written in the gauge $\mathbf{A} = \frac{1}{2}(-Hy, Hx)$.

Equation (A3) cannot be solved exactly for arbitrary values of $\Phi$ and its solutions can be obtained only for rational $\Phi/\pi$, which gives rise to the familiar ultrametricity and the "Hofstädter butterfly" (Ref. 9). By way of example, we shall write down the values of the partition function corresponding to the fluxes $\Phi = 0$ and $\Phi = \pi$:

$$\begin{aligned} &Z(\Phi=0, 2L) = (C_{2L}{}^L)^2, \\ &Z(\Phi=\pi, 2L) = \sum_{m=0}^{L} C_L{}^m C_{2m}{}^m C_{2L-2m}^{L-m}. \end{aligned} \quad \text{(A4)}$$


[1] V. L. Berezinskiĭ, Zh. Eksp. Teor. Fiz. **59**, 907 (1970); **61**, 1144 (1971) [Sov. Phys. JETP **32**, 493 (1971); **34**, 610 (1972)].
[2] J. Villain, J. Phys. (Paris) **36**, 581 (1975).
[3] J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973).
[4] A. M. Polyakov, Phys. Lett. B **59**, 79, 82 (1975).
[5] B. Nienhuis, J. Stat. Phys. **34**, 731 (1984); *Phase Transitions and Critical Phenomena* (ed. by C. Domb and J. L. Lebowitz), Vol. 11, Academic Press, New York (1987); M. den Nijs, Phys. Rev. B **27**, 1674 (1983).
[6] B. Duplantier, Preprint No. SPhT/89-162, CENS, Jif-sur-Yvette, France (1989).
[7] F. Tanaka, Prog. Theor. Phys. **68**, 148, 164 (1982).
[8] D. C. Khandekar and F. W. Wiegel, J. Phys. (Paris) **50**, 263 (1989).
[9] D. R. Hofstädter, Phys. Rev. B **14**, 2239 (1976); J. Guillement, B. Helffer, and P. Treton, J. Phys. (Paris) **50**, 2019 (1989).
[10] P. G. de Gennes, *Scaling Concepts in Polymer Physics*, Cornell University Press, Ithaca, NY (1979).


Translated by A. Tybulewicz