Formation of the angular spectrum of wind waves

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We study one of the central problems of the theory of wind waves, namely, the problem of the formation of the angular spectrum of the swell. We consider the evolution of a random field of weakly nonlinear surface waves in the energy-carrying range taking into account a new mechanism (in the context of this problem), namely the induced scattering by subsurface flow produced by the wind. The term "induced scattering" as applied to the present problem means the transformation of the wave field due to resonance absorption (or emission) of difference harmonics of the field in critical layers in the drift flow. This mechanism makes a contribution to the kinetic equation for spectral density $N_{\mathbf{k}}$ of the wave action which is proportional to $N_{\mathbf{k}}^2$ and to the small parameter of the problem, the ratio of the drift velocity to the phase velocity of the wind waves considered, and also to a large parameter determined in our model by the Reynolds number. The particular features of this mechanism consist in that it causes a strong angular redistribution of the wave action practically without affecting directly the dynamics of the spectrum as regards the absolute magnitude of the wavenumber. We obtain for the description of the evolution of the angular spectrum through the action of the induced scattering a differential equation and we construct its explicit solutions. We show that the action of the induced scattering gives rise to the rapid formation of narrow spectral distributions which are, as a rule, bimodal. The preferred direction is given by the wind not directly, but through the direction of the subsurface drift flow. The results are in qualitative agreement with data from field observations.

1. INTRODUCTION

The waves which are produced on the surface of a fluid by the wind have been a classical topic in the theory of nonlinear wave processes, starting from the time when the theory originated a century ago. Many ideas and methods from the general theory of nonlinear waves first appeared and were developed in connection with waves on water. The problem of constructing a qualitative and quantitative theory of the angular spectrum of the swell has recently been in the forefront of these problems.

It is known from observational data that wind waves propagate in directions which are close to the wind direction and that they have a very narrow angular spectrum in the energy-carrying range.[1–3] There is, however, a problem: "Why do the waves propagate strictly along the wind and why do they have such a narrow spectrum?" In other words, it took a long time to realize that there was the problem of the mechanisms for the formation of such spectra. It was established only relatively recently that in the energy-carrying range the processes by which the wind acts directly on the waves are negligibly weak compared to the weakly nonlinear wave–wave interactions and, hence, the narrow angular distribution of the waves along the wind must be explained by some other factors.[4]

This problem is undoubtedly also of appreciable independent interest, but for now it has become part of the "bottleneck" of the weak turbulence theory of the swell. The main difficulty of the weak turbulence theory, based upon an analysis of the kinetic equation, is that within its framework it has been possible neither to explain the fact of the existence of a narrow angular spectrum, nor, even worse, to obtain the quantitative characteristics of the angular distributions. On the other hand, it has been found that the intensity of the nonlinear transport of the wave action to low frequencies and large scales and, thereby, the whole course of the temporal evolution to a large extent are determined just by the integral width of the angular spectrum.[3,4] We note that according to the results of the analysis of the kinetic equation the shape of the frequency spectra (or the spectra with respect to the absolute magnitudes of the wavenumbers), corresponding to spectra with a constant action flux (close to the ones realized in the inertial range),[4] depend on the parameters of the angular distribution much more weakly than the flux of the action along the spectrum.

The possibility for the existence of stationary spectra, narrow in angle, in the framework of the four-wave kinetic equation was studied in Ref. 5. It was shown, firstly, that for spectra which are narrow in angle there is no transport of action along the spectrum (to first order in the angular width) and, secondly, that the width of a narrow angular distribution, corresponding to a stationary frequency spectrum, increases exponentially with time. From another point of view (as the problem of the stability of isotropic Kolmogorov spectra) this hypothesis was considered in Ref. 6 where it was shown that such spectra are stable with respect to anisotropic perturbations. These results indicated the necessity to look for other physical mechanisms guaranteeing, on the one hand, the existence of the observed narrow angular distributions and, on the other hand, the negligible effect on the shape of the frequency spectra, which agree well with experiments and which are constructed on the basis of the solutions of the kinetic "four-wave" equation.

The dynamics of a random field of weakly nonlinear surface waves will be considered in the present paper taking into account a new mechanism (in the context of the present problems), namely, induced scattering by subsurface shear flow.

This kind of mechanism is well known in the theory of

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plasma turbulence. It is connected with including into the discussion new degrees of freedom. In typical cases this nonlinear interaction of the waves with the plasma particles is the induced scattering of waves by electrons or ions.[7] The main effect to which such an interaction gives rise is the narrowing of the spectrum in angle, the formation of "jet spectra."[8] In our opinion the scattering of waves by drift flow, a shear flow produced by the wind, plays an analogous role in the theory of wind waves.

The drift flow is the same unavoidable consequence of the action of the wind, as the wind waves themselves, and, generally speaking, posing the problem about the simultaneous evolution of the waves and the flow would be the most correct procedure. However, the difference in the characteristic time scales enables us to restrict ourselves in the first stage to analyzing the problem of the evolution of the waves for a given shear flow. The term induced scattering, as applied to the present problem, means the transformation of the wave field due to the interaction between difference harmonics of the field and critical layers in the drift flow. One of the components of this mechanism, the generation by fixed difference harmonics of the perturbations of a given drift flow with a given very simple model profile, has been studied before in connection with Langmuir circulation.[9] (The peculiar motions of the flow excited by the difference harmonics of surface waves were interpreted as a Langmuir circulation.) We note that a significant difficulty of the induced scattering mechanism was noted in Ref. 10 where, without calculations or estimates, it was stated *a priori* that its effect is negligibly small. We obtain in the present paper a quantitative description of this mechanism and we show that it plays a dominating role in the formation of the angular spectra of the swell. (In a recent paper[11] the formation of the angular spectrum caused by induced scattering of the waves by perturbations of the air flow was studied. We shall discuss in the Conclusion the possible role of this effect.)

The paper is constructed as follows: we give in §2 a statement of the problem; §3 contains the derivation of the kinetic equation which takes into account the induced scattering of the waves by the shear flow; in §4 we use it to study the evolution of spectra which are narrow in angle; and in the Conclusion we give a discussion of the results.

2. STATEMENT OF THE PROBLEM

We consider the evolution of an ensemble of free (i.e., not directly interacting with the wind) gravitational waves on the surface of an ideal deep fluid of unit density on the background of a horizontally uniform flow with a vertical shear velocity. We assume the shear flow U generated by the wind to be given and stationary,[1)] and also to have a single direction[2)] (along the wind) and decrease monotonically with the depth z, i.e., $U=\{U(z),0,0\}$ and $U'<0$ (the x-axis is directed along the wind and the flow, and in the unperturbed state the fluid occupies the lower half-space $z>0$). To describe the motions of the fluid we take as the initial set the Euler equations for the velocity perturbation vector $\mathbf{u}=\{u,v,w\}$ and the pressure P:

$$\begin{gathered} D_t u+wU'+P_x=f^{(1)}=-(\mathbf{u}\nabla)u, \\ D_t v+P_y=f^{(2)}=-(\mathbf{u}\nabla)v, \\ D_t w+P_z=f^{(3)}=-(\mathbf{u}\nabla)w, \\ u_x+v_y+w_z=0, \end{gathered} \tag{2.1}$$

where

$$D_t=\partial_t+U\partial_x,\quad U'=\partial_z U,$$

with the standard boundary conditions

$$\eta_t+((\mathbf{u}+\mathbf{U})\nabla)\eta=w,\quad P=0 \tag{2.2}$$

at the free surface $z=\eta(x,y,t)$ and

$$w=0 \tag{2.3}$$

at the bottom or at infinity.

For simplicity we restrict our discussion to the case of an infinite depth of fluid. We emphasize that there is no description of the air motion in the present formulation of the problem and it enters the problem only indirectly through the magnitude of the drift flow $\mathbf{U}$ and the wave action flux along the spectrum to large scales.

We assume that initially some set of gravitational waves given by the spatial spectrum (for definiteness, of the wave action $N_\mathbf{k}$), is excited at the surface of the fluid. The problem consists, firstly, in deriving an equation to describe the evolution of the swell spectrum (the kinetic equation), taking into account the processes of induced scattering of surface waves by the drift flow perturbations, and, secondly, in using it to study the dynamics of the swell spectrum, in the first place, its angular spectrum.

The geophysical problem contains a number of natural small parameters, the use of which considerably simplifies the investigation. First of all, we define the nonlinearity parameter of the surface waves in the usual way:

$$\varepsilon=u/C,$$

ε characterizes the smallness of the horizontal velocity u of the particles in the wave as compared to the phase velocity C of the surface wave. We also introduce the small parameter μ,

$$\mu=U_{max}/C,$$

which characterizes the smallness of the drift flow compared to the same phase velocity C. (Typical for the ocean are the values $\varepsilon\sim10^{-1}$; $\mu\sim5\times10^{-2}$–10^{-1}.) In our investigation we also use the narrowness of the angular spectrum of the surface swell, characterized by the small parameter δ. We do not, however, fix the order of this small parameter beforehand.

The presence of these natural small parameters makes it possible in principle to construct a consistent asymptotic procedure for deriving the kinetic equation.

3. KINETIC EQUATION

1. The aim of the present section is to derive (in the framework of the usual hypotheses about the statistics of the surface wave field) a kinetic equation for the spectral density $N_\mathbf{k}$ of the wave action of the following structure:

$$\dot{N}_\mathbf{k}=I_1[N_\mathbf{k}]+N_\mathbf{k}\int G_{\mathbf{k}\mathbf{k}_1}N_{\mathbf{k}_1}\,d\mathbf{k}_1=I_1+I_2. \tag{3.1}$$

The first term on the right-hand side (I_1) is the "Boltzmann" collision integral, describing the change in $N_\mathbf{k}$ as the result of resonance four-wave interactions of the spectral components of the surface swell. The actual form of the collision integral I_1, first obtained in Ref. 12, is as yet unimportant for us. We only stress that I_1 is independent of the presence of the flow. Our problem is thus reduced essentially to finding only the form of the second term (I_2) on the right-hand side of Eq. (3.1), which describes the change in $N_{\mathbf{k}}$ due to the induced scattering processes of the waves. We can treat the induced scattering as follows. Taking the shear flow into account gives rise to the appearance of new eigenmodes of fluid motions forming resonance triads with the spectral components of the surface swell.

Let us elaborate on this crucial point. Each pair of spectral components of the surface swell with wavevectors, say, $\mathbf{k}_1$ and $\mathbf{k}_2$, generates, due to the nonlinearity of the Euler equations and the boundary conditions at the free surface difference harmonics, perturbations of the medium with a space-time structure of the form

$$\exp\{i[(\mathbf{k}_1\pm\mathbf{k}_2)\mathbf{x}-(\omega(\mathbf{k}_1)\pm\omega(\mathbf{k}_2))t]\}.$$

For the "difference" harmonics with a positive x-component of the phase velocity c $[c=(\omega_1-\omega_2)/(k_{1x}-k_{2x})$ less than the maximum flow velocity $U_{\max}$ there occurs a critical layer (i.e., a layer where $c=U(z_c)$]. The resonant interaction of difference harmonics with the flow causes an energy exchange between the waves and the flow and gives rise to a transformation of the wave spectrum due to this interaction.

The problem of deriving the kinetic equation thus splits naturally into two stages: the first one is the solution of the purely dynamic problem of calculating the difference harmonics or, in other words, the calculation of the induced low-frequency motions; the second is the calculation of the total effect of the energy exchange between the waves and the flow, caused by the existence of critical layers, on the evolution of the spectrum of the wave field.

2. For convenience, in order to solve the problem we rewrite the initial set (2.1) and the boundary conditions (2.2) in the following equivalent form:

$$D_t\Delta w-U''\partial_x w=\mathcal{F}_{Ra}, \tag{3.2}$$

$$\begin{gathered}\mathcal{F}_{Ra}=\partial_z[\nabla_h(\mathbf{u}\nabla)\mathbf{u}_h]-\Delta_h(\mathbf{u}\nabla)w,\\ (D_t)^2w'-U'\partial_x D_t w+g\Delta_h w=\mathcal{F}_{surf},\\ \mathcal{F}_{surf}=g\Delta_h\{\hat{L}[\mathbf{u}\nabla)\eta]-(\hat{L}-1)w\}\\ +D_t\{\hat{L}[\nabla_h(\mathbf{u}\nabla)\mathbf{u}_h]+(\hat{L}-1)[D_t w'+U'\partial_x w]\},\end{gathered} \tag{3.3}$$

where

$$\mathbf{u}_h=\{u,v\},\quad \nabla_h=\{\mathbf{i}\partial_x,\mathbf{j}\partial_y\},\quad \Delta_h=\nabla_h^2,$$
$$\hat{L}=(1+\eta\partial_z+{}^1/_2\eta^2\partial_{zz}^2+\ldots).$$

Equation (3.2) is closed only to first order, so that for an analysis of the nonlinear terms we must consider it together with the initial set (2.1).

We look for the solution of the initial system (2.1) with the boundary conditions (2.2), (2.3) in the form of an asymptotic series in ε:

$$\mathbf{u}=\varepsilon(\mathbf{u}_{1(0)}+\mu\mathbf{u}_{1(1)}+\mu^2\mathbf{u}_{1(2)}+\ldots)+\varepsilon^2\mathbf{u}_2+\ldots, \tag{3.4}$$

$$\mathbf{u}_{1(0)}=\nabla\varphi_1.$$

One sees easily that to lowest order the rotational and potential components are separated: the motion is a superposition of shear flow and potential wave motion. The wave motion, strictly speaking, is a superposition of Fourier harmonics of the form

$$\varphi_{\mathbf{k}}=A_{\mathbf{k}}e^{-Kz}e^{i\theta}+\text{c.c.}, \tag{3.5}$$

where

$$\theta=\mathbf{kx}-\omega(\mathbf{k})t,\quad K=|\mathbf{k}|,\quad \omega=(gK)^{1/2},$$

and $A_{\mathbf{k}}$ is the complex amplitude of the potential. It is more convenient at this stage to work with the amplitudes $A_{\mathbf{k}}$ and not to change to canonical variables $a_{\mathbf{k}}$. In the normalization used in what follows we have $N_{\mathbf{k}}=|C||A_{\mathbf{k}}|^2$.

However, even to first order in μ the wave motion ceases to be potential. We find the first-order rotational correction $\mathbf{u}_{1(1)}(\mathbf{k})$ to the potential wave with wavevector $\mathbf{k}$ and frequency ω. Substituting the expansion (3.4) into Eq. (3.2) to order $O(\varepsilon\mu)$ we get an equation determining the vertical structure $W_{1(1)}(z)$ of the rotational correction $w_{1(1)}$ to the vertical velocity of the wave: $w_{1(1)}=W_{1(1)}\exp(i\mathbf{kx})$ (we no longer indicate that the quantities refer to the kth harmonic):

$$W''_{1(1)}-K^2W_{1(1)}=\frac{U''}{\omega}k_x\varphi_z=-\frac{U''}{\omega}k_xKAe^{-Kz}. \tag{3.6}$$

One must, generally speaking, require from the solutions of Eq. (3.6) that they satisfy well defined boundary conditions at the bottom and at the free surface, but in the context of the present problem the form of $W_{1(1)}$ is important for us only apart from the solution of the homogeneous equation. We therefore choose for the sake of convenience

$$\tilde{W}_{1(1)}=Ke^{\tilde{z}}\int_z^\infty\tilde{U}'e^{-2\tilde{z}}dz,$$

$$\tilde{z}=zK,\quad \tilde{U}'=k_xU'/\omega,\quad \tilde{W}=W/A. \tag{3.7}$$

We find the vertical structure of the rotational correction to the Fourier component of the horizontal velocity $\mathbf{u}_{1(1)h}(\mathbf{k})$ directly from the linearized Euler equations [taking into account Eq. (3.7) for the vertical velocity]:

$$u_{1(1)h}\approx\frac{ik}{K^2}W'_{1(1)}(z)=\frac{ik}{K}\left[W_{1(1)}-AU'\frac{k_x}{\omega}e^{-Kz}\right]. \tag{3.8}$$

Here and below we use the same notation for the corrections to the horizontal velocity, differing by the factor $\exp(i\mathbf{kx})$. Since the parameter μ is small from the homogeneous Rayleigh equation we can also easily find the linear corrections to higher order [through successive solutions of an inhomogeneous equation of the kind (3.6) where on the right-hand side the corrections to the vertical velocity occur from the preceding approximation]. However, these corrections are important only in dynamic problems which are not considered in the present paper.

3. We find the field of the induced low-frequency motions arising to second order in ε. To do this it is sufficient to calculate the low-frequency motions produced by an arbitrary pair of harmonics with amplitudes A_1 and A_2 $(A_i=A(\mathbf{k}_i,\omega(\mathbf{k}_i)),\ i=1,2)$. We introduce the notation

$$\varkappa=\mathbf{k}_1-\mathbf{k}_2,\quad \sigma=\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)=\omega_1-\omega_2,\quad c=\sigma/\varkappa_x. \tag{3.9}$$

We emphasize that we are interested in the induced low-frequency motion only on those space-time scales for which the existence of critical layers in the given flow is possible. The range of interest to us is distinguished by the obvious condition

$$c \leq U_{max}. \tag{3.10}$$

We note that the contribution to the $\varkappa$-harmonic comes from all pairs $\mathbf{k}_1$, $\mathbf{k}_2$ satisfying the condition $\mathbf{k}_1 - \mathbf{k}_2 = \varkappa$ by virtue of the definition of $\varkappa$. Hence, the total difference field is given by a convolution of the form

$$f_1 \times f_2^* = \int f_1 f_2^* \delta(\mathbf{k}_1 - \mathbf{k}_2 - \varkappa)\, d\mathbf{k}_1.$$

However, for the sake of convenience we shall work with components generated by some selected pair of harmonics.

To find the induced motions $\mathbf{u}_2(\varkappa, z)$ we substitute into (3.2) the solution in the form of a sum of a potential wave and the first-order rotational correction $\mathbf{u}_{1(1)}$. The problem of finding the vertical component W_2 requires, generally speaking, the solution of the Rayleigh equations with a right-hand side (we drop the indices)

$$(U-c)(W''-\varkappa^2 W) - U''W = F(z, \mathbf{k}_1, \mathbf{k}_2, A_1, A_2^*) \tag{3.11}$$

with inhomogeneous boundary conditions at the free surface ($z=0$),

$$-\varkappa_x^2 (U-c)^2 W' + U'\varkappa_x^2 (U-c) W - g\varkappa^2 W = \mathscr{F}_{surf}, \tag{3.12a}$$

where

$$\mathscr{F}_{surf} = iA_1A_2^* \frac{K_1K_2}{\omega_1\omega_2}\left\{(\omega_1 K_1 - \omega_2 K_2) - (\mathbf{k}_1\mathbf{k}_2)\left(\frac{\omega_1}{K_1} - \frac{\omega_2}{K_2}\right)\right\},$$

and zero at infinity ($z = \infty$),

$$W = 0. \tag{3.12b}$$

Here and henceforth $W(z)$ is the vertical structure w_2: $w_2 = W\exp(i\varkappa \mathbf{x})$. We note that in the range of interest to us ($c/U_{max} \lesssim \mu$) the contribution from the first and second terms in (3.12a) is negligibly small compared to that from the third one. Their ratio is $\sim (U^2/gh)\delta^2 \ll 1$ (here h is the characteristic vertical scale of the flow). Thus, the boundary condition at the free surface can be significantly simplified:

$$W_2 = -(\mathscr{F}_{surf}/g\varkappa^2)_{z=0}. \tag{3.12a$'$}$$

We must note that the value of W_2 at the surface, given by (3.12a), is $\lesssim \varepsilon^2\mu\delta^2$, notwithstanding the fact that $\mathscr{F}_{surf}$ in that approximation is independent of the presence of a flow and is given by formula from the potential theory. The additional small factor μ arises because of restrictions such as (3.10) on the frequency of the induced motions: $\sigma = c\varkappa_x \sim \mu\delta^2\omega$.

We reduce the boundary value problem (3.11), (3.12) to a problem with zero boundary conditions by splitting off the "potential" part $\hat{W}$. Splitting W into two components: $\tilde{W}$, $\hat{W}$ ($W = \tilde{W} + \hat{W}$), where

$$\hat{W} = -(\mathscr{F}_{surf}/g\varkappa^2)\, e^{-\varkappa z},$$

we get for $\tilde{W}$ a boundary value problem of the form

$$(U-c)(\tilde{W}'' - \varkappa^2\tilde{W}) - U''\tilde{W} = \frac{i\mathscr{F}_{Ra}}{\varkappa_x} + \frac{U''\mathscr{F}_{surf}}{g\varkappa^2} = F.$$

If we consider a general situation it is necessary to take into account both terms on the right-hand side. Up to this moment we have nowhere used the assumption that the angular spectrum is narrow. Bearing in mind that we have $\delta \ll 1$ or $\varkappa \ll K$, we can neglect the contribution from the "surface" nonlinearity since we have $\mathscr{F}_{surf} \propto \varkappa^4$ and $\mathscr{F}_{Ra} \propto \varkappa^2$, and moreover the component of the solution caused by the surface nonlinearity vanishes to first approximation in $\varkappa$ in the critical layer. Correspondingly, in what follows we identify the field $\tilde{W}$ with W (we omit the tilde). The boundary value problem (3.11), (3.12) can thus finally be written in the form

$$(U-c)(W''-\varkappa^2 W) - U''W = F, \quad F = i\mathscr{F}_{Ra}/\varkappa_x, \tag{3.11$'$}$$

$$W(0) = W(\infty) = 0. \tag{3.12$'$}$$

We emphasize the following important fact: the vertical velocities of the induced motions given by the boundary value problem (3.11), (3.12), although we have $\mathscr{F}_{Ra} \propto \varepsilon^2\mu$, are of order ε^2 rather than $\varepsilon^2\mu$. We evaluate $\mathscr{F}_{Ra}$, restricting ourselves to the first term of the expansion in $\varkappa^2$:

$$\mathscr{F}_{Ra} \approx \frac{\varkappa^2}{2C} A_1A_2^* \chi(K, z), \qquad \chi(K, z) = (U''' - 4KU'' + 8K^2U')\, e^{-2Kz}. \tag{3.13}$$

In the two limiting cases we have

$$\mathscr{F}_{Ra} = \frac{\varkappa^2}{2C} A_1A_2^* U''' \quad \text{when} \quad K \ll \frac{U'}{U}, \tag{3.13a}$$

$$\mathscr{F}_{Ra} = \frac{4\varkappa^2}{C} A_1A_2^* K^2 U' e^{-2Kz} \quad \text{when} \quad K \gg \frac{U'}{U}. \tag{3.13b}$$

Since $\varkappa^2$ is small we can easily obtain a solution of the inhomogeneous Rayleigh equation (3.11) with an arbitrary right-hand side F in the "long-wavelength" approximation as a series in $\varkappa^2$. The main terms of the expansion of the solution of the homogeneous equation have the form[14]

$$W^{(0)} = U - c, \quad W^{(1)} = (U-c)\int_0^z \frac{dx}{(U-c)^2}$$

(we discuss below the correct way to go round the singularity in the integral). Expressions for the next terms in the series were obtained in Ref. 13 and are, for instance, given in Ref. 14. We restrict ourselves, however, to the first terms of the expansion and give the solution of the inhomogeneous equation satisfying the selected boundary conditions (we note that the fundamental solutions $W^{(0)}$, $W^{(1)}$ are normalized so that their Wronskian equals unity):

$$W(z) = -(U-c)\left[\int_0^z R(\zeta)F(\zeta)\,d\zeta + R(z)\int_z^\infty F(\zeta)\,d\zeta\right], \tag{3.14}$$

where

$$R(z) = \int_0^z \frac{dx}{(U-c)^2}.$$

For our purposes it turns out to be sufficient to use the long-wavelength asymptotic forms of the Rayleigh equation which do not satisfy the boundary conditions at infinity. Equally suitable solutions which satisfy both boundary conditions (3.12$'$) can easily be obtained by the method of matched asymptotic expansions [the solution (3.14) is matched with an exponent of the form $\exp(-|\varkappa|z)$].

The horizontal components of the velocity $\mathbf{u}_2$ can be expressed in terms of w_2 and the right-hand sides of the appropriate Euler equations. We give the expression for the x-component of the horizontal velocity $\mathbf{u}_2$ used below (we omit the indices):

$$u=\frac{i}{\varkappa^2}\left[\varkappa_x W'+\frac{\varkappa_y^2}{\varkappa_x}\frac{U'W}{U-c}+\frac{f}{\varkappa_x(U-c)}\right], \tag{3.15}$$

where

$$f=(\varkappa_x f_{2(1)}^{(2)}-\varkappa_y f_{2(1)}^{(1)})\varkappa_y \approx o(\varkappa_x\varkappa_y^2 A_1 A_2^*),$$

while W is given by Eq. (3.14).

We note two different features of the velocity field $\mathbf{u}_2$ which is given by Eqs. (3.14), (3.15). Firstly, the velocity field $\mathbf{u}_2$ is singular in the critical layer $z=z_c$ ($U(z_c)=c$). In its vicinity

$$W\approx W_c+g_1(z_c)(z-z_c)\ln(z-z_c), \tag{3.16}$$

$$u\approx g_2(z_c)/(z-z_c),$$

where

$$W_c=\frac{1}{U'}\int_{z_c}^{\infty}F\,dz,\quad g_1=\left[\frac{F(z)}{U'}-\frac{U''}{2U'^2}\int_z^{\infty}F\,d\zeta\right]_{z=z_c},$$

$$g_2=\frac{iW_c}{\varkappa_x}.$$

The ambiguity of the solutions which is connected with the singularity $z=z_c$ is removed by giving a "rule for going around the singularity" which will be discussed somewhat later. Secondly, the velocity field has a singularity also in k-space. As we let $\varkappa_x\to 0$ the amplitude of the perturbations tends to infinity ($W\propto\varkappa_x^{-1}$). This is connected with the resonance with the eigenmode of the shear motion which is often identified with Langmuir circulation.[9]

To remove the singularity of the $\mathbf{u}_2(\varkappa)$ field in the limit $\varkappa_x\to 0$ we must go beyond the framework of our original assumptions. Among the factors limiting $\mathbf{u}_2$ we must, apparently, first distinguish the viscosity. Taking the viscosity into account the Rayleigh equation (3.11′) for W_2 changes to the well known Orr-Sommerfeld equation which in the long-wavelength approximation in which we are interested has the form

$$(U-c)W''-U''W=\frac{i\nu}{\varkappa_x}W^{\mathrm{IV}}+\frac{i\mathcal{F}_{Ra}}{\varkappa_x}.$$

One sees easily that the effect of even a small viscosity always dominates for sufficiently small $\varkappa_x$ in the vicinity of the critical layer where the behavior of the solutions of the Orr–Sommerfeld equation has been studied in quite some detail.[14] An analysis of the complex structure of the field in the vicinity of the critical layer goes beyond the framework of the present paper. For us it is important only that there exists a viscous cut-off scale $\varkappa_x^*$. As a rough upper estimate for $\varkappa_x^*$ we can take

$$\varkappa_x^*\sim h^{-1}(Re)^{-1}=(hU(0)h/\nu)^{-1},$$

where Re is the Reynolds number and h the characteristic vertical scale for changes in U. For typical parameters of the upper layer of the ocean ($\nu\sim 10^{-2}\,\mathrm{m}^2/\mathrm{s}$), $U(0)\sim 1$ m/s] we have $Re\sim 10^3$–10^4, $\varkappa_x^*\sim 10^{-4}$–$10^{-6}\,\mathrm{m}^{-1}$. We shall in what follows take the effect of viscosity into account only for determining the cut-off scale $\varkappa_x^*$. We note that the problem of the primary mechanism for the cut-off of induced low-frequency motions remains to a large extent open and requires a special study. We do not exclude the possibility that other physical mechanisms (time-dependence, Rayleigh viscosity, and so on) may be more efficient for limiting the long-wavelength divergence. However, in the context of our problem it is only important that there exists a cut-off scale $\varkappa_x^*$ which is small compared to $\varkappa$. Differences in the nature of the cut-off only affect the numerical value of the coefficient.

4. The usual mathematical procedure for deriving kinetic equations assumes as a first step the use of some asymptotic procedure to obtain a dynamic integro-differential equation for the canonical field variables $a_{\mathbf{k}}$. For a medium with forbidden three-wave processes we have[15]

$$\dot a_{\mathbf{k}}=\int T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}a_{\mathbf{k}_1}^*a_{\mathbf{k}_2}a_{\mathbf{k}_3}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3.$$

We emphasize that due to the singular nature of the perturbation introduced by the flow the assumption that the shear flow is small (of order μ) does not enable us to write the interaction coefficient as a series in μ,

$$T=T_{(0)}+\mu T_{(1)}+\dots,$$

where $T_{(0)}$ is the coefficient when there is no flow.

Obtaining the complete expression for T or, equivalently, obtaining the dynamic equation for $a_{\mathbf{k}}$ to order $O(\varepsilon^3)$ requires, in particular, the calculation of the combinational harmonics to next order (in ε) compared to the ones we have calculated and not only the "difference," but also the "sum" combinational harmonics. To realize the aims of the present paper, these involved calculations can be avoided if we take into account that for the kinetic description of the field the effects of the symmetric and of the antisymmetric parts of T are significantly different. The antisymmetric part of T has the meaning of a nonlinear damping/growth rate and it shows up in the kinetic equation in lowest order (quadratic in the wave action). This enables us to neglect the symmetric component of T.

The special feature of the present paper which enables us in calculating the antisymmetric part of T to restrict ourselves to the main terms in the series for the difference harmonics is connected with the fact that, since the flow velocity is much smaller than the characteristic phase velocities of the surface waves, the critical layer (in terms of which this kind of interaction occurs) is realized only for a narrow range of difference harmonics (to be more precise, for a fraction of order μ of the phase volume). This last important fact gives rise, on the one hand, to the fact that the contribution to the kinetic equation from the antisymmetric part of T, which corresponds to the induced scattering process and which does not contain the small parameter μ, is found to be of order μ; on the other hand, notwithstanding that the symmetric part of T differs considerably from $T_{(0)}$ in some part of phase space, in kinetics these differences show up only in the next order and hence the term describing the usual resonance four-wave processes remains the same as when there is no flow. Strictly speaking, the kinetic equation describing the induced scattering process can most simply be obtained on the basis of the expressions for $\mathbf{u}_2$ which we have already found starting from energy considerations, which are often applied in similar situations in plasma physics.[7]

We consider the energy exchange between the set of surface waves and the average flow. Separating in the standard way the stationary flow $U(z)$ and the fluctuating (wave) component we get a well known relation which couples the change in the energy of the flow E_{flow} and of the waves E_{waves} (per unit area of a horizontal surface element) to the xz-component of the Reynolds stress tensor τ:

$$\dot{E}_{\text{flow}} = -\int_0^\infty \tau U'\, dz = -\dot{E}_{\text{waves}}. \tag{3.17}$$

According to the definition we have

$$\tau = -\langle uw \rangle,$$

where the angle brackets indicate ensemble averaging, which by virtue of the ergodic hypothesis can be replaced by integration over x-space. In turn, we replace the integration over x-space by integration over $\mathbf{k}$-space. For the present case of horizontally uniform flow the nonzero contribution to τ, as we have already stressed, can give only difference harmonics. The increase in the flow energy due to the elementary interaction process of the flow with some selected pair of spectral components of the surface swell with wavenumbers $\mathbf{k}_1$ and $\mathbf{k}_2$ is connected with the redistribution of the wave action $N_{\mathbf{k}}$ among the harmonics:

$$-\delta E_{\text{flow}}^{(1,2)} = \delta E_{\text{waves}}^{(1,2)} = \omega_1 \delta N_{\mathbf{k}_1} + \omega_2 \delta N_{\mathbf{k}_2} = (\omega_1 - \omega_2)\delta N_{\mathbf{k}_1}. \tag{3.18}$$

Conservation of the total action ($\delta N_{\mathbf{k}_1} = -\delta N_{\mathbf{k}_2}$) follows directly from the conservation of the transverse component $\varkappa(\varkappa = \mathbf{k}_1 - \mathbf{k}_2)$ of the momentum $\mathbf{p}^{(1,2)}$ ($\mathbf{p}^{(1,2)} = N_{\mathbf{k}_1}\mathbf{k}_1 + N_{\mathbf{k}_2}\mathbf{k}_2$) in an interaction with a difference harmonic with wavevector $\varkappa$:

$$[\delta \mathbf{p}^{(1,2)} \varkappa] = 0, \quad [(\mathbf{k}_1 \delta N_{\mathbf{k}_1} + \mathbf{k}_2 \delta N_{\mathbf{k}_2}),$$

$$(\mathbf{k}_1 - \mathbf{k}_2)] = [\mathbf{k}_1 \mathbf{k}_2](\delta N_{\mathbf{k}_1} + \delta N_{\mathbf{k}_2}) = 0.$$

Combining Eqs. (3.17) and (3.18) for the energy increase we obtain an equation for the change in the action $N_{\mathbf{k}_1}$ of the spectral component with wavevector $\mathbf{k}_1$:

$$\dot{N}_{\mathbf{k}_1} = \frac{1}{\omega_1 - \omega_2}\int_0^\infty \tau(\mathbf{k}_1, \mathbf{k}_2) U'(z)\, dz = -\frac{1}{\omega_1 - \omega_2}\int_0^\infty \langle u_\varkappa w_\varkappa \rangle U'\, dz. \tag{3.19}$$

Here $\tau(\mathbf{k}_1, \mathbf{k}_2)$ is the component of the Reynolds wave stresses due to the pair of harmonics $\mathbf{k}_1$ and $\mathbf{k}_2$ while we must take $u_\varkappa$ and $w_\varkappa$ in the sense discussed in §3.3.

Since to lowest order the wave action in the flow is additive, the required kinetic equation can be obtained from (3.19) by integration over $\mathbf{k}_2$. The problem has thus been reduced to evaluating the explicit form of $\langle u_\varkappa w_\varkappa \rangle$.

The main term of τ has the form

$$\tau = -2[u_2 w_2^*], \tag{3.20}$$

where $u_2(z)$ and $w_2(z)$ are given by Eqs. (3.14) and (3.15).

When there are no critical layers the components of the vertical and the horizontal velocities are shifted in each harmonic over $\frac{1}{2}\pi$ in phase and for such difference harmonics τ vanishes identically. To find the nonzero value of τ caused by the singularity of the critical layer, it is most important to select the correct branch of the many-valued solutions of (3.14) and (3.15) (choice of the rule for going around the singularity). The starting point for the choice of the rule for going around the singularity is the following indication. The wave motion must be a solution of a Cauchy problem, i.e., must "start" at some time. One must therefore consider only motions which grow exponentially with time (letting this growth rate tend to zero in the answer). Thus, for a wave of the form

$$\exp(-i\omega_R t + i\varkappa_x x + \omega_i t) \sim \exp(i\theta), \qquad \omega = \omega_R + i\omega_i$$

the imaginary correction to the frequency must be positive. The requirement $\omega_i > 0$ can be reformulated in terms of c_i:

$$c_i \operatorname{sign} \varkappa_x > 0.$$

The exponential growth with time must occur also for the conjugate solution which is proportional to $\exp(-i\theta)$, from which it follows that we have $\omega_i^* = -\omega_i$, $c_i = -c_i^*$. Determining in this way the rule for going around the singularity we can write down the main "working" formula:

$$\frac{1}{U-c} = \mathscr{P}\frac{1}{U-c} + i\pi\delta(U-c)\operatorname{sign}\varkappa_x = \mathscr{P}\frac{1}{U-c} + \frac{i\pi}{|U'|}\delta(z-z_c)\operatorname{sign}\varkappa_x, \tag{3.21}$$

$$\frac{1}{U-c^*} = \mathscr{P}\frac{1}{U-c} - \frac{i\pi}{|U'|}\delta(z-z_c)\operatorname{sign}\varkappa_x, \tag{3.22}$$

where $\mathscr{P}$ is the principal value symbol and U_c' the value of the first derivative of $U(z)$ at the point $z = z_c$.

Using (3.22) we rewrite Eq. (3.14) for $w_2(z,\varkappa,K)$, separating the $\varkappa$ from the K and z-dependence and splitting off the function $\mathscr{W}$. The latter is the solution of the form (3.14) for $F(\zeta) = \chi(z,K)$, where $\chi(z,K)$ is given by Eq. (3.13). We note that for the evaluation of τ we need know only the value $\mathscr{W}_c$ of $\mathscr{W}(z)$ in the critical layer which is given by the relatively simple expression:

$$\mathscr{W}_c = \frac{1}{U_c'}\int_{z_c}^\infty \chi(z,K)\, dz = \frac{\psi_c}{U_c'}. \tag{3.23}$$

We turn to a direct calculation of τ following the definition (3.20) and using Eqs. (3.13) to (3.15) for u_2 and w_2 and taking into account the auxiliary Eqs. (3.21) to (3.23). One sees easily that the main contribution[3)] comes from the terms caused by the product of the second term in Eq. (3.15) for u_2 and w_2 given by (3.14):

$$\langle uw \rangle = \left(\frac{\varkappa^2}{\varkappa_x}\right)^2 \frac{1}{\varkappa_x}\left(\frac{1}{2C}\right)^2 |A_1|^2 |A_2|^2 \times \left\langle \frac{iU'}{U-c}\mathscr{W}\mathscr{W}^* - \frac{iU'}{U-c^*}\mathscr{W}^*\mathscr{W} \right\rangle$$

$$= -\frac{2\pi}{\varkappa_x}\left(\frac{\varkappa^2}{\varkappa_x}\right)^2\left(\frac{1}{2C}\right)^2 |A_1|^2 |A_2|^2 \frac{U'}{|U_c'|}\operatorname{sign}\varkappa_x \cdot \delta(z-z_c)|\mathscr{W}_c|^2.$$

Using the normalization $|A_i|^2 = |C|N_i$, we have

$$\langle uw\rangle = \frac{\pi}{2}\frac{\varkappa^4}{\varkappa_x^3}N_1N_2\,\mathrm{sign}\,\varkappa_x\cdot\delta(z-z_c)\,\frac{U'}{|U_c'|}\frac{\psi_c^2}{U_c'^2},$$

where

$$\psi_c = \int_{z_c}^{\infty}\chi(z,K)\,dz,$$

whence we finally get the required expression for τ:

$$\tau = -\langle uw\rangle = -\frac{\pi}{2}\frac{\varkappa^4}{\varkappa_x^3}N_1N_2\,\mathrm{sign}\,\varkappa_x\cdot\delta(z-z_c)\,\frac{U'}{|U_c'|^3}\psi_c^2. \tag{3.24}$$

Thus, Eq. (3.19) for $N_{\mathbf{k}}$ takes, if we use (3.24), the form

$$\dot{N}_{\mathbf{k}_1} = -\frac{\int_0^\infty U'\langle uw\rangle dz}{\omega_1-\omega_2}$$

$$= \frac{\pi}{2}\left(\frac{\varkappa}{\varkappa_x}\right)^4 N_{\mathbf{k}_1}N_{\mathbf{k}_2}\frac{\psi_c^2}{|U_c'|}\frac{\mathrm{sign}\,\varkappa_x}{U_c} \equiv G_{\mathbf{k}_1\mathbf{k}_2}N_{\mathbf{k}_1}N_{\mathbf{k}_2}. \tag{3.25}$$

Using the definition (3.25) of $G_{\mathbf{k}_1\mathbf{k}_2}$ and (3.13) we get an explicit expression for $G_{\mathbf{k}_1\mathbf{k}_2}$:

$$G_{\mathbf{k}_1\mathbf{k}_2} = \frac{\pi}{2U_c}\left(\frac{\varkappa}{\varkappa_x}\right)^4 N_{\mathbf{k}_1}N_{\mathbf{k}_2}\,\mathrm{sign}\,\varkappa_x\frac{1}{|U_c'|}$$

$$\times\left[\int_{z_c}^{\infty}(U'''-4KU''+8K^2U')e^{-2Kz}\,dz\right]^2. \tag{3.25a}$$

The interaction coefficient $G_{\mathbf{k}_1\mathbf{k}_2}$ contains a strong nonintegrable singularity as $\varkappa_x \to 0$ due to the long-wavelength divergence discussed in §3.3. Remaining within the framework of the nonviscous theory we introduced, parametrizing the effect of viscosity, a cut-off function in the shape of the Heaviside function $H(\varkappa_x)$:

$$\tilde{G}_{\mathbf{k}_1\mathbf{k}_2} \equiv H(|\varkappa_x|-|\varkappa_x^*|)G_{\mathbf{k}_1\mathbf{k}_2} \equiv \begin{cases} G_{\mathbf{k}_1\mathbf{k}_2}, & |\varkappa_x| \geqslant |\varkappa_x^*|, \\ 0, & |\varkappa_x| \leqslant |\varkappa_x^*|, \end{cases} \tag{3.25b}$$

where, for definiteness, $|\varkappa_x^*|$ is given by the estimate

$$|\varkappa_x^*| \sim h^{-1}(Re)^{-1}. \tag{3.26}$$

We note that the depth of the critical layer $z_c(\mathbf{k}_1,\mathbf{k}_2)$ is a symmetric function of $\mathbf{k}_1$ and $\mathbf{k}_2$ for any profile $U(z)$ while the kernel $G_{\mathbf{k}_1\mathbf{k}_2}$ is antisymmetric ($G_{\mathbf{k}_1\mathbf{k}_2} = -G_{\mathbf{k}_2\mathbf{k}_1}$).

Integrating over $\mathbf{k}_2$ and using the resonance condition

$$\sigma = \varkappa_x U_c \quad (\sigma = \omega_1-\omega_2)$$

gives us the required kinetic equation (here and below we drop the "tilde")

$$\dot{N}_{\mathbf{k}_1} = N_{\mathbf{k}_1}\int G_{\mathbf{k}_1\mathbf{k}_2}\delta(\sigma-\varkappa_x U_c)N_{\mathbf{k}_2}d\mathbf{k}_2 \tag{3.27}$$

for the description of the induced scattering processes.

5. For well defined ratios of the small parameters occurring in the problem (we shall determine these ratios below) a contribution to the evolution of the wave field may be given not only by the induced scattering processes but also by resonance wave interactions, which are normal for a medium with a nondecay spectrum, of the form

$$\omega+\omega_1 = \omega_2+\omega_3, \tag{3.28a}$$

$$\mathbf{k}+\mathbf{k}_1 = \mathbf{k}_2+\mathbf{k}_3. \tag{3.28b}$$

Taking these two kinds of nonlinear interactions simultaneously into account we are led to the following kinetic equation:

$$\dot{N}_{\mathbf{k}_1} = 2\pi\int |T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,\delta(\omega+\omega_1-\omega_2-\omega_3)$$

$$\cdot[N_{\mathbf{k}_2}N_{\mathbf{k}_3}(N_{\mathbf{k}}+N_{\mathbf{k}_1})-N_{\mathbf{k}}N_{\mathbf{k}_1}(N_{\mathbf{k}_2}+N_{\mathbf{k}_3})]\,d\mathbf{k}\,d\mathbf{k}_2\,d\mathbf{k}_3$$

$$+N_{\mathbf{k}_1}\int G_{\mathbf{k}_1\mathbf{k}_2}N_{\mathbf{k}_2}\delta(\sigma-\varkappa_x U_c)\,d\mathbf{k}_2. \tag{3.29}$$

We do not give the complete expression for $T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ since it is cumbersome and since the second term is obtained in the small angle approximation; in what follows we only use an expansion of T in small angles, obtained with the necessary degree of accuracy in Ref. 5.

From the general properties of the kinetic equations (3.27) and (3.29) we note the conservation of total wave action within the framework of (3.29)

$$\partial_t\int N_{\mathbf{k}}\,d\mathbf{k} = 0. \tag{3.30}$$

As to Eq. (3.27) we note that such a kind of equation often arises in plasma theory in wave-particle interaction problems and their properties have been rather well studied.[7,8]

4. EVOLUTION OF THE ANGULAR SPECTRUM OF THE SWELL

1. Our final aim is a study of the evolution of the spectrum of the wave field in the framework of the kinetic equation (3.29), i.e., taking simultaneously the two kinds of nonlinear interactions into account. In the present section we concentrate on one aspect of the evolution of the field, the transformation of its angular spectrum.

We start with an analysis within the framework of Eq. (3.27). We consider the consequences of the resonance condition $\sigma = \varkappa_x U_c$. Expanding the first term of the dispersion relation $\omega_i = (gK_i)^{1/2}$ we find

$$\sigma = \omega_1-\omega_2 \approx \frac{1}{2}C\left[\varkappa_x+\frac{1}{2K}(k_{1y}^2-k_{2y}^2)\right] = \varkappa_x U_c.$$

We rewrite this equation in a clearer form:

$$\varkappa_x+\frac{1}{2K}(k_{1y}^2-k_{2y}^2) = 2\varkappa_x\frac{U_c}{C}. \tag{4.1}$$

Two important conclusions follow immediately from the form (4.1): the first one is the resonance surface $\mathbf{k}_1(\mathbf{k}_2)$ close to the circle $|\mathbf{k}| = \mathrm{const}$ and, hence,

$$\varkappa_x = \frac{1}{2K}\frac{(k_{2y}^2-k_{1y}^2)}{1-2U_c/C} \approx \frac{1}{2K}(k_{2y}^2-k_{1y}^2)(1+O(\mu)), \tag{4.1'}$$

and the second the difference in the frequencies of the interacting pair of waves which has the same sign as $\varkappa_x$ (see Figure 1):

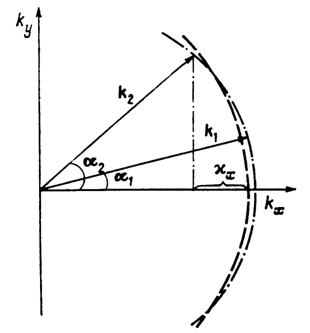

FIG. 1. Sketch of the resonance surface (dot-dash curve) given by the resonance condition (4.1). The isofrequency surface (dashed curve) close to the resonance surface is the circle $\omega = (gK)^{1/2}$.

$$\operatorname{sign}\sigma=\operatorname{sign}\varkappa_x=\operatorname{sign}(k_{2y}^2-k_{1y}^2). \tag{4.1''}$$

The induced scattering process thus primarily transforms, the angular spectrum of the swell. If we consider the elementary interaction of a pair of waves, we see easily from (3.27) and (4.1′), (4.1″) that if the kernel $G_{\mathbf{k}_1\mathbf{k}_2}\operatorname{sign}\varkappa_x$ is positive the wave with the lower frequency will transfer action to the wave with the higher frequency, where by virtue of (4.1″) action is transferred from the wave with the larger transverse wavenumber to the wave with the smaller transverse wavenumber. If $G_{\mathbf{k}_1\mathbf{k}_2}$ sign $\varkappa_x$ retains its positive sign for any pair [which occurs in our problem independently of the form of $U(z)$] we can conclude that the angular spectrum is narrowed. We note, however, that for narrowing (broadening) of the angular spectrum the positivity (negativity) of the quantity $G_{\mathbf{k}_1\mathbf{k}_2}\sin\varkappa_x$ is sufficient only in some average sense which we shall determine somewhat further on. [The latter case has a meaning only for nonmonotonic functions $U(z)$ which are not considered in the present paper.]

It is convenient for a description of the dynamics of the angular spectrum to change to polar coordinates K, α

$$\sin\alpha=k_y/K\approx\alpha. \tag{4.2}$$

The presence of a δ-function in the collision integral and the simple form of the resonance surface enable us to simplify the kinetic equation significantly, carrying out a single integration over one of the components of $\mathbf{k}_2$. Since for fixed $\mathbf{k}_1$, $\mathbf{k}_2$ we simultaneously give the position z_c of the critical layer it is advisable to replace the integration over k_{2x} by an integration over z_c. Using (4.1′) and (4.2) we have

$$dk_{2x}=-d\varkappa_x=\frac{2}{K}\,|k_{1y}^2-k_{2y}^2|\,\frac{U_c'}{C}\,dz_c$$
$$=\frac{2K}{C}\,|\alpha_1^2-\alpha_2^2|\,U_c'\,dz_c,\qquad dk_{2y}=Kd\alpha_2.$$

As a result the kinetic equation (3.27) is transformed into an equation for the angular spectrum

$$\dot N_{\alpha_1}=8\pi\frac{K^2}{C}DN_{\alpha_1}\int\limits_{-\pi/2}^{\pi/2}H\frac{\alpha_2-\alpha_1}{(\alpha_2+\alpha_1)^3}N_{\alpha_2}\,d\alpha_2, \tag{4.3}$$

where $N_{\alpha i}=N(K,\alpha_i)$ and H is the regularizing function defined by (3.25b), while all characteristics of the flow are accumulated in the integral interaction coefficient D:

$$D=-\int\limits_0^\infty dz_c\frac{U_c'}{|U_c'|}\frac{\psi_c^2}{U_c}$$
$$=-\int\limits_0^\infty dz_c\left\{\frac{U_c'}{|U_c'|}\frac{1}{U_c}\left[\int\limits_{z_c}^\infty (U'''-4KU''+8K^2U')\,e^{-2Kz}\,dz\right]^2\right\}. \tag{4.4a}$$

We introduce the notation

$$D_1=8\pi K^2D/C. \tag{4.4b}$$

The sign of D_1 determines the qualitative nature of the evolution of the angular spectrum: for $D_1>0$ the spectrum has a tendency to narrow, and for $D_1<0$ to broaden. We emphasize that the coefficient D which is a functional of $U(z)$ is always positive [for any monotonic profile $U(z)$]. The sign of D_1 and thereby the qualitative nature of the wave evolution is uniquely determined by the orientation of the waves with respect to the flow: for waves propagating along the flow ($C>0$) the angular spectrum narrows and for waves propagating against the flow ($C<0$, $D_1<0$) the spectrum broadens.

We summarize the preliminary analysis of the dynamics of the angular spectrum: we have reduced the problem of a quantitative description of the evolution of the angular spectrum to an analysis of the relatively simple Eq. (4.3); on the other hand, the problem of a qualitative description was reduced to solely determining the sign of the interaction coefficient $D_1(K)$ given by (4.4a,b).

2. For a quantitative description of the evolution of the angular spectrum it is more convenient to use the normalized angular dependence $n(\alpha)$:

$$n(\alpha)=N(K,\alpha)/N_\Sigma(K),\quad N_\Sigma=\int N(K,\alpha)\,d\alpha.$$

Since $\dot N_\Sigma$ vanishes in the framework of (4.3) we find

$$\dot n(\alpha)=\hat D(K)n(\alpha)\int\limits_{-\pi/2}^{\pi/2}H(|\varkappa_x(\xi)|-|\varkappa_x^*|)\,\frac{\alpha-\xi}{(\xi+\alpha)^3}\,n(\xi)\,d\xi$$
$$=\hat D(K)n(\alpha)\int\frac{2\alpha-\beta}{\beta^3}\,n(-\alpha+\beta)\,\tilde H(|\beta|-|\beta^*|)\,d\beta, \tag{4.5}$$

where

$$\hat D=D_1N_\Sigma/C,\quad \beta=\alpha+\xi,$$

and $\varkappa^*$ is the cut-off scale.

The quantitative characteristics of the evolution of the angular spectrum depend strongly on the regularizing function $\tilde H$ or, more precisely, on the cut-off scale $\varkappa_x^*$. Using (4.1′) and (4.2), we write the Heaviside function (3.25b) in terms of α:

$$\tilde H=\begin{cases}1, & |(\alpha_1-\alpha_2)(\alpha_1+\alpha_2)|\geqslant 2(\varkappa_x^*/K),\\ 0, & |(\alpha_1-\alpha_2)(\alpha_1+\alpha_2)|<2(\varkappa_x^*/K),\end{cases}$$

whence in the small β region ($\beta=\alpha_1+\alpha_2$) which is of interest to us we get

$$|\beta^*|=|\varkappa_x^*/K\alpha|. \tag{4.6}$$

Using (4.6) and retaining in the integro-differential Eq. (4.5) the main terms in $(\varkappa_x^*/K)^{-1}$ we get the differential equation

$$\dot n(\alpha)-\hat D\left(\frac{K}{\varkappa_x^*}\right)[|\alpha|n(\alpha)n(-\alpha)-2\alpha^2 n(\alpha)n'(-\alpha)]=0. \quad (4.7)$$

We note that in connection with the approximate nature of the transition from (4.5) to (4.7) the peripheral part of the solution may be significantly distorted which, in particular, gives rise to a violation of the normalization condition $\int n d\alpha = 1$ for solutions of (4.7). We introduce the dimensionless time

$$\tilde t = t g^{1/2}/DK^{5/2}N_\Sigma(K/\varkappa_x^*)$$

and restrict the discussion to the evolution of symmetric angular distributions $(n(\alpha) = n(-\alpha))$, whereupon in the $\alpha > 0$ region Eq. (4.7) takes the form

$$\dot n = \alpha n^2 - 2\alpha n n', \quad n = n(\alpha, \tilde t), \quad n' = \partial_\alpha n. \quad (4.8)$$

Equation (4.8) is a hyperbolic first-order equation and can easily be integrated using the method of characteristics. It is more convenient to write the solution of the set of two ordinary differential equations which is equivalent to (4.8):

$$\dot n = \alpha n^2, \quad \dot\alpha = 2\alpha^2 n, \quad (4.9)$$

with initial conditions for $\tilde t = 0$

$$n = n_0(\alpha_0, 0), \quad \alpha = \alpha_0 \quad (4.10)$$

in the variables

$$\tilde\alpha = \alpha/\alpha_0, \quad \tilde n = n(\alpha, \tilde t)/n_0(\alpha_0, 0), \quad \tau = n_0\alpha_0\tilde t.$$

The initial conditions on the different characteristics (4.10) and the solution of Eqs. (4.9) corresponding to them take on a universal form:

$$\tilde n|_{\tau=0} = 1, \quad \tilde\alpha = 1|_{\tau=0}, \quad (4.11)$$

$$\tilde n = \frac{1}{1-\tilde\alpha\tau} \quad \tau = \frac{\ln\tilde\alpha}{\tilde\alpha}. \quad (4.12)$$

The region $\tau > 0$ and $\tilde\alpha > 1$ corresponds to solutions describing the narrowing of the angular spectrum for waves moving along the flow.[4)] In that case n increases along all characteristics with the characteristic dimensionless time which is $\propto (n_0\alpha_0)^{-1}$.

Thus, on the "zero" characteristic ($\alpha_0 = 0$) the initial value n_0 is conserved [in the framework of (4.8)]. The maximum value of n at each time τ is realized on the characteristic α_{0m} corresponding to the maximum value of $n_0\alpha_0$. This means, in particular, that smooth single-humped initial distributions $n(\alpha_0)$ evolve for $t > 0$ into narrow two-hump distributions. At time $\tau_* = e^{-1}$, n becomes infinite in the point $\tilde\alpha_* = e$ [in the variables $\tilde t$, α: $\tilde t_* = (en_0\alpha_{0m})^{-1}$, $\alpha_* = \alpha_{0m}e$] and Eq. (4.8) ceases to be applicable.

3. An analysis of the evolution of the wave field in the framework of the kinetic equation (4.5), neglecting the usual four-wave interactions, is justified for a well defined class of initial conditions, for instance, when the initial nonlinearity of the waves is sufficiently weak. However, it follows from (4.12) that during the evolution (for $C > 0$) there is a concentration of wave action in **k**-space and, hence, at sufficiently long times it becomes as a matter of principle necessary to take the "Boltzmann" collision integral into account. A detailed study of the dynamics of the field on the basis of the "complete" kinetic equation (3.29) goes beyond the framework of the present paper. Here we restrict ourselves to giving the equation for describing the evolution of a narrow spectrum. We use the results of Ref. 5 where expansions of the "Boltzmann" integral for small angles were obtained and analyzed. The fact that to a first approximation $\dot N_\Sigma$ vanishes enables us to write the "complete" Eq. (3.29) in terms of the normalized angular spectrum $n(\alpha)$:

$$\dot n = \hat D(K/\varkappa_x^*)[|\alpha|n(\alpha)n(-\alpha)-2\alpha^2 n(\alpha)n'(-\alpha)] + D_N\alpha^2(-\ln\alpha^2)n^3(\alpha),$$
$$D_N = 16\pi^2 g^2 K^7 N_\Sigma^3.$$

The equation for the equilibrium form of the angular spectrum $n(\alpha)$ follows immediately from the condition $\dot n = 0$:

$$S[n(-\alpha)-2|\alpha|n'(-\alpha)]-|\alpha|\ln(\alpha^2)n^2(\alpha)=0,$$
$$S = \hat D(K/\varkappa_x^*)/D_N. \quad (4.13)$$

A rough estimate of the equilibrium width of the angular spectrum $n(\alpha)$ on the basis of (4.13) for typical values of the parameters enables us to reach important conclusions. Firstly, the characteristic equilibrium width of $n(\alpha)$ turns out to be significantly smaller than the observed one.[3] Thus, the mechanism studied guarantees a rather efficient narrowing of the angular spectrum which not only compensates for the opposite action of the normal four-wave interactions, but also for the effect of factors which are not taken into account in this paper.

Indeed, we take as the characteristic time for the transformation of the angular spectrum due to induced scattering the time T_* for the appearance of the first singularity in the framework of (4.8):

$$T_*^{-1} = eDK^{5/2}N_\Sigma(K/\varkappa_x^*) \sim O(\mu\varepsilon^2(K/\varkappa_x^*)). \quad (4.14)$$

The largest uncertainty in the estimate (4.14) comes from the large parameter $K/\varkappa_x^*$. Even considerably reducing it by an order of magnitude and putting it of order unity, we get a characteristic time which is $\propto (\mu\varepsilon^2)^{-1}$ which is less than or of order ε^{-4}, the times for the usual four-wave interactions. A rough estimate of this large parameter $(K/\varkappa_x^* \sim Re)$ based upon the idea of the viscous nature of the cut-off $\varkappa_x$ and about the turbulent viscosity in the upper layer gives a characteristic time for the formation of the spectrum on the order of minutes. The problem of a more precise estimate of $\varkappa_x^*$ becomes in this context extremely difficult and requires the analysis of other physical mechanisms which goes beyond the framework of the present paper.

In the light of these remarks the problem remains open of the mechanism for the narrowing of the angular spectra and of other physical mechanisms (differing from resonance interactions) which compensate for the tendency to narrow.

5. CONCLUSION

The main results of this paper can be summarized as follows. We have shown that, firstly, the mechanism of induced scattering of wind waves by the subsurface drift flow is undoubtedly an important factor in the evolution of the wave field and must always be taken into account for a kinetic description of the ocean swell. The contribution from the induced scattering to the collision integral of the kinetic equation is proportional to the square of the wave action, $N_{\mathbf{k}}^2$, to the small parameter μ (more precisely to U'/ω), and also to the large parameter $K/\varkappa_x^*$ which for typical conditions of the ocean is much larger than the contribution from the resonant four-wave interactions of the gravitational waves. Secondly, the effect of this mechanism gives rise to the formation of narrow angular spectra without practically affecting the shape of the frequency spectra which are determined by the usual four-wave interactions. The main direction of propagation of the waves is thus determined by the direction of the drift flow and not directly by the wind. Two-hump distributions that qualitatively agree with the results of field observations[1,3] turn out to be typical in the framework of our model. The characteristic time T for the formation of the angular spectrum increases as a power law with increasing wavenumber. For typical ocean condition T lies in the range of minutes.

Let us discuss the problem of the possible role of other physical mechanisms in the formation of the angular spectrum. Induced scattering of surface waves by the wind flow has been specially studied from this point of view in Ref. 11. The contribution from this mechanism to the collision integral, proportional to $N_{\mathbf{k}}^2$ and to the ratio of the air and the water density, turns out to be much smaller than that caused by the scattering of the waves by the flow (according to the calculations of Ref. 11 the characteristic time T is ~ 10 yr). Moreover, the width of the stationary angular distribution increases in this model with decreasing frequency which contradicts the observational data. It was noted in Ref. 16 that the effect of a nonstationary wave field (the development of the swell) can give rise to a narrowing of the angular distributions. One may expect that this mechanism will dominate in the region where the swell is significantly nonstationary, on the leading slope of the frequency spectrum. (No numerical estimates are given in Ref. 16.)

There are therefore grounds for assuming that the induced scattering of waves by the drift flow is the main factor guaranteeing the formation of narrow angular distributions in the whole energy-carrying range of the wind swell (with the possible exception of the leading slope of the spectrum). An indirect confirmation of this statement is the qualitative agreement of the main conclusions of this paper with the experimental data. A quantitative comparison of the present theory and the experiments is prevented, firstly, by the absence of reliable data about the vertical structure of the subsurface drift flow and its connection with the development of the swell and, secondly, by the absence of clarity regarding the mechanism of the long-wavelength cut-off and, as a result, the uncertainty of its estimate.

[1)] We distinguish the problem of the evolution of the wavefield itself and assume the flow to be given and stationary; this enables us to separate the characteristic times: for the establishment of the drift flow τ_d ($\tau_d \sim 10^3$–10^4 s) and for the wave processes.

[2)] The neglect of the rotation of the direction of the drift flow with depth is justified by the fact that the characteristic spatial (Ekman) scale Λ is considerably larger than the thickness of the layer of effective interactions. For typical conditions of the upper layer of the ocean we have $\Lambda \sim 40$ m.

[3)] The main omitted term is of order $O(\varkappa_x)$ (in relation to the main term).

[4)] Waves propagating against the flow correspond to $\tau < 0$, $\tilde{\alpha} < 1$. In that case (4.11) and (4.12) describe a broadening of the spectrum. Along all characteristics n decreases and the characteristics themselves get concentrated towards the ordinate.

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