Theory of quasiballistic transport in microstructures in a quantizing magnetic field

É. M. Baskin and M. V. Éntin

Institute of Semiconductor Physics, Siberian Branch of the Academy of Sciences of the USSR
(Submitted 19 July 1990)
Zh. Eksp. Teor. Fiz. **98**, 2150–2157 (December 1990)

A solution is found for the problem of the motion of 2*D* electrons in microstructures with a smooth potential in a quantizing magnetic field. The conductances (in units of e^2/h) of two-poles, three-poles, etc., are generally rational numbers. The establishment of the chemical potential near the Fermi seas of the electrodes and in the sample itself is studied.

I. We investigate the quantization of the resistance of microstructures with a smooth potential relief $V(x,y)$ in a quantizing magnetic field under ballistic and quasiballistic conditions. The present study is based on the assumption of semiclassical dynamics for the center of the electron orbit in the quantizing magnetic field.[1] If the length scale l of the potential variation is much larger than the magnetic length scale $a=(c\hbar/eH)^{1/2}$, the motion of an electron can be described[1)] as a drift of the center of the orbit along the contour lines of the potential $V(x,y)$ at a velocity

$$\dot{\mathbf{r}}=(c/eH^2)[\nabla V\mathbf{H}].$$

The coordinates x,y are then canonical variables, and classical motion occurs along contour lines of the potential

$$V(x,y)+(N+{}^1/_2)\hbar\omega_H=E.$$

Quantization within a single Landau band can be carried out on the basis of the Hamiltonian

$$\mathcal{H}=V(x,y)+(N+{}^1/_2)\hbar\omega_H, \quad y=-ia^2\partial/\partial x. \tag{1}$$

In principle, the Hamiltonian (1) can be used to find both the energy levels with semiclassical accuracy and the probability for tunneling between different semiclassical states. The Bohr-Sommerfeld rule implies quantization of the areas between neighboring contour lines:

$$\Delta S_{n,\,n+1}=2\pi a^2, \quad E_{nN}=V(x,y)+(N+{}^1/_2)\hbar\omega_H,$$

where n specifies the particular sublevel of Landau level N. Finite contour lines determine the discrete spectrum, and infinite ones the continuum. Corrections to E_{nN} for tunneling between semiclassical trajectories are found with the help of the Hamiltonian (1) in the usual way.[2]

II. We consider a sample with several junctions. The current is carried through the junctions exclusively by electrons on contour lines which start from or end at the junctions (the direction of a contour line is determined by the drift velocity on this line). Each Landau level N has its own system of Fermi-level lines. Contour lines cannot intersect. We say that a junction is "ideal" if the electrons leaving it and going into the sample conserve the chemical potential μ of the bulk part of the electrode. A sample, on the other hand, is "ideal" if there is no mixing of electrons between different contour lines. The contour lines going into a junction are infinite; i.e., they correspond to a continuous energy spectrum. Corresponding to an energy interval ΔE is a band of width $\Delta E/|\nabla V(\mathbf{r})|$ at the point $\mathbf{r}$. Multiplying it by the electron velocity $c|\nabla V(\mathbf{r})|/eH$ and by the electron density in the Landau level $(2\pi a^2)^{-1}$, we find the current from the energy band ΔE near the given contour line:

$$\Delta I=e\frac{c|\nabla V(\mathbf{r})|}{eH}\frac{\Delta E}{|\nabla V(\mathbf{r})|}\frac{1}{2\pi a^2}=\frac{e}{h}\Delta E.$$

For electrons leaving the junction, the width of the energy band is determined by the chemical potential μ of the junction. We can make use of this circumstance to formulate the following "Kirchhoff's laws": Contour lines of the Fermi level are unidirectional quantum conductors which connect nodes on a plane. The contour lines of the Fermi level for a given Landau level alternate in direction with any curve which intersects them. Each node is a point which current goes into and comes out of and which has its own potential $\varphi_i=\mu_i/e$. The current on the line $i\to j$ is determined by the potential of node i:

$$I_{ij}=\frac{e^2}{h}\sum_\alpha N_{ij}^\alpha\varphi_i,$$

where N_{ij}^α is the number of contour lines of the Fermi level which connect node i with node j and which correspond to Landau level α. The total current leaving node i is

$$I_i=\sum_j(I_{ij}-I_{ji}).$$

The number of contour lines emerging from a node must be the same as the number going into it:

$$\sum_j(N_{ij}^\alpha-N_{ji}^\alpha)=0.$$

This condition means that the numbers of "right banks" is equal to the number of "left banks." We can now write I_i as

$$I_i=\frac{e^2}{h}\sum_j N_{ji}(\varphi_i-\varphi_j), \quad N_{ji}=\sum_\alpha N_{ji}^\alpha. \tag{2}$$

This result explicitly eliminates the origin of the energy scale and shows that the current is determined exclusively by the electrons in bands of width equal to the potential difference between junctions.

A change in the sign of the magnetic field is equivalent to the replacement $N_{ji}(H)\to N_{ij}(-H)$. Generally speaking, we thus have $I_i(H)\neq I_i(-H)$. Only in an axisymmetric arrangement would we have $N_{ij}=N_{ji}$ and $I_i(H)=I_i(-H)$.

This classical situation corresponds to Landauer's quantum-mechanical formula,[3] in which the transmission coefficients are identically equal to either one or zero. Accordingly, the conductances are quantized exactly.

 0038-5646/90/121208-05$03.00

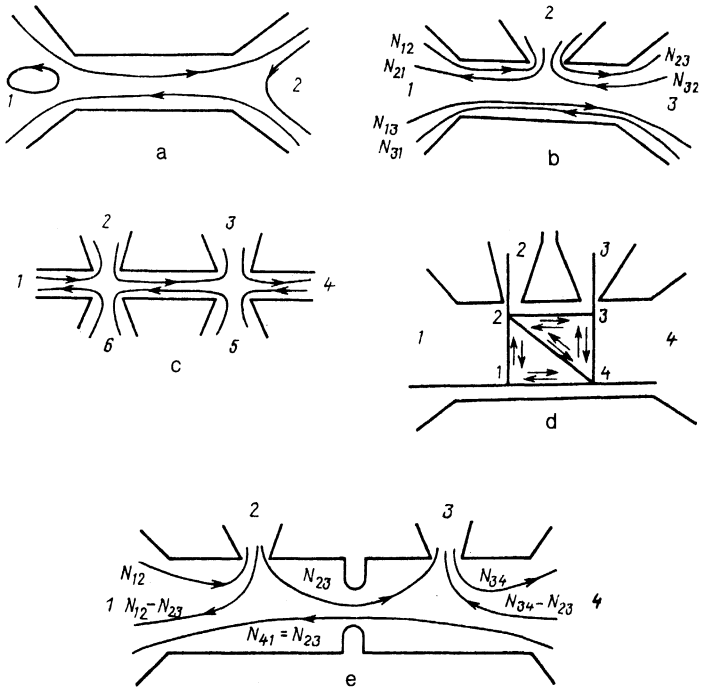

FIG. 1. a—A two-pole. The closed contour line and the line which emerges from electrode 2 and goes back into it do not contribute to the conductance. b—A three-pole. General model. c—A long Hall sample with an ideal middle. All the contour lines emerging from junction 2 reach junction 3, and those coming from junction 5 reach junction 6. There are no contour lines connecting 1 and 4; there are no "counterflows" 4–1, etc. d—A four-pole. A topological model of this four-pole is shown inside it. Also possible is another model, symmetric with respect to the one shown, in which diagonal 2–4 is replaced by diagonal 1–3. e—A sample with a "bottleneck." Only the contour lines 2–3 and 4–1 pass through the constriction.

III. Let us consider the conductances of some very simple arrangements.

a) For a two-pole (Fig. 1a),

$$G=(e^2/h)N, \tag{3}$$

where N is the number of contour lines connecting the junctions.

b) For a three-pole (Fig. 1b),

$$I_2=(e^2/h)\,[N_{32}(\varphi_2-\varphi_3)+N_{12}(\varphi_2-\varphi_1)],$$
$$I_3=(e^2/h)\,[N_{13}(\varphi_3-\varphi_1)+N_{23}(\varphi_3-\varphi_2)].$$

If junction 3 is disconnected ($I_3=0$), we have

$$\varphi_3=\frac{\varphi_1 N_{13}+\varphi_2 N_{23}}{N_{13}+N_{23}},$$

and the two-point conductance is

$$G_{12}=\frac{e^2}{h}\frac{N_{12}(N_{13}+N_{23})+N_{32}N_{13}}{N_{13}+N_{23}}. \tag{4}$$

The replacement of H by $-H$ interchanges i and j. In the particular case of "one-armed" junction (i.e., a single contour line goes into and comes out of each Landau level) which are identical in the sense that at equilibrium they have identical numbers of filled Landau levels N, the potential φ_3 is equal to φ_1 or φ_2, depending on the sign of H, and we have $G_{12}=e^2N/h$. Here and below, the theory is linear; i.e., $\mu_i-\mu_j\ll\hbar\omega_H$.

c) For a Hall sample (Fig. 1c) which is also long, the interjunction distances L_{23} and L_{65} are much larger than the width of the sample, W. The numbers of contour lines are $N_{12}=N_{23}=N_{34}=\ldots=N$. We then have

$$G_{14}=e^2N/h, \quad \varphi_2=\varphi_3=\varphi_1, \quad \varphi_6=\varphi_5=\varphi_4.$$

d) For a four-pole (Fig. 1d), electrodes 2 and 3 are "potentiometric" ($I_2=I_3=0$), and we have $G_{23}=I_{14}/(\varphi_2-\varphi_3)$.

$$I_{14}=\frac{e^2}{h}\left[N_{41}-\frac{N_{21}(N_{43}N_{32}+N_3N_{42})}{N_{23}N_{32}-N_2N_3}\right](\varphi_1-\varphi_4),$$

$$G_{23}=\frac{e^2}{h}\frac{N_{41}(N_2N_3-N_{23}N_{32})+N_{21}(N_{43}N_{32}+N_3N_{42})}{N_{12}N_{43}}, \tag{5}$$

$$N_2=N_{21}+N_{23}+N_{24}, \quad N_3=N_{32}+N_{34}.$$

To determine how the conductance of a four-pole depends on the magnetic field, we need to specify the nature of the potential in the structure. There can evidently be at least three saddle points of the potential in a sample; these points are bifurcation points for the contour lines (three cuts are sufficient to disconnect all the junctions from each other). For definiteness we place these points at junctions 2 and 3 and between them: V_{d2}, V_{d3}, and V_d.

If the Fermi level is higher than all the saddle points, and if the current junctions are identical, we have $N_{12}=N_{23}=N_{34}=N_{41}=N$ (and otherwise $N_{ij}=0$). In this case we have

$$I_{14}=e^2N(\varphi_1-\varphi_4)/h, \quad \varphi_2-\varphi_3=0, \quad G_{23}=\infty.$$

We assume the following hierarchy of saddle points. $V_{d2}<V_{d3}<V_d<\hbar\omega_H$. As H is increased, the contour lines $4\to1$, $2\to3$ are first "short-circuited" to $4\to3$ and $2\to1$, so we have $N_4=N-1$, $N_{23}=N-1$, and $N_{43}=N_{21}=1$. In this case we have

$$V_d>\mu-(N+{}^1/_2)\hbar\omega_H>V_{d3}$$

and

$$G_{23}=(e^2/h)N(N-1), \quad I_{14}=(e^2/h)\,(N-1)\,(\varphi_1-\varphi_4).$$

A further increase in H [$V_{d3}>\mu-(N+1/2)\hbar\omega_H>V_{d2}$] leads to short-circuiting of the lines $4\to3$ and $3\to4$ to $4\to4$ and $3\to3$; G_{23} becomes infinite, while I_{14} does not change. This pattern repeats itself when the next Landau level crosses the equilibrium Fermi level μ.

e) A particular case of a four-pole is a "bottleneck" between two identical Fermi seas (Fig. 1e). From (5) we find the known expression[4]

$$G_{23}=\frac{e^2}{h}\left(\frac{1}{N_{23}}-\frac{1}{N_{12}}\right)^{-1}, \quad N_{12}=N_{34}. \tag{6}$$

We wish to call attention to two circumstances. (1) The conductances are generally not multiples of e^2/h, but they are rational numbers. (2) The conductance of a sample depends on the presence of potentiometric electrodes. For example, even in the case of a three-pole the conductance is described by (4) instead of simple expression (3) for a two-pole. The reason is a mixing of the chemical potentials of the contour lines in a junction. If contour lines with different chemical potentials do not go into a junction, the presence of this electrode will have no effect on the conductance. A potentiometric electrode is "connected" to the corresponding current junction.

The expressions derived in this section of the paper contain an implicit dependence on the magnetic field through the numbers N_{ij}. In samples with a regular potential, the result is determined unambiguously if the contour topology is known. If the potential has no extrema, N_{ij} may be related to the number of filled Landau levels in different parts of the sample. If the potential is random, i.e., has a well-developed structure, one must resort to percolation arguments.

IV. Let us examine the conductance of a two-pole with a random potential (Fig. 2a). If the range of this random potential is $w \ll \hbar\omega_H$, the dependence $G(\mu)$ consists of repeating series of jumps of magnitude e^2/h. Figure 2b shows a dependence of this sort, corresponding to the potential in Fig. 2a. The filling of each successive Landau level leads to an upward shift of the series as a whole by e^2/h.

The statistics of G is determined by percolation arguments. We assume a square sample of side $L \gg l$ but with $L \ll L_c = l\tau_\mu^{-\nu}$, where $\tau_\mu = (|\mu - V_c|/w)^{-\nu}$, $\nu = 1.33$, V_c is the percolation level in the random potential, and L_c is the correlation radius at the level $V(x,y) = \mu$. For such a square, the contour lines of $V(x,y) = E$, where E is found from the condition $l(|E - V_c|/w)^{-\nu} > L$, are percolative. The mean value of the percolation threshold in a finite $L \times L$ square is

$$V_L = V_c \pm w\tau_L, \quad l\tau_L^{-\nu} = L.$$

The quantity $\pm w\tau_L$ determines the times at which percolation arises and disappears and therefore the width of the series. The number of jumps in each series is of order unity. This conclusion follows from the similarity of a square of size $L \ll L_c$ and an infinite square, in which there is a single infinite contour line. The fluctuations in the width of the series are on the order of the width itself, since the fluctuations of the percolation threshold in a finite system of size L are on the order of τ_L (Ref. 5).

For a two-pole in the form of an interlayer between tv · Fermi seas (the width is $W \gg L$), the picture changes. The number of contour lines intersecting an interlayer of this sort is approximately W/L, since each $L \times L$ square can be treated independently. A contour line going into a square will, with a probability $\approx 1/4$, emerge at the opposite side. With the same probability, it will reach a neighboring square through its lateral face. Consequently, the probability for the appearance of a contour line which traverses a distance much greater than L in the interlayer is exponentially small. The number of jumps in a series is thus $W/L \gg L$ and has a dispersion of order $(W/L)^{1/2}$.

In the opposite case of a highly elongated neck ($W \ll L$),

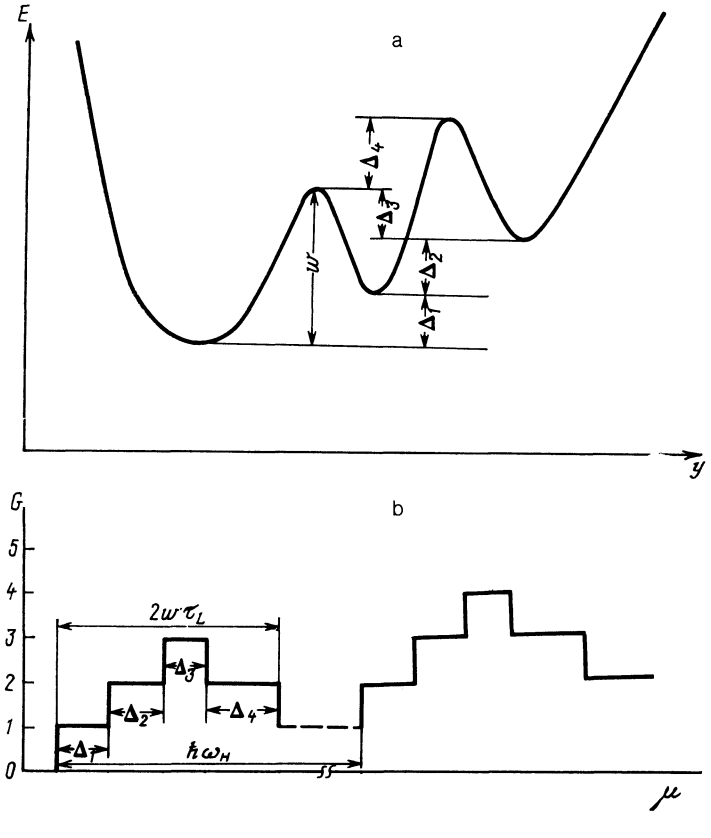

FIG. 2. a—Intersection of the random potential $V(x,y)$ with an x = const plane; b—conductance of a two-pole, in units of e^2/h, as a function of μ.

the probability for a contour line to pass through the neck from one electrode to the other is exponentially small. However, the edge contour lines are conserved. We are thus left with exclusively regular jumps on the $G(\mu)$ curve.

V. We now drop our earlier assumption that the junctions and the sample are ideal, and we consider a more general case.

1. We first consider the establishment of the chemical potential in the junctions. It is frequently assumed that junctions are "dirty," i.e., that the mean free path in them is comparable to the magnetic length scale.[4] In this case, the model of a motion along contour lines is itself incorrect. Since motion along contour lines is ballistic, and since the effect of the magnetic field inside a junction can be ignored, that approach is in a sense equivalent to the flow of gas out of an aperture into vacuum. In this case there will evidently be a jump in the chemical potential in the junction itself, and the conductance of the electrode will differ from the ideal conductance given by expression (2). Actually, the bridge is usually made of the same material as the junction region itself. Consequently, the picture of a motion along contour lines is also valid in this region.

Nevertheless, hops between contour lines may occur (and may be either elastic or inelastic). We introduce a mean free path of an electron with respect to such hops, l_p. The motion is quasiballistic if $l_p \gg l$. The chemical potential is established over distances $\sim l_p$. Far from the percolation threshold, the length of a contour line is on the order of the distance between its ends. The chemical potential in an electrode equalizes in a region of area $\sim \alpha l_p^2$, where α is the vertex angle of the junction cone.

The number of states in this region which are participating in a "mixing" with states which are going into the junction through contour line $j \to i$ can be estimated in the following way. The number of closed contour lines in this region is $\sim \alpha l_p^2/l^2$. Their total perimeter is $\alpha l_p^2/l$, while their area S is determined by the energy width of the band ΔE for the electrons participating in the establishment of the chemical potential (i.e., in the mixing):

$$S = \frac{\alpha l_p^2}{l} \frac{\Delta E}{|\nabla V|}, \quad |\nabla V| \sim \frac{w}{l}.$$

Here $\Delta E = |\mu_i - \mu_j|$ if $|\mu_i - \mu_j| \gg T$, or $\Delta E \sim T$ in the opposite limit (T is the temperature in energy units). The number of states is then

$$\frac{S}{2\pi a^2} = \frac{\alpha l_p^2}{2\pi a^2} \frac{\Delta E}{w}.$$

The number of states on contour line $j \to i$ in the mixing region in band ΔE is $l_p l \Delta E / 2\pi a^2 w$. The increment $[(\mu_i' - \mu_i)_j]$ in the chemical potential of electrode i as a result of mixing with states on contour line $j \to i$ is determined by the relative number of these states:

$$(\mu_i' - \mu_i)_j = (\mu_j - \mu_i) \frac{l}{\alpha l_p + l}, \tag{7}$$

i.e., by the ratio of l_p to the perimeter (total length) of the mixing contour lines in the junction. We see that the change in the chemical potential of the junction is small if $l/l_p \ll 1$ (for $\alpha \sim 1$).

It might seem that the situation would change radically if the chemical potential μ were near the percolation level ($\tau_\mu \ll 1$). In this case the contour line would be of a fractal nature;[6] i.e., the length along a contour line, $\mathscr{L}$, would increase far more rapidly than the distance between its ends, L (or far more rapidly than the mean diameter, in the case of a closed line). For $L \ll L_c$ we have

$$\mathscr{L} = l\left(\frac{L}{l}\right)^{d_f}, \quad d_f > 1. \tag{8}$$

The area of the electrode region in which the mixing occurs depends on the relation between l_p and $\mathscr{L}_c = l(L_c/l)^{d_f}$. If $l_p \gg \mathscr{L}_c$, the size of the mixing region is found from the condition $l_p = \mathscr{L}_c L / L_c$. In this case we have $L_c \ll L \ll l_p$. As μ approaches V_c, the condition $l_p \gg \mathscr{L}_c$ is of course violated. We then find from (8)

$$L = l(l_p/l)^{1/d_f} \ll \mathscr{L}_c \ll l_p.$$

The total length of the contour lines of the Fermi level in Landau level N is found on the basis of the following arguments. This length is the perimeter of all finite clusters for which the condition $V(x,y) + (N + 1/2)\hbar\omega_H = \mu$ holds. We know from lattice problems of percolation theory[7] that at the percolation threshold x_c the mean perimeter p_s of a cluster of s nodes has the behavior $(1 - x_c)s/x_s$. For a continuous random potential we would have

$$x_c = \int_{-\infty}^{V_c} F(V)\,dV,$$

where $F(V)$ is the distribution function of the values of V. In two dimensions we would have $x_c = 1/2$. The total perimeter of finite clusters in a volume L^d (d is the dimensionality of the space) is $P = L^d \Sigma_s p_s n_s$ (n_s is the number of clusters of s nodes per lattice site). For P we find $P = L^d(1 - x_c)$. In our case we find $P = \frac{1}{2} l (L/l)^2$. Consequently, the relative number of edge states is

$$\frac{l_p}{P} = \begin{cases} \dfrac{l}{l_p}\left(\dfrac{L_c}{l}\right)^{2(d_f - 1)}, & l_p \gg \mathscr{L}_c \\ \left(\dfrac{l}{l_p}\right)^{2/d_f - 1}, & l_p \ll \mathscr{L}_c \end{cases}$$

We see that the behavior $l_p \sim p$ prevails only for $d_f > 2$, and this condition cannot hold for $d = 2$. Consequently, electrons can be injected into a sample with a chemical potential equal to that of the electrode except under the condition $l_p \lesssim l$.

2. We now consider the establishment of the chemical potential in a quasi-1D sample. Clearly, the equalization of chemical potentials within a sample is particularly important in long structures. Let us assume that the potential V has a flat bottom and steep walls and depends only on the transverse coordinate. In this case the distance between contour lines in one direction is much smaller than the distance between contour lines going in different directions. An equilibrium is established between contour lines which are spatially close together. The system is then characterized by only two chemical potentials, which vary along the sample as a result of transitions between lines going in different directions. This change is described by the equations

$$\frac{d\mu_{1,2}}{dx}=\frac{1}{l_p}(\mu_2-\mu_1)$$

with the boundary conditions

$$\mu_1(0)=\mu_1^{(0)}, \quad \mu_2^{(0)}=\mu_2(0)=\mu_1^{(0)}-e\varphi.$$

Solutions are

$$\mu_1(x)=\mu_1^{(0)}\left(1-\frac{x}{l_p}\right), \quad \mu_2(x)=\mu_2^{(0)}\left(1-\frac{x}{l_p}\right)\left(1-\frac{L}{l_p}\right)^{-1}.$$

In this case we have

$$G_{12}=\frac{e^2}{h}N\left(1+\frac{L}{l_p}\right)^{-1}.$$

For potentiometric probes we have

$$G_{23}=\frac{e^2}{h}N\frac{l_p}{L_{23}},$$

and for Hall electrodes $G_{\text{Hall}}=e^2N/h$. In a sample with an irregular potential, the length $\mathscr{L}$ of the contour line plays the role of L.

These expressions describe the transition from the ballistic regime to the ordinary Drude regime. Strictly speaking, this situation is realized at high temperatures, at which the hopping processes are inelastic.

What happens at low temperatures? If an electron experiences an impurity potential (which is not necessarily smooth) in addition to the potential $V(y)$, then all states of the electrons in a long sample will be localized in the absence of a magnetic field. The reason for this localization is the well-known divergence of the quantum-mechanical corrections to the conductance in $1D$ systems. Imposing a magnetic field suppresses the quantum-mechanical corrections at fields greater than[8] $H_c=3^{1/2}hc/(eWL)$ and disrupts the localization.

In a $1D$ system without a magnetic field, the highly localized case can also occur, if the Fermi level intersects the bottoms of the quantum-size subbands. The pronounced localization occurs because the density of states becomes infinite, since the mean free path decreases, and the condition $p_Fl_p\gg 1$ is violated. The imposition of a weak magnetic field cannot disrupt the pronounced localization (as follows from the small value of the change in the Gell-Mann–Low function β under the influence of a weak magnetic field and from the negative value of β in the case of pronounced localization). The situation apparently remains the same as the magnetic field is strengthened. Unfortunately, it is not clear to us at this point whether weak localization occurs in a quantizing magnetic field in a $1D$ system. In particular, we cannot rule out the possibility of a situation in which regions of localized and delocalized states alternate in energy, as in the $2D$ case.[9]

We wish to thank Z. D. Kvon for stimulating discussions.

[1)] A more precise condition for the applicability of the drift approximation for electrons in Landau level N is $|\nabla V|\ll m\omega_H^2r_H$, where $\omega_H=eH/mc$ and $r_H=a(N+1/2)^{1/2}$, according to Ref. 1.

[1] É. M. Baskin, L. I. Magarill, and M. V. Éntin, Zh. Eksp. Teor. Fiz. **75**, 723 (1978) [Sov. Phys. JETP **48**, 365 (1978)].
[2] J. K. Jain and S. Kivelson, Phys. Rev. B **37**, 4111 (1988).
[3] R. Landauer, Philos. Mag. **21**, 863 (1970).
[4] M. Büttiker, Phys. Rev. B **38**, 9375 (1988).
[5] M. E. Levinshteĭn, M. S. Shur, B. I. Shklovskiĭ, and A. L. Éfros, Zh. Eksp. Teor. Fiz. **69**, 386 (1975) [Sov. Phys. JETP **42**, 197 (1975)].
[6] B. M. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, San Francisco, 1982.
[7] D. Stauffer, Phys. Rep. **54**, 1 (1979).
[8] B. L. Al'tshuler and A. G. Aronov, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 515 (1981) [JETP Lett. **33**, 499 (1981)].
[9] D. E. Khmel'nitskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 454 (1983) [JETP Lett. **38**, 552 (1983)]; H. Levine, S. B. Libby, and A. M. M. Pruisken, Phys. Rev. Lett. **51**, 1915 (1983).

Translated by D. Parsons