Generation of an observable turbulence spectrum and solitary dipole vortices in rotating gravitating systems

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A nonlinear equation has been obtained, describing the dynamics of disturbances of small, but nonvanishing amplitude of a rigidly rotating gaseous system of nonuniform density. Both in the case of short-wave disturbances (of wavelength λ much smaller than the Jeans wavelength λ_J, i.e., $\lambda \ll \lambda_J$) and in the opposite limiting case of long-wavelength disturbances $\lambda \gg \lambda_J$ the equation reduces to the well-known nonlinear equation for Rossby waves with a vector nonlinearity i.e., the Charney-Obukhov (or Hasegawa-Mima) equation. The weak turbulence spectrum of this equation leads to known observable relations among the fundamental parameters of the gas population of our Galaxy, in particular to the star mass spectrum discovered by Saltpeter in 1955. The stationary solution of this equation, due to Larichev-Reznik is obtained in the form of a solitary vortex (the modon) with a circular separatrix. The equation of two-dimensional gravidynamics derived here for the cases $\lambda \gg \lambda_J$ and $\lambda \ll \lambda_J$ is analogous to the nonlinear equation which describes the dynamics of an incompressible fluid in the β plane with a rigid lid. This nontrivial effect of the β plane is a consequence of a recently established [A. M. Fridman, Sov. Phys. Doklady (1989)] analogy between the nonlinear equations for Rossby waves on the β plane with a rigid lid, and those in the f plane with a free surface [provided the conditions of the experiment described in Izv. AN SSSR, Ser. Fiz. atm. i okeana **23**, 170 (1987) are satisfied and viscosity is neglected]. The conclusion discusses possible observable manifestations of modons: double galaxies, double galactic nuclei, and double stars.

1. INTRODUCTION

The analogy between the fundamental equations for particles coupled by Coulomb interaction in a magnetic field—the Boltzmann-Vlasov equations—and the fundamental equations describing particles interacting according to Newton's law—the kinetic equations (in a rotating coordinate system) and the Poisson equation, has served as a basis for the construction of a stability theory for gravitating systems (Refs. 1–3). The theory made ample use of the methods developed for collective processes in plasma physics. If one can consider completed[1)] the linear stability theory of gravitating systems, the foundations of the theory of nonlinear waves[8] and of turbulence[9] have been laid relatively recently. Until now the nonlinear theory was developed only for the model of an infinitely thin gravitating disk, i.e., for the case of large-scale disturbances, with sizes exceeding by much the thickness of the disk. In these first papers used was made of Jeans perturbations of nonvanishing amplitude and it was shown for the first time that these can propagate as solitons of the envelope.[8] The theory of weak Jeans turbulence[9] was developed by analogy with Ref. 10, and the spectrum $E_k \propto k^{-7/4}$ was derived, a spectrum which coincides practically with the "proper" Kolmogorov spectrum (Refs. 11, 12).[2)]

In a recent years numerous observational investigations of the turbulence spectrum of the gaseous component of the galaxy (see, e.g., Ref. 13) have noted a deviation of that spectrum from a proper Kolmogorov spectrum. Thus, the dependence of the velocity v of the center of mass of a gas accumulation on the parameter l is the following:[13]

$$v_l \propto l^{1/2}, \tag{1}$$

and, correspondingly, for the gas density ρ the observations[13] yield:

$$\rho_l \propto l^{-1}. \tag{2}$$

The deviations from a proper Kolmogorov spectrum ($v \propto l^{1/3}$) exceed the observation errors by so much that the observers speak with certainty of a spectrum which deviates from that obtained in Refs. 11 and 12. Moreover, in some theoretical papers in place of the hypothesis of Refs. 11, 12 according to which the energy flux is constant across the spectrum, other assumptions are proposed. Thus, in Ref. 14 two "fundamental hypotheses" are advanced: first, constancy along the spectrum of the flux of angular momentum of gas accumulations, which together with the second hypothesis that the characteristic time for the displacement of a gas accumulation coincides with the Jeans time, leads to a constant pressure $P(l) \equiv \rho(l) v_l^2 = \text{const}$. The authors of Ref. 14 identify the observed dependence (1) with the spectrum $E_k \propto k^{-1}$, noting that it is shallower than the Kolmogorov spectrum $E_k \propto k^{-5/3}$. Not touching upon the two "fundamental hypotheses" of the authors of Ref. 14, we point out the erroneous nature of their ensuing conclusions, consisting in the fact that in place of the relation $E_k \propto v_k^2$ one must write:[15]

$$\int_k^\infty E_k \, dk \propto v_k^2,$$

from which one gets the spectrum[16]

$$E_k \propto k^{-2}, \tag{3}$$

which is steeper than a proper Kolmogorov spectrum.[11,12]

We show below how the spectrum (3) is obtained from the original equations of hydrodynamics for a rotating gravitating compressible system, within the framework of the theory of weak turbulence. We restrict our attention to the derivation of a nonlinear equation (neglecting terms of third

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order of smallness in the amplitude of perturbations), from which we obtain in the two-limiting cases ($\lambda \gg \lambda_J$, $\lambda \ll \lambda_J$, where λ_J is the Jeans scale of disturbances) a nonlinear equation for the gravitational potential Ψ and the density ρ, respectively.

We first consider the nonlinear equation for the gravitational potential Ψ. It contains a so-called "vectorial nonlinearity"[17] and is known in hydrodynamics as the Charney-Obukhov equation (Refs. 18–20), and in plasma physics as the Hasegawa-Mima equation (Ref. 21). In the linear approximation this equation describes gravitating Rossby waves.[22] We propose to name them "gravitating" because, in distinction from their analogs in hydrodynamics—Rossby waves (Refs. 23–25),[3)] and in plasma physics—drift waves (Refs. 27, 28), the existence of gravitating Rossby waves is due the perturbed gravitational potential.

The equation which we derive for nonlinear gravitating Rossby waves describes both turbulent pulsations and stationary dipole vortices. By means of known methods of weak turbulence theory (i.e., with the help of the kinetic equation for the waves, see, e.g., Ref. 29) an equation similar to our Charney-Obukhov (or Hasegawa-Mima) equation was investigated in Ref. 30 and a turbulence spectrum was derived there. Here we have made use of this spectrum and have shown that it corresponds to the spectrum (3) and determines not only the observable relations (1), (2), but also the Salpeter spectrum of the stars with respect to their masses:[31]

$$n \propto m^{-4/3}, \tag{4}$$

where n is the number of stars with masses between m and ∞ per unit volume.

A stationary solution of the nonlinear equation obtained here for gravitating Rossby waves describes solitary dipole vortices. This solution is analogous to that obtained in Ref. 32 for dipole vortices in an incompressible fluid on the β-plane. Below we prove the uniqueness of the solution obtained in Ref. 32 under conditions which are usually satisfied in real systems, and under the assumption that the separatrix of the dipole vortex is circular (see also Ref. 33). The formation of solitary dipole vortices is natural in a medium described by an equation with only a vectorial nonlinearity. An exception is the case of one vortex considered in Ref. 34, when the center of the vortex is located on the line of tangential discontinuity of the velocity, in the presence of two opposite flows. As was correctly noted by the authors of Ref. 34, if the direction of one of the flows is changed to its opposite, so that in accord with Ref. 32, the total flow becomes homogeneous, then the direction of rotation of one half of the vortex is also changed. As a result one obtains a dipole vortex.[32] A note on the possible observational manifestations of dipole-vortical structures in astronomical objects can be found in the Conclusion.

The second nonlinear equation for the density obtained below is analogous to the first one (for the potential Ψ)—it also contains a vectorial nonlinearity.

The plane of the paper is the following: In Sec. 2 the problem is formulated. Sec. 3 contains a derivation of the fundamental nonlinear equations. In Sec. 4 the stationary solutions of the two nonlinear equations are found from the basic solution in the two limiting cases $\lambda \gg \lambda_J$ and $\lambda \ll \lambda_J$. The same section also contains the proof of the uniqueness of the solution describing the solitary dipole vortices. In Sec. 5 the weak turbulence spectrum is represented. In the concluding Sec. 6 possible astrophysical applications are discussed.

2. THE MODEL. THE INITIAL EQUATIONS

We consider a gravitating system rotating with constant angular velocity Ω_0, with nonhomogeneous density $\rho_0' \neq 0$. (Stationary quantities will be denoted with the subscript "0" and the accent denotes differentiation with respect to the radial coordinate r). This is the way the central parts of galaxies and many polytropic rotating gas configurations are built.[35]

The basic equations are the equations of hydrodynamics in a coordinate system which rotates with angular velocity Ω_0 (Ref. 15), and Poisson's equation

$$\frac{d}{dt}\tilde{\mathbf{v}}_\perp = 2[\tilde{\mathbf{v}}_\perp, \boldsymbol{\Omega}_0] - \nabla_\perp \tilde{\chi}, \tag{5}$$

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v}_\perp = 0, \tag{6}$$

$$\Delta_\perp \Psi = 4\pi G \rho, \tag{7}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\mathbf{v}_\perp \cdot \nabla_\perp), \qquad \Delta_\perp \equiv \frac{1}{r}\frac{\partial}{\partial r} r + \frac{1}{r}\frac{\partial}{\partial \varphi}, \tag{8}$$

and any function A can be represented in the form of a sum:

$$A(r, \varphi, t) = A_0(r) + \tilde{A}(r, \varphi, t). \tag{9}$$

In writing Eq. (5), use was made of the equilibrium condition for the system with respect to r (assuming that $\boldsymbol{\Omega}_0$ is along the z axis):

$$\Omega_0^2 r = \frac{\partial \chi_0}{\partial r}. \tag{10}$$

The function χ is defined in the following manner (the remaining notation is commonly used[15]):

$$\frac{\partial \chi}{\partial x_i} = \frac{\partial \Psi}{\partial x_i} + \frac{1}{\rho}\frac{\partial P}{\partial x_i}. \tag{11}$$

Assuming that the gas is barotropic, $\rho = \rho(P)$, the last term in Eq. (11) can be represented in the form

$$\frac{1}{\rho(P)}\frac{\partial P}{\partial x_i} = \frac{\partial f(P)}{\partial x_i}, \tag{12}$$

where the function $f(P)$ is determined from the equation

$$\frac{df(P)}{dP} = \frac{1}{\rho(P)}.$$

Finally, we obtain from (11) and (12)

$$\chi = \Psi + f(P). \tag{13}$$

It follows from the form of the fundamental equations that we have restricted our attention to two-dimensional disturbances situated in the (r,φ) plane transverse to the rotation axis (the z axis).

3. DERIVATION OF THE FUNDAMENTAL NONLINEAR EQUATIONS

Taking the cross product of Eq. (5) by $\mathbf{e}_z \equiv \boldsymbol{\Omega}_0/\Omega_0$, we obtain

$$\tilde{\mathbf{v}}_\perp=\tilde{\mathbf{v}}_\chi+\tilde{\mathbf{v}}_I', \tag{14}$$

where

$$\tilde{\mathbf{v}}_\chi=\frac{1}{2\Omega_0}[\mathbf{e}_z,\nabla_\perp\tilde{\chi}], \tag{15}$$

$$\tilde{\mathbf{v}}_I'=\frac{1}{2\Omega_0}\left[\mathbf{e}_z,\frac{d\tilde{\mathbf{v}}_\perp}{dt}\right]. \tag{16}$$

We substitute the expression (14) for $\tilde{\mathbf{v}}_\perp$:

$$\tilde{\mathbf{v}}_I'=\frac{1}{4\Omega_0^2}\left[\mathbf{e}_z,\frac{d}{dt}[\mathbf{e}_z,\nabla_\perp\tilde{\chi}]\right]+\frac{1}{4\Omega_0^2}\left[\mathbf{e}_z,\frac{d}{dt}\left[\mathbf{e}_z,\frac{d\tilde{\mathbf{v}}_\perp}{dt}\right]\right].$$

Substituting the expression (14) into the latter equation, we see that it is proportional to $\propto\Omega_0^{-3}$. We omit this term, since we are interested only in "slow" motions, $(1/\Omega_0)(d/dt)\ll 1$. We note that

$$\left|\frac{\tilde{\mathbf{v}}_I'}{\tilde{\mathbf{v}}_\chi}\right|\sim\frac{d}{\Omega_0 dt}\ll 1; \tag{17}$$

consequently

$$\frac{d}{d}\approx\frac{\partial}{\partial t}+(\tilde{\mathbf{v}}_\chi,\nabla_\perp)\equiv\frac{d_0}{dt}, \tag{18}$$

whence,

$$\tilde{\mathbf{v}}_I=\frac{1}{4\Omega_0^2}\left[\mathbf{e}_z,\frac{d_0}{dt}[\mathbf{e}_z,\nabla_\perp\tilde{\chi}]\right]. \tag{19}$$

Making use of Eqs. (15) and (19) we calculate div $\tilde{\mathbf{v}}_\chi$ and div $\tilde{\mathbf{v}}_I$

$$\operatorname{div}\tilde{\mathbf{v}}_\chi=0, \tag{20}$$

$$\operatorname{div}\tilde{\mathbf{v}}_I=-\frac{1}{4\Omega_0^2}\left(\frac{\partial}{\partial t}+\frac{1}{2\Omega_0}[\nabla_\perp\tilde{\chi},\nabla_\perp]_z\right)\Delta_\perp\tilde{\chi}. \tag{21}$$

Substituting (20) and (21) into the continuity equation (6) and making use of the Poisson equation (7), we finally obtain the fundamental nonlinear equation:

$$\left(\frac{\partial}{\partial t}+\frac{1}{2\Omega_0}[\nabla_\perp\tilde{\chi},\nabla_\perp]_z\right)\Delta_\perp\tilde{\Psi}-\frac{(\omega_0^2)'}{2\Omega_0}\frac{1}{r}\frac{\partial\tilde{\chi}}{\partial\varphi}$$
$$-\frac{\omega_0^2}{4\Omega_0^2}\left(\frac{\partial}{\partial t}+\frac{1}{2\Omega_0}[\nabla_\perp\tilde{\chi},\nabla_\perp]_z\right)\Delta_\perp\tilde{\chi}=0, \tag{22}$$

where $\omega_0^2\equiv 4\pi G\rho_0$ is the square of the Jeans length and key$(\omega_0^2)'\equiv d\omega_0^2/dr$.

We successively consider the two opposite limiting cases $\tilde{\Psi}\gg\tilde{f}$ and $\tilde{\Psi}\ll\tilde{f}$.

1) $\tilde{\Psi}\gg\tilde{f}$. This case corresponds to the existence of long-wave disturbances $\lambda\gg\lambda_J$, which is possible only for configurations which are sufficiently oblate along the rotation axis, with $h\ll R$ (where h and R are the thickness and radius of the system, respectively), so that $\lambda_J\sim h$ (Ref. 2).

With the condition 1), Eq. (22) takes the form:

$$\left(\frac{\partial}{\partial t}+\frac{1}{2\Omega_0}[\nabla_\perp\tilde{\Psi},\nabla_\perp]_z\right)\Delta_\perp\tilde{\Psi}+\frac{1}{\alpha-2}\frac{(\omega_0^2)'}{\Omega_0}\frac{1}{r}\frac{\partial\tilde{\Psi}}{\partial\varphi}=0, \tag{23}$$

where

$$\alpha\equiv\omega_0^2/2\Omega_0^2. \tag{24}$$

We note that in the special case of a cold cylinder of homogeneous density rotating with angular velocity Ω_0 we have $\alpha=1$.[1]

We investigate the linear approximation to Eq. (23). Setting

$$\tilde{\Psi}(r,\varphi,t)\propto\exp[i(k_r r+m\varphi-\omega t)], \tag{25}$$

we obtain the following expression for gravitating Rossby waves:

$$\omega=-\frac{1}{\alpha-2}\frac{(\omega_0^2)'}{\Omega_0}\frac{k_\varphi}{k_\perp^2},\qquad k_\varphi\equiv\frac{m}{r},\qquad k_\perp^2\equiv k_r^2+k_\varphi^2. \tag{26}$$

Since in real systems $(\omega_0^2)'\sim\rho_0'<0$, the direction of the azimuthal component of the phase velocity of gravitating Rossby waves depends on the sign of the difference $\alpha-2$: for $\alpha-2>0$ the velocity is directed along the direction of rotation ("eastward") and for $\alpha-2<0$ it is directed "westward."

The necessary condition (17) is valid either in the case of an anisotropic spectrum, $k_\varphi\ll k_r$, or in the case when the gravitating disk is at the boundary of a gravitational instability (which is true for galactic disks[36] as well as for some planetary rings[6]).

2) $\tilde{\Psi}\ll\tilde{f}$. This condition is more universal than the preceding one, since it is valid for short-wavelength pulsations, $\lambda\ll\lambda_J$ which exist in a system of arbitrary geometry. It is obvious that the turbulence in gas clouds can be created only by disturbances with $\lambda\ll\lambda_J$. In the α model of accretion disks (Ref. 37, 38) a fundamental role is attributed to such disturbances with $\lambda\lesssim h\sim\lambda_J$ in the creation of turbulent viscosity.

Under the condition 2) the equation (22) takes the form

$$\left(\frac{\partial}{\partial t}+\frac{1}{2\Omega_0}[\nabla_\perp\tilde{f},\nabla_\perp]_z\right)\Delta_\perp\tilde{f}+2\Omega_0\frac{\rho_0'}{\rho_0}\frac{1}{r}\frac{\partial\tilde{f}}{\partial\varphi}=0. \tag{27}$$

In the linear approximation, assuming that $\tilde{f}$ has a form analogous to Eq. (25), we obtain from Eq. (27)

$$\omega=-2\Omega_0\frac{k_\varphi}{k_\perp^2}\frac{d\ln\rho_0}{dr}, \tag{28}$$

which is an expression for the frequency of the Rossby wave. We note that since in real objects $d\ln\rho_0/dr<0$, the azimuthal component of the phase velocity of (short-wavelength) Rossby waves in rotating gravitating systems is directed in the direction of motion (to the "east").

As we see, the equations (23) and (27) obtained from the fundamental equation (22) in the two opposite limiting cases ($\lambda\gg\lambda_J$ and $\lambda\ll\lambda_J$) differ only in the coefficients.

4. SOLITARY DIPOLE VORTICES

We subject equations (23) and (27) to a series of transformations of the independent variables. First, we transform to a local Cartesian coordinate system:

$$\frac{\partial}{\partial r}\equiv\frac{\partial}{\partial x},\qquad\frac{1}{r}\frac{\partial}{\partial\varphi}\equiv\frac{\partial}{\partial y}. \tag{29}$$

Assume now that in the system of three variables x, y, t the last two are related by the following condition: $y=\eta+ut$. In terms of such a system of two independent variables (let these be x and $\eta=y-ut$), the equations (23) and (27) take the form

$$\hat{D}(\xi_i)\Delta_\perp\xi_i=\Lambda_i\frac{\partial\xi_i}{\partial\eta},\qquad i=1,\ 2. \tag{30}$$

Here we have introduced the notations:

$$\hat{D}(\xi_i) = -\frac{\partial}{\partial \eta} + \frac{1}{U_i}[\nabla_\perp \xi_i, \nabla_\perp]_z, \tag{31}$$

$$\xi_1 = \tilde{\Psi}, \quad U_1 = 2u\Omega_0, \quad \Lambda_1 = -\frac{2(\omega_0^2)'}{(\alpha-2)U_1}, \tag{32}$$

$$\xi_2 = \tilde{\xi}, \quad U_2 = 2u\Omega_0, \quad \Lambda_2 = -\frac{4\Omega_0^2}{U_2}\frac{\rho_0'}{\rho_0}.$$

It is easy to see that the equation (30) can be written in the form

$$J(\xi_i - U_i x, \quad \Delta_\perp \xi_i + U_i \Lambda_i x) = 0, \tag{33}$$

where J is the Jacobian. The last equation can be rewritten as:

$$\Delta_\perp \xi_i + \Lambda_i U_i x = F(\xi_i - U_i x), \tag{34}$$

where F is an arbitrary function.

We are interested only in local solutions, i.e., $\xi_i \to 0$ for $\eta \to \infty$ for arbitrary x. In a neighborhood of the point at infinity in η (for $\eta \to \infty$) it follows from Eq. (34) that

$$F(-U_i x) = \Lambda_i U_i x. \tag{35}$$

Consequently, the function F in Eq. (34) must be linear in a region sufficiently remote from the point $x = \eta = 0$. We choose this function to be linear in the whole (x,η) plane:

$$\Delta_\perp \xi_i + U_i \Lambda_i x = -k^2(\xi_i - U_i x), \quad r<a, \tag{36}$$

$$\Delta_\perp \xi_i + U_i \Lambda_i x = p^2(\xi_i - U_i x), \quad r>a, \tag{37}$$

where $k > 0$, $p > 0$. The meaning of splitting the x, η region into the interior and the exterior of a circle of radius a centered at $x = \eta = 0$ will be explained later.

Introducing polar coordinates r, φ ($x = r\cos\varphi$, $\eta = r\sin\varphi$) we get in place of Eqs. (36) and (37)

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + k^2\right)\xi_i = U_i(k^2 - \Lambda_i) r\cos\varphi, \quad r<a, \tag{38}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} - p^2\right)\xi_i = 0, \quad r>a. \tag{39}$$

The last equation turns out to be homogeneous, since we must require $\xi_i \to 0$ as $r \to \infty$. This condition implies

$$p^2 = -\Lambda_i. \tag{40}$$

We look for the general solution of Eq. (38) in the form $\xi = \tilde{\xi} + \xi^*$ (we omit the subscripts i henceforth, where ξ^* is a particular integral and $\tilde{\xi}$ is the general solution of the homogeneous equation, which is well known:

$$\tilde{\xi}(r,\varphi) = \sum_{m=0}^{\infty} (A_m \cos m\varphi + B_m \sin m\varphi) J_m(kr), \quad r<a.$$

We look for a particular solution of the nonhomogeneous equation (38) in the form

$$\xi^* = r(a\sin\varphi + b\cos\varphi). \tag{41}$$

Substituting (41) into (38), we obtain:

$$a = 0, \quad b = U\left(1 - \frac{\Lambda}{k^2}\right). \tag{42}$$

Finally, we find the general solution of Eq. (38):

$$\xi_I = \sum_{m=0}^{\infty} (A_m \cos m\varphi + B_m \sin m\varphi) J_m(kr) + U\left(1 - \frac{\Lambda}{k^2}\right) r\cos\varphi,$$

$$r<a, \tag{43}$$

where $J_m(z)$ is a Bessel function of the first kind.

The general solution of Eq. (39) has the form

$$\xi_{II} = \sum_{m=0}^{\infty} (C_m \cos m\varphi + D_m \sin m\varphi) K_m(pr), \quad r>a, \tag{44}$$

where $K(z)$ is a Macdonald function (modified Bessel function).

The continuity equation and the equation of motion contains the first derivatives of ρ and v with respect to r, which must be continuous for $r = a$. It follows from Eqs. (5)–(7) and (15) that ξ and its first two derivatives must be continuous with respect to r.[4] This is why one can write Eqs. (36) and (37) so that the left-hand and right-hand sides contain homogeneous functions F and f which are continuous at the point $r = a$:

$$F(\xi, \xi_r', \xi_{rr}'', \xi_{\varphi\varphi}'', r, \varphi) = \begin{cases} -k^2 f(r,\varphi), & r<a, \\ p^2 f(r,\varphi), & r>a. \end{cases} \tag{45}$$

Subtracting the first equation of (45) from the second we obtain at the point $r = a$

$$(p^2 + k^2) f(a, \varphi) = 0. \tag{46}$$

Since we have required before $p > 0$, $k > 0$, it follows from (36), (37), and (46) that:

$$(\xi - Ux)|_{r=a} = 0,$$

or

$$\xi_I(a,\varphi) = \xi_{II}(a,\varphi) = Ua\cos\varphi.$$

Substituting the solutions (43) and (44) we find that $B_m = D_m = 0$ for all m; $A_m = C_m = 0$ for all m except $m = 1$:

$$A_1 = U\Lambda a / k^2 J_1(ka), \quad C_1 = Ua / K_1(pa).$$

Thus, we finally obtain:

$$\xi = \begin{cases} \Omega_0 a u \left[\dfrac{r}{a} + \dfrac{\Lambda}{k^2}\left(\dfrac{J_1(kr)}{J_1(ka)} - \dfrac{r}{a}\right)\right]\cos\varphi, & r<a, \\ \Omega_0 a u \dfrac{K_1(pr)}{K_1(pa)}\cos\varphi, & r>a. \end{cases} \tag{47}$$

We have made use of the fact that according to Eq. (32) $U_2 = \Omega u$. [Thus, Eq. (47) describes $\tilde{\xi}_2$; ξ_1 differs from it by a factor $(\alpha - 2)/(\alpha - 1)$.]

From Eq. (14) we get $v_\perp = v_\chi + v_l'$ (here and in the sequel the "tilde" over v will be omitted for simplicity); from Eq. (17) we obtain $|v_\chi| \gg |v_l'|$, consequently $v_\perp \approx v_\chi$. From Eq. (15) we obtain $\mathbf{v}_\chi = [\mathbf{e}_z, \nabla_\perp \tilde{\chi}]/2\Omega_0$, i.e.,

$$v_r = -\frac{1}{2\Omega_0 r}\frac{\partial \tilde{\chi}}{\partial \varphi}, \quad v_\varphi = \frac{1}{2\Omega_0}\frac{\partial \chi}{\partial r}. \tag{48}$$

Let R denote a characteristic length scale for the variation of the stationary parameters (the size of the system) in a direction perpendicular to the axis of rotation, and $a/R \ll 1$. Then in the solution (47) the quantity Λ can be considered constant to accuracy $a/R \ll 1$, and one obtains from (47) and (48):

$$2v_r = \begin{cases} \left[1-\frac{\Lambda}{k^2}\left(1-\frac{a}{r}\frac{J_1(kr)}{J_1(ka)}\right)\right] u\sin\varphi, & r<a, \\ \frac{a}{r}\frac{K_1(pr)}{K_1(pa)} u\sin\varphi, & r>a, \end{cases} \quad (49)$$

$$2v_\varphi = \begin{cases} \left[1-\frac{\Lambda}{k^2}\left(1-ka\frac{J_1'(kr)}{J_1(ka)}\right)\right] u\cos\varphi, & r<a, \\ pa\frac{K_1'(pr)}{K_1(pa)} u\cos\varphi, & r>a. \end{cases} \quad (50)$$

It follows from Eq. (49) that the continuity of v_r at $r=a$ is automatic. The continuity of v_φ at $r=a$ is valid at those points $z_0 \equiv ka$ which are roots of the following equation (recall that p is fixed: $p^2 = -\Lambda$):

$$\frac{s_0}{z_0}\left(1-z_0\frac{J_1'(z_0)}{J_1(z_0)}\right) = s_0\frac{K_1'(s_0)}{K_1(s_0)} - 1.$$

Here we have introduced the notation $s_0 \equiv pa$. After elementary transformations we obtain:[17]

$$\frac{s_0}{z_0}K_1(s_0)J_2(z_0) = -J_1(z_0)K_2(s_0), \quad (51)$$

or

$$K_1(s_0)J_3(z_0)+J_1(z_0)K_3(s_0)=0.$$

This equation has a countable set of positive roots $z_{0n} = z_{0n}(s_0)$, $n = 1, 2, \ldots$. The first three roots ($n = 1, 2, 3$) $z_{0n} = z_{0n}(s_0)$ are shown in Fig. 1 (Ref. 39).

We investigate the structure of the vector field $\mathbf{v}_\perp$. For this purpose we calculate $\operatorname{curl}_z \mathbf{v}_\perp$ for $r<a$. We substitute into the equation

$$\operatorname{curl}_z \mathbf{v}_\perp = \frac{1}{r}\frac{\partial}{\partial r}(rv_\varphi) - \frac{1}{r}\frac{\partial v_r}{\partial\varphi}$$

the expressions for v_r and v_φ from Eqs. (49) and (50) for $r<a$. As a result we obtain

$$\operatorname{curl}_z \mathbf{v}_\perp = \frac{\Lambda a}{2J_1(ka)}\left[J_1''(kr)+\frac{1}{kr}J_1'(kr) - \frac{1}{k^2r^2}J_1(kr)\right] u\cos\varphi,$$

or, after simplifications:

$$\operatorname{curl}_z \mathbf{v}_\perp = -\frac{1}{2}\Lambda a\frac{J_1(kr)}{J_1(ka)} u\cos\varphi, \quad r<a. \quad (52)$$

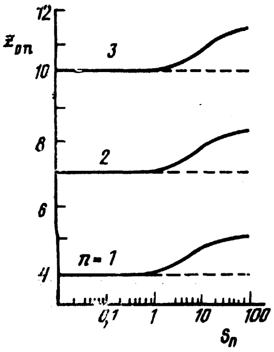

FIG. 1. The first three roots ($n = 1,2,3$) of the equation (51) in the form of the function $z_{0n} = z_{0n}(s_0)$.

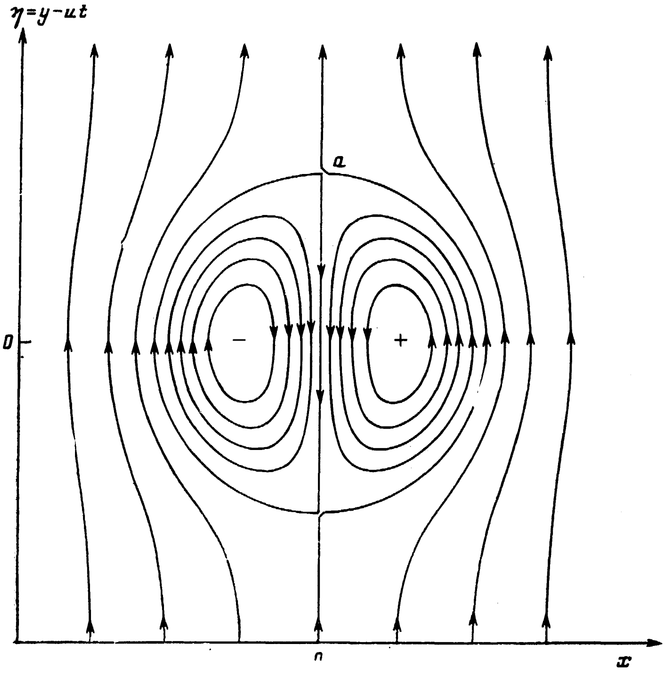

FIG. 2. The structure of the solitary dipole vortex (modon) in the case of a circular separatrix.

It follows from Eq. (52) that $\operatorname{curl}_z \mathbf{v}_\perp$ changes sign when r varies in a circle of radius a, i.e., for $r<a$, at the points where $J_1(kr) \equiv J_1(z) = 0$; the roots of this equations are (Ref. 40): $z_k \approx 4, 7, 10, \ldots$ ($k = 1, 2, 3, \ldots$). Since $r<a$, $z_k < z_{0n}$. If, for some reason, a root z_{01} of the equation (51) is preferred over the others (obviously, in such cases the selection is made by Nature—we discuss this problem in the conclusion), then the inequality $z_k < z_{01} < z$ (see Fig. 1) can be satisfied only for $k = 1$. Consequently for $r<a$, $J_1(kr)$ can undergo only one sign change: at the point $z = z_1$. This corresponds to the structure illustrated in Fig. 2. We note that compatibility of equation (40) and the condition $p>0$ is realized only for $\Lambda_i = \Lambda_1$ for $u>0$, if $\alpha<2$ and $u<0$ if $\alpha>2$; for $\Lambda_i = \Lambda_2$ it is realized for $u<0$. Under these conditions the structure represented in Fig. 2 represents a solitary dipole vortex (a modon). It is also called an isolated vortex, since the amplitude of the vortex decays exponentially for $r>a$ [since it is described by the modified Bessel function (Macdonald function) K_1].

In order to discuss the opposite cases: $\Lambda_i = \Lambda_1$ for $u<0$ if $\alpha<2$ and $\Lambda_i = \Lambda_2$ for $u>0$ if $\alpha>2$, one must change the sign of the right-hand side of Eq. (37)

$$\Delta_\perp\xi + U\Lambda x = -p^2(\xi - Ux), \quad r>a. \quad (37a)$$

The solution of this equation no longer depends on the Macdonald function K_1, but on the cylinder function of the second kind (called the Neumann function N_1 by some and the Weber function Y_1 by others), which decays like $r^{-1/2}$ as $r\to\infty$. Such a solution was known for a long time (Refs. 24, 25); in this case the dipole vortex does not behave like an isolated vortex (Ref. 26) but like a long-range "force center" (compare with the Hertz dipole).

To conclude this section we make some remarks. First, we call attention to the fact that the solution (47) obtained by us is the unique solution of the equations (36) and (37) obtained from the general solutions (43) and (44) by junc-

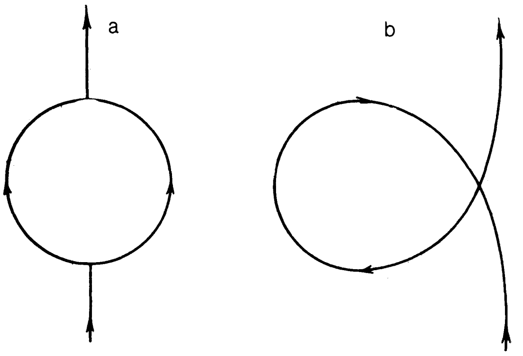

FIG. 3. Shapes of separatrices: a—circular, b—loop-shaped.

tion along a circle of radius a under the assumption that the function and its first derivatives are continuous.[5] Second, one can now understand the reason for the decomposition of the (x,η) plane into the two regions: the interior one $(r<a)$ and the exterior one $(r>a)$. The linear dependence of the functions $\xi_i - Ux$ and $\Delta_\perp \xi_i + U\Lambda x$ proved in Ref. 39 on the basis of the finiteness of the solution defines an equation involving cylinder functions. Since the general solution of this equation becomes infinite either at the point $r=0$ or for $r\to\infty$, the requirement of boundedness of the solutions forces us to split it into two parts, in each of which the solution is that cylinder function which remains bounded in that region. From here follows the expression of the linear dependence in the form of the two equations (36) and (37). Finally and third, comparing our equation (33) with Eq. (4) of Ref. 32, equation which describes dipole vortices in an incompressible fluid in the β plane with a rigid lid, we see their complete analogy.

5. THE WEAK TURBULENCE SPECTRUM

In Ref. 30 is was shown that equations of the type (23) and (27) describe the Kolmogorov power spectrum of weak turbulence first derived in the papers of Zakharov in 1965 for a compressible weakly dispersive fluid (for references see Ref. 10). Under the assumption of practically one-dimensional disturbances, $k_x \gg k_y$, the following spectral dependences were obtained in Ref. 30 for the energy density

$$W_k^{(1)} \propto k_y^{-3/2} k_x^{-2}, \qquad W_k^{(2)} \propto k_y^{-3/2} k_x^{-3}. \tag{53}$$

There arises the question of the dependence of the turbulence spectrum W_k of the Rossby waves on k_z. When Rossby waves are generated in shallow water, the absence of such a dependence seems natural, on account of the shallow-water approximation. However, drift waves in a plasma of size L along the z axis which are much larger than the transverse dimension d (i.e., $L \gg d$) have a similar dispersion law. It follows from the theory of generation of drift waves (Ref. 48) and from experiments (Refs. 49–51) that: 1) $k_z \ll k_y$ (recall that we have assumed that $k_y \ll k_x$); 2) there is practically no dependence of the turbulent diffusion coefficient on the z coordinate. Let us explain the last fact, since in this case this is very important. In order to determine the mechanism of the diffusion across a magnetic field, in Ref. 51 oscillograms were taken of the density oscillations in plasma column and of the current of the measuring device. In the absence of any instability (small oscillation amplitudes) the current was small in magnitude and constant in time. In a developed instability the current of the measuring current had the form of peaks which were correlated with the density oscillations, and the current maxima were observed to be in phase with the density maxima. Thus, the plasma flow across the field had the character of "gushes." The "gush" occurred practically simultaneously along the whole plasma filament (i.e., there was no phase shift at different points of the z axis) as it propagated along with the wave in the azimuthal direction. Since the diffusion coefficient was close to the Bohm value, a turbulent diffusion was observed.

Thus, according to Refs. 48–51 (at the limit of applicability of the theory, for $k_x \sim k_y = k_\perp$) we obtain

$$W_k \propto k_\perp^{-(\alpha_1+\alpha_2)} k_z^{-\alpha_3},$$

where $\alpha_3 \ll \alpha_2, \alpha_1$. Going to spherical coordinates in k-space $k_\perp = k\sin\theta$, $k_z = k\cos\theta$, we obtain

$$W_k \propto k^{-(\alpha_1+\alpha_2+\alpha_3)} f(\theta) \propto k^{-(\alpha_1+\alpha_2)} f(\theta),$$

where

$$f(\theta) = (\sin\theta)^{-(\alpha_1+\alpha_2)} (\cos\theta)^{-\alpha_3}.$$

Integrating W_k with respect to the angle θ from θ_0 to $\pi/2$, where θ_0 is determined from the condition of applicability of drift theory,[48] $\cot\theta_{0,\min} \approx k_\perp / k_{z,\max}$, $k_{z,\max} \sim \omega_* / V_{Ti}$, is the thermal velocity of the ions, and ω_* is the drift frequency, we obtain

$$W_k \propto k^{-(\alpha_1+\alpha_2)}.$$

According to Eq. (53), it follows that

$$W_k^{(1)} \propto k^{-3,5}, \quad W_k^{(2)} \propto k^{-4,5}. \tag{54}$$

We note that the spectra (53) agree with two out of the three obtained in Refs. 44–46 for the case of short-wavelength Rossby waves. (It is explained in Ref. 30 why the third spectrum obtained in these papers cannot be realized.) Earlier, the second spectrum in Eq. (54) was derived in Ref. 47 by dimensional estimates for Rossby waves in the β-plane. Obviously Eq. (54) should be understood as $W_k \propto k^\gamma$, where $\gamma \in (3.5, 4.5)$, i.e., $\gamma \approx 4$. Indeed, the numeri-

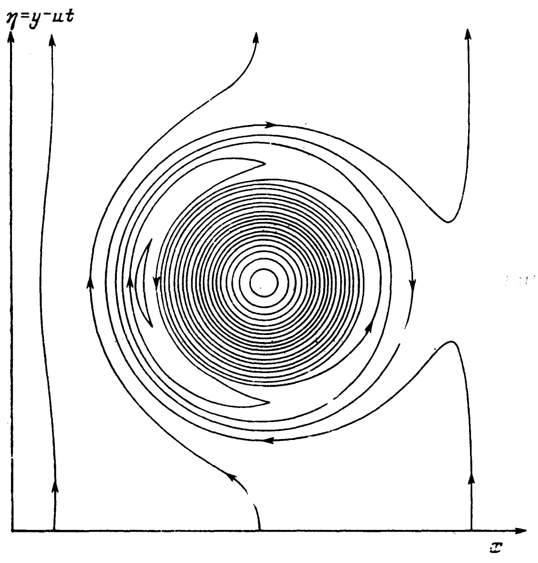

FIG. 4. The structure of a modon for the case of a loop-shaped separatrix.

cal solutions of equations of the same type as the one obtained in the present papers, Eqs. (23), (27), have shown (Ref. 43) that the shape of the spectra is similar to this for $\gamma \approx 4$.

6. ASTROPHYSICAL APPLICATIONS

6.1. The observed relations between the parameters of gas structures and the Salpeter mass spectrum[16]

We define the function $E(k)$ from the condition

$$E(k)\,dk = W_k k^2 dk. \tag{55}$$

According to Ref. 15

$$\int_k^\infty E(k)\,dk \propto v_\lambda^2, \tag{56}$$

where $\lambda \sim 1/k$. Since $W_k \propto k^{-4}$, it follows that $E(k) \propto k^{-2}$, we obtain from Eq. (56) (Ref. 16)

$$v_\lambda \propto \lambda^{1/2}. \tag{57}$$

Comparing the dependence we have obtained with the observed one (1), we see that they are completely identical: exactly such a spectrum is observed in rotating gaseous gravitating systems (Ref. 13). We now utilize the Navier-Stokes equation assuming that all its terms have the same order of magnitude: $v\nabla v \sim \nabla \Psi$ or $v_\lambda^2/\lambda \sim \lambda \Psi/\lambda^2 \lambda\, 4\pi G \rho_\lambda$. Making use of the spectrum (57) we obtain

$$\rho_\lambda \propto \lambda^{-1}. \tag{58}$$

Comparing Eqs. (58) and (2) we see that we have obtained the observed dependence (2) of the density of a gas element of size λ (Ref. 13). From the Navier-Stokes equation we obtain in order of magnitude $v\nabla v \sim \Delta P/\rho$, or $v_\lambda^2/\lambda \sim P_\lambda/\rho_\lambda \lambda$. Making use of Eqs. (57), (58), we obtain $P_\lambda \equiv n_\lambda m_\lambda v_\lambda^2 = \text{const}$ (m_λ is the mass of a gas element of size λ, n_λ is the number of such masses per unit volume). From the last equality we obtain, taking account of $m \sim \rho \lambda^3 \sim \lambda^2$, i.e., $\lambda \propto m^{1/2}$, that

$$n_\lambda \propto (m_\lambda v_\lambda^2)^{-1} \propto m_\lambda^{-3/2}, \tag{59}$$

which coincides with the observed Salpeter mass spectrum (4) (Ref. 31). The theoretical spectra (54) also explain the derivations from the average (i.e., $m \sim m_\odot$) of the distribution of the numbers of starts with respect to masses, Eq. (4), for the region of small ($m \ll m_\odot$) and large ($m \gg m_\odot$) masses (for details see Ref. 16).

There naturally arises the question of the possible generation of a quasi-two-dimensional spectrum in interstellar clouds, which are three-dimensional objects. According to observations (Refs. 52, 53), the mean rotation time of a molecular cloud is $\approx 7 \times 10^6$ years, which is much smaller than the lifetime of the cloud (over its life the cloud makes ≈ 30–50 revolutions). Consequently, the hypothesis of "fast" rotation (17) can be satisfied, which implies the equation for quasi-two-dimensional Rossby waves (22).

6.2. Possible observable manifestations of the generation of modons[54]

Making use of the results of Sec. 4, we compare the direction of the velocity u_m of the motion of the modons (solitary dipole vortices) with the y-component $u_{ph,y}$ of the phase velocity of Rossby waves in two limiting cases: $\lambda \gg \lambda_J$ and $\lambda \ll \lambda_J$. In the calculations we shall take into account the fact that in all observable systems the density decreases with the radius. We consider the two limiting cases.

1) $\lambda \gg \lambda_J$:

$$u_{ph,y} \equiv \frac{\omega}{k_\varphi} = -\frac{1}{\alpha-2}\frac{(\omega_0^2)'}{k_\perp^2} = \begin{cases} >0 & \text{for } \alpha>2, \\ <0 & \text{for } \alpha<2, \end{cases} \tag{60}$$

$$u_m = \begin{cases} <0 & \text{for } \alpha>2, \\ >0 & \text{for } \alpha<2 \end{cases} \tag{61}$$

2) $\lambda \ll \lambda_J$:

$$u_{ph,y} \equiv \frac{\omega}{k_\varphi} = -2\Omega_0 \frac{d\ln \rho_0/dr}{k_\perp^2} > 0, \tag{62}$$

$$u_m < 0. \tag{63}$$

From Eqs. (60)–(63) it can be seen that in both limiting cases

$$\frac{u_m}{u_{ph,y}} < 0. \tag{64}$$

The physical meaning of the latter condition is the following. The modons, being stationary structures, cannot lose energy by radiating Rossby waves (on account of a Cherenkov resonance). The condition (64) ensures that this is so.[26]

In order to determine the sign of the difference $\alpha - 2$ along the equatorial radius of a rotating gas cloud, in Ref. 55 a series of models of rotating axially symmetric polytropes was calculated for the values of the polytrope exponent $n = 0.5$, 1.0, 1.5. The distribution of $\alpha - 2$ turned out to depend only on n and the ratio of the rotation velocity Ω_0 to the critical angular velocity Ω_{cr} (i.e., the limiting value for which the effective gravitational acceleration at the equator falls to zero; when this velocity is exceeded an outflow of matter starts at the equator—Ref. 35). It was shown in Ref. 55 that there always exists a radius r_0 at which the quantity $\alpha - 2$ changes sign (with $\alpha - 2 > 0$ in the interior region of the cloud, $r < r_0$, and $\alpha - 2 < 0$ in the exterior region). The quantity r_0/R, where R is the radius of the cloud, is a monotonically decreasing function of the ratio Ω_0/Ω_{cr}. For $(\Omega_0/\Omega_{cr})_{max} = 1$ we have $r_0/R = 0.83$, 0.73, 0.63 (for $n = 0.5$, 1.0, 1.5, respectively). For $\Omega_0/\Omega_{cr} < 1$ the values of r_0/R are larger than the listed values.

Thus, one may assert that the direction of the azimuthal component of the phase velocity of gravitating Rossby waves in the main part of the system coincides with the direction of the analogous component of ordinary hydrodynamic Rossby waves (namely in the direction of rotation, i.e., to the "east"). The velocity of the modons points in the same direction in the internal region ($r < r_0$) of the rotating system.

In conclusion we recall that in the analytic method of obtaining stationary solutions describing modons we have used two assumptions. The first is the circular form of the separatrix, $r = a$. The second is that only the first root of Eq. (51) is realized in nature, leading to only one change of sign of curl $\mathbf{v}_\perp$ for $r < a$ (i.e., the dipole structure of the solitary vortex). As follows from Fig. 5,[6)] such forms are actually observed in different experiments in rotating shallow water. The reason of the appearance of such selection rules is one of the problems for future investigations.

Until now the astrophysical literature has discussed

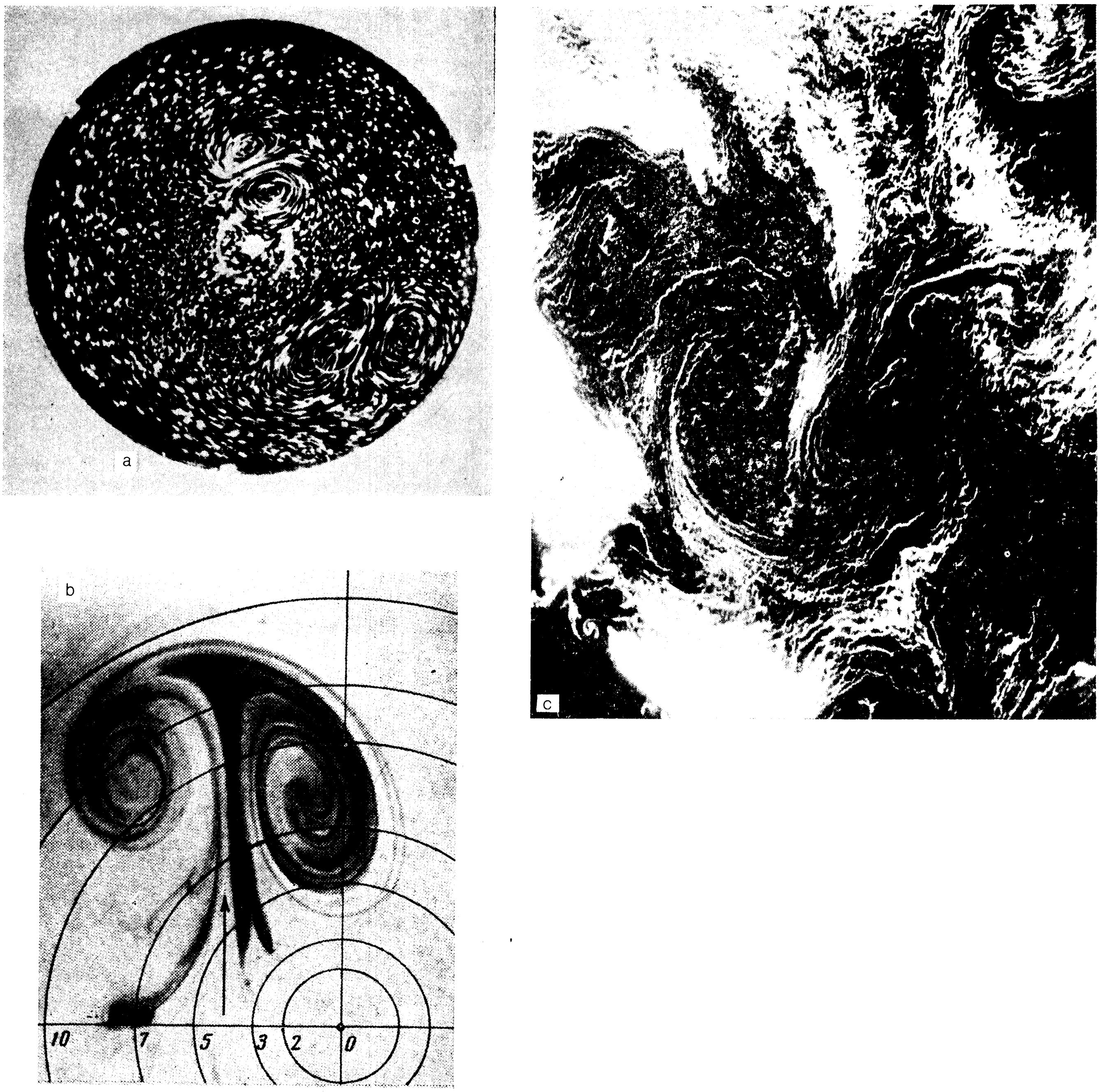

FIG. 5. a—The generation of a modon in a device with rotating shallow water in a vessle with paraboloidal bottom.[26] The rotation of the vessel is counterclockwise (to the "east"). Two modons can be seen: one moves in the direction of rotation ("eastward") with almost vanishing speed (it is situated near the driving disk—the white circle). The second modon propagates against the direction of rotation of the vessel, with speed $v_m > v_R$ (v_R is the Rossby velocity); b—the generation of mushroom-like dipole vortices in a rotating basin with shallow water, produced by a jet of air which is not related to the rotating system;[71] similar structures occur when a counter-current short-duration water jet is injected.[71] In Ref. 73 it is proved that these structures are Larichev-Reznik modons;[32] c—mushroom-shaped vortex structures in the ocean.

monopole vortices (Refs. 26, 56–60), whose observational discovery has either stimulated the development of the theory (e.g., the article 56), or was itself the result of verification of laboratory and theoretical investigations (such as the recent paper 60).[7)]

At the same time, at least in the last three decades, the same astrophysical literature contains numerous examples of observations of pairs of nearby objects. The hierarchy of such objects with respect to their scale is extremely wide: pairs of galaxies (Refs. 61, 62), double galactic nuclei (Refs. 63–66), double stars.[31] The determination of the direction of rotation of spectrally double stars is a problem, which is not solvable in the near future. An analogous problem for eclipsed double stars is now being planned jointly with the Special Astrophysical Observatory (USSR Academy of Sciences) and the Shternberg Observatory (Moscow State University).

We note that the most active among the stars are close by pairs. A remarkable example of such a pair is the object *SS* 433.[67] It was already noted in the literature (Refs. 68–70), the properties of the *SS* 433 system remind one in miniature of those of active galactic nuclei. As regards the deter-

FIG. 6. The isodense picture of the galaxy Markaryan 266 with two nuclei, rotating in the opposite directions.[64]

mination of the direction of rotation of pairs of galaxies and double galactic nuclei, in all cases when these directions are determined they are substantially more often opposite to each other. Figure 6, taken from Ref. 64, shows the picture of isodenses for the galaxy Markaryan 266 with two nuclei rotating in opposite directions. Two-dimensional analogs of the above-mentioned objects may be solitary vortices in laboratory experiments in shallow water[26,71] and in the ocean[72] (Fig. 5); The structure and dynamics of vortices may differ substantially as a function of the conditions of the experiment.[73]

In conclusion we express our sincere gratitude to A. M. Balk, V. E. Zakharov, V. D. Larichev, A. B. Mikhaĭlovskiĭ, S. S. Miseev, M. V. Nezin, and G. M. Reznik for discussion of some aspects of the theory, and to A. I. Ginzburg, B. V. Komberg, A. R. Petrosyan, A. V. Tutukov, É. E. Khachikyan, A. M. Cherepashchuk, and B. M. Shustov for a discussion of the astrophysical applications.

[1)] In addition to the creation of the stability theory of isolated rotating figures (Refs. 1–6), there has been great progress also in the construction of a linear theory of embedded figures of rotation.[7]

[2)] Following Ref. 10, we understand by Kolmogorov spectrum any power-law spectrum of turbulence derived from the assumption of constant flow of energy along the spectrum. We designate as "properly" Kolmogorov the spectrum $E_k \propto k^{-5/3}$, derived in Refs. 11, 12 from dimensional estimates making use of the mentioned hypothesis of constancy of the energy flow.

[3)] These waves were investigated by Margulis in 1893 and then by a series of authors before the publication of the Rossby paper (for references, see Ref. 26). One of the authors of the present paper (A.M.F.) first heard about the same dispersion law for Rossby waves and drift waves from M. A. Leontovich in 1965, long before the publication of the papers by the Japanese authors on this theme (see also Ref. 26).

[4)] This constitutes the distinction between a gravitating compressible medium with $\Delta_\perp \Psi = 4\pi G\rho$ from the case of an incompressible nongravitating fluid,[32] where the second derivative (vorticity) may have a finite discontinuity.

[5)] For the first time the uniqueness of the solution for the two-layer flow was considered by a different method in Ref. 33. The case when the separatrix is not a circle $r = a$ (Fig. 3a), but a loop (Fig. 3b) is shown in Fig. 4 (Ref. 41). The case of discontinuous conditions on a separatrix is considered in Ref. 42.

[6)] The photographs in Fig. 5 have been kindly provided by M. V. Nezlin and A. I. Ginzburg.

[7)] Monopole vortices, in distinction from dipole vortices, are described by a fundamentally different equation—with a so-called "scalar" nonlinearity (we recall that the modons are described by an equation with a vectorial nonlinearity). Such as, e.g., the Korteveg–de Vries equation.

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