Anomalous pairings in the problem of the fractional quantum Hall effect

Yu. A. Bychkov

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences, Chernogolovka
(Submitted 2 July 1990)
Zh. Eksp. Teor. Fiz. **99**, 268–273 (January 1991)

An attempt is made to investigate the problem of the fractional quantum Hall effect (FQHE) on the basis of methods of quantum field theory. It is argued that the use of the anomalous one-particle Green function, corresponding to matrix elements between states with different values of the total angular momentum, and the Green function analogous to the F-function in the theory of superconductivity leads to the result that the pair correlation function acquires properties analogous to those which obtain for the Laughlin trial function. By means of Bogolyubov transformations, equations describing the system in the generalized Hartree–Fock approximation are obtained.

1. INTRODUCTION

In the theory of the fractional quantum Hall effect (FQHE) the work of Laughlin[1] is of fundamental significance. The trial function introduced in Ref. 1 is used in one form or another in most of the investigations devoted to the theory of the FQHE. It is evident that the most delicate element in the trial-function method is the question of the introduction of the elementary excitations. In view of this, it is very tempting to study the problem of the FQHE with the aid of well developed methods from quantum field theory. The degeneracy of the spectrum of two-dimensional electrons in a magnetic field in a given Landau level makes the problem of allowing for the interaction between the particles extremely complicated. Therefore, it is of interest to determine in the language of Green functions even the most general properties of two-dimensional interacting electrons in a strong magnetic field H. In a previous paper[2] we succeeded in establishing an interesting distinctive feature of the diagrams describing the interaction of the electrons in a magnetic field. It was found that in the two-dimensional case it is possible to exclude from the analysis all diagrams containing polarization loops by means of a renormalization, performed in explicit form, of the two-particle interaction potential.

An important feature of the Laughlin trial function is the zeros of high order in the difference of the coordinates of an arbitrary pair of particles. This trial function corresponds to the previously known Jastrow method in the theory of nuclear interactions. However, a method for deriving approximations of the Jastrow type on the basis of a diagram technique is not known. In Ref. 2 the author proposed for the vertex part a concrete form that makes it possible to construct a two-particle correlation function containing zeros of high order in the difference of the particle coordinates. On the other hand, Girvin[3] proposed an alternative form of two-particle correlator that leads to zeros of arbitrary order. At the same time, it seems to be extremely important to operate only with anomalous pairings for the one-particle Green function and with pairings of the Cooper type in the theory of superconductivity. The character of the vertex part introduced in Ref. 2 suggests the thought that anomalous one-particle Green functions corresponding to matrix elements that are nondiagonal in the total angular momentum may be present. However, the form proposed by Girvin[3] for the vertex part suggests that it is possible to introduce pairings of the Cooper type. It should be noted that the idea of the importance of nondiagonal matrix elements in the theory of the FQHE has been developed in one form or another in a whole series of papers.[4–6]

In the present paper, we consider the two types of possible anomalous pairings in the theory of the FQHE, leading to zeros of high order in the two-particle correlation function. An analysis is given of certain properties of the anomalous Green functions. By means of Bogolyubov transformations, equations describing the system in the generalized Hartree-Fock approximation are obtained.

2. ANOMALOUS PAIRINGS

The Hamiltonian of the interaction of polarized electrons (the spin index is omitted below) has the form

$$\hat{H}_{int}=\frac{1}{2}\iint d\mathbf{r}_1\,d\mathbf{r}_2\,\hat{\psi}^+(\mathbf{r}_1)\hat{\psi}^+(\mathbf{r}_2)\hat{\psi}(\mathbf{r}_2)\hat{\psi}(\mathbf{r}_1)V(|\mathbf{r}_1-\mathbf{r}_2|),\quad (1)$$

where

$$\hat{\psi}(\mathbf{r})=\sum_m (2\pi m!2^m)^{-1/2}z^m\exp(-|z|^2/4)\hat{a}_m,\qquad (2)$$

$z=x+iy$, and $\hat{a}_m$ is the annihilation operator for particles in the state with angular momentum m in the symmetric gauge for the vector potential. It is assumed that the particles are in the zeroth Landau level, and that the interaction of the particles does not lead to transitions of particles from one Landau level to another.

The fundamental idea of Laughlin is that a two-dimensional electron gas in a strong magnetic field with filling factor $\nu=2\pi l_H^2 n_s=1/p\,[l_H+(c\hbar/eH)^{1/2}$ is the magnetic length, n_s is the electron density, and p is an odd number] possesses an especially stable ground state. The pair correlation function then has zeros of order no lower than $2p$ in the difference $|z_1-z_2|$ of the coordinates of the two electrons. In Ref. 2 it was shown that the two-particle correlator

$$n^{(2)}(\mathbf{r}_1,\mathbf{r}_2)=\langle\hat{\psi}^+(\mathbf{r}_1)\hat{\psi}^+(\mathbf{r}_2)\hat{\psi}(\mathbf{r}_2)\hat{\psi}(\mathbf{r}_1)\rangle\qquad (3)$$

corresponding to a certain trial function can have zeros of order no lower than $2(2k_0+1)$ in the coordinate difference $|z_1-z_2|$ if the correlator appearing in it satisfies the condition

 0038-5646/91/010151-03$03.00

$$\langle \hat{a}_{m_1}{}^+\hat{a}_{m_2}{}^+\hat{a}_{m_2'}\hat{a}_{m_1'}\rangle \propto \delta_{m_2',m_2+l}\delta_{m_1',m_1-l}(-1)^l\frac{1}{(k_0+l)!(k_0-l)!} \times\left[\frac{(m_2+l)!(m_1-l)!}{m_2!m_1!}\right]^{1/2} \quad (4)$$

and $|l|\leqslant k_0$.

In (4) we have not antisymmetrized over the states, since it was shown in Ref. 2 that the effective interaction potential automatically takes into account only states that are antisymmetric under permutation of the particles.

The factorization of the correlator (4) in the variables m_1 and m_2 argues in favor of the existence of anomalous averages

$$g_{mm'}=\langle \hat{a}_m{}^+\hat{a}_{m'}\rangle,\quad m\neq m'. \quad (5)$$

The anomalous pairings $g_{mm'}$ for $m\neq m'$ arise in the form of matrix elements between states with different total angular momenta. Here it should be noted that the operator of magnetic translation commutes with the Hamiltonian of the system and at the same time does not commute with the operator of the total angular momentum of the system (see Ref. 2). Thus, the anomalous averages $g_{mm'}$ can exist as matrix elements between states with the same energy. It should be noted that the condition $m'=m+l$ with $|l|\leqslant k_0$ requires the presence of correlation only between closest orbits of an electron in a magnetic field, and does not require long-range order.

The other way of introducing anomalous pairings arises from analysis of the two-particle correlation function proposed by Girvin.[3] Analyzing data from calculations of the energy gaps by the Monte Carlo method, he proposed for the correlation function the following form:

$$n^{(2)}(\mathbf{r}_1,\mathbf{r}_2)\propto 1-\exp\left(-\frac{r^2}{2}\right)-2\exp\left(-\frac{r^2}{4}\right)\sum_l{}'\frac{C_l}{l!}\left(\frac{r^2}{4}\right)^l, \quad (6)$$

where $r=|z_1-z_2|$. In (6) the summation is over odd values of the quantity l. By selecting values of the parameters C_l it is possible to obtain zeros of any order in the variable $|z_1-z_2|$. According to Ref. 7, the last term in the expression (6) corresponds to the following anomalous pair correlator:

$$\langle \hat{a}_{m_1}{}^+\hat{a}_{m_2}{}^+\hat{a}_{m_2'}\hat{a}_{m_1'}\rangle =\delta_{m_1+m_2,m_1'+m_2'}\frac{1}{2^{m_1+m_2-1}} \times\sum_l{}'\frac{C_lI_l^{m_1m_2}I_l^{m_1'm_2'}}{l!(m_1+m_2-l)!}(m_1!m_2!m_1'!m_2'!)^{1/2}, \quad (7)$$

where

$$I_l^{m_1m_2}=(-1)^{m_2}\sum_n(-1)^nC^n_{m_1+m_2-l}C_l^{m_2-n} \quad (8)$$

and C_n^m are binomial coefficients.

The factorization in the correlator (7) into factors with the variables m_1, m_2 and m_1', m_2' suggests the possible existence of anomalous pairings of the Cooper type

$$f_{mm'}=\langle \hat{a}_m\hat{a}_{m'}\rangle\propto\left[\sum_l A^{(l)}_{m+m'-l}I_l^{mm'}-(m\leftrightarrow m')\right]. \quad (9)$$

In the derivation of the correlator (7) from the anomalous average (9) it should be taken into account that the matrix element for an isotropic interaction potential differs from zero when both the angular momentum of the relative motion (in the variable z_1-z_2) and the angular momentum of the center of mass (in the variable z_1+z_2) are conserved. In this case the anomalous pairing of the type (9) generalizes the right-hand side of the relation (7).

Thus, we arrive at the conclusion that in the study of the problem of a system of interacting electrons in a strong magnetic field there are grounds for introducing two types of anomalous averages:

$$g_{mm'}=\langle \hat{a}_m{}^+\hat{a}_{m'}\rangle,\quad m\neq m',\qquad f_{mm'}=\langle \hat{a}_m\hat{a}_{m'}\rangle. \quad (10)$$

In connection with such anomalous pairings, we can introduce Bogolyubov transformations involving Fermi quasiparticle operators $\hat{b}_m$ and $\hat{b}_m^+$ by means of the relations

$$\hat{a}_m=\sum_n(U_{mn}\hat{b}_n+V_{mn}\hat{b}_n{}^+),\qquad \hat{a}_m{}^+=\sum_n(V_{mn}{}^*\hat{b}_n+U_{mn}{}^*\hat{b}_n{}^+), \quad (11)$$

or, in a more convenient vector form,

$$\hat{\mathbf{a}}=\mathbf{U}\hat{\mathbf{b}}+\mathbf{V}\hat{\mathbf{b}}^+,\quad \hat{\mathbf{a}}^+=\mathbf{V}^*\hat{\mathbf{b}}+\mathbf{U}^*\hat{\mathbf{b}}^+, \quad (12)$$

where the quantities **U** and **V** are matrices.

In order that the annihilation and creation operators for the quasiparticles satisfy commutation relations, it is necessary that the conditions

$$\mathbf{U}\mathbf{U}^++\mathbf{V}\mathbf{V}^+=1,\quad \mathbf{U}\mathbf{V}^T+\mathbf{V}\mathbf{U}^T=0 \quad (13)$$

be fulfilled, where $\mathbf{A}^T$ is the transposed matrix and $\mathbf{A}^+$ is the adjoint matrix.

It follows from the relations (12) and (13) that

$$\mathbf{b}=\mathbf{U}^+\hat{\mathbf{a}}+V^T\hat{a}^+,\quad \hat{\mathbf{b}}^+=\mathbf{V}^+\hat{\mathbf{a}}+\mathbf{U}^T\hat{\mathbf{a}}^+. \quad (14)$$

By again making use of the commutation relations, we obtain one more condition:

$$\mathbf{U}^+\mathbf{U}+\mathbf{V}^T\mathbf{V}^*=1. \quad (15)$$

Thus, the matrices **U** and **V** of the Bogolyubov transformations should satisfy the relations (13) and (15).

3. THE GENERALIZED HARTREE-FOCK APPROXIMATION

The complete Hamiltonian of interacting two-dimensional electrons in a magnetic field is

$$\hat{H}=\varepsilon\sum_m\hat{a}_m{}^+\hat{a}_m+\frac{1}{2}\sum_{m_1+m_2=m_1'+m_2'}\tilde{U}_{m_1m_2,m_2'm_1'}\hat{a}_{m_1}{}^+\hat{a}_{m_2}{}^+\hat{a}_{m_2'}\hat{a}_{m_1'},$$

where the matrix element $\tilde{U}_{m_1m_2,m_2'm_1'}$ of the effective interaction potential is antisymmetric both in the initial and in the final indices.[2]

In the generalized Hartree-Fock approximation with allowance for the anomalous averages (10) the Hamiltonian (16) goes over into the following Hamiltonian:

$$\hat{H}' = \sum_{m,m'} (\varepsilon\delta_{m,m'} + \alpha_{mm'})\hat{a}_m{}^+\hat{a}_{m'} + \frac{1}{2}\sum_{m,m'} (\beta_{mm'}\hat{a}_m\hat{a}_{m'} + \beta^*_{mm'}\hat{a}_{m'}{}^+\hat{a}_m{}^+), \quad (17)$$

where the elements of the matrices α and β are

$$\alpha_{mm'} = 2\sum_{m_1,m_1'} \bar{U}_{mm_1,m_1'm'}\langle\hat{a}_{m_1}{}^+\hat{a}_{m_1'}\rangle,$$
$$\beta_{mm'} = \sum_{m_1,m_2} \bar{U}_{m_1m_2,mm'}\langle\hat{a}_{m_1}{}^+\hat{a}_{m_2}{}^+\rangle. \quad (18)$$

If we introduce the well known temperature Green functions[8]

$$G(\mathbf{r}, \mathbf{r}'; \tau-\tau') = -\langle T\hat{\psi}(\mathbf{r}, \tau)\hat{\psi}^+(\mathbf{r}', \tau')\rangle,$$
$$F(\mathbf{r}, \mathbf{r}'; \tau-\tau') = \langle T\hat{\psi}(\mathbf{r}, \tau)\hat{\psi}(\mathbf{r}', \tau')\rangle, \quad (19)$$

the Hamiltonian (17) leads to equations for the matrix temperature Green functions in the frequency representation:

$$(i\omega-\varepsilon-\alpha)\mathbf{G}+\beta^+\mathbf{F}^+=1,$$
$$(i\omega+\varepsilon+\alpha^T)\mathbf{F}^++\beta\mathbf{G}=0. \quad (20)$$

The relation between these Green functions and the matrices of the Bogolyubov transformations generalizes the corresponding relations in the theory of superconductivity:[8]

$$\mathbf{G} = \mathbf{U}\frac{1}{i\omega-\mathbf{E}}\mathbf{U}^+ + \mathbf{V}\frac{1}{i\omega+\mathbf{E}}\mathbf{V}^+,$$
$$\mathbf{F}^+ = -\mathbf{V}^*\frac{1}{i\omega-\mathbf{E}}\mathbf{U}^+ - \mathbf{U}^*\frac{1}{i\omega+\mathbf{E}}\mathbf{V}^+, \quad (21)$$

where the diagonal matrix $\mathbf{E}$ describes the energy levels of the quasiparticles. These levels can be found, in principle, from the equations

$$\mathbf{UE} = (\varepsilon+\alpha)\mathbf{U}+\beta^+\mathbf{V}^*,$$
$$\mathbf{VE} = -(\varepsilon+\alpha)\mathbf{V}-\beta^+\mathbf{U}^* \quad (22)$$

with allowance for the conditions (13) and (15) imposed on the matrices $\mathbf{U}$ and $\mathbf{V}$.

The equations (22) are essentially nonlinear, since, according to (18), the matrices α and β appearing in them are themselves functionals of the matrices $\mathbf{U}$ and $\mathbf{V}$. This circumstance, together with the complicated matrix character of the equations, has prevented us from finding the exact solution for the Green functions, even in the Hartree–Fock approximation.

In view of this, it is important to elucidate general properties of the system with allowance for the anomalous pairings. In particular, a very important question is the uniformity of the system. It is possible to require that the average of the commutator of the particle-density operator with the operators of infinitesimal magnetic translations be equal to zero. The latter operators are[2]

$$\hat{t}_+ = \sum_m [2(m+1)]^{1/2}\hat{a}^+_{m+1}\hat{a}_m,$$
$$\hat{t}_- = \sum_m (2m)^{1/2}\hat{a}^+_{m-1}\hat{a}_m; \quad (23)$$

the operator $\hat{t}_+$ raises, and the operator $\hat{t}_-$ lowers, the total angular momentum of the system (both operators commute with the total Hamiltonian of the system, and do not commute with each other). A distinctive feature of the magnetic-translations operators is the fact that $\hat{t}_-$ lowers the order of the zeros of the wave function, while $\hat{t}_+$ does not alter the character of the wave function. In view of this, it is natural to regard the following condition as a characteristic of the uniformity of the system:

$$\langle[\hat{\psi}^+(\mathbf{r})\hat{\psi}(\mathbf{r}), \hat{t}_+]\rangle = 0. \quad (24)$$

It is easy to convince oneself that the relation (24) is fulfilled if the equality

$$(m+1)^{1/2}\langle\hat{a}^+_{m+1}\hat{a}_{m'}\rangle = (m')^{1/2}\langle\hat{a}_m{}^+\hat{a}_{m'-1}\rangle \quad (25)$$

is valid. Here it should be noted that the relation (25) corresponds to the assumption that the correlator

$$\langle\hat{a}_m{}^+\hat{a}_{m'}\rangle \propto (m'!/m!)^{1/2},$$

as follows from the expression (4). Thus, there is every reason to assume that the introduction of anomalous pairings does not lead to violation of the uniformity of the system.

I thank S. V. Iordanskiĭ for discussion of the questions considered.

[1] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[2] Yu. A. Bychkov, Fiz. Tverd. Tela (Leningrad) **31**, No. 7, 56 (1989) [Sov. Phys. Solid State **31**, 1130 (1989)].
[3] S. M. Girvin, Phys. Rev. B **30**, 558 (1984).
[4] S. M. Girvin and A. H. MacDonald, Phys. Rev. Lett. **58**, 1252 (1987).
[5] E. H. Rezayi and F. D. M. Haldane, Phys. Rev. Lett. **61**, 1985 (1988).
[6] N. Read, Phys. Rev. Lett. **62**, 86 (1989).
[7] A. H. MacDonald and S. M. Girvin, Phys. Rev. B **33**, 4414 (1986).
[8] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, *Quantum Field Theoretical Methods in Statistical Physics* (Pergamon Press, Oxford, 1965) [Russ. original, Fizmatgiz, Moscow, 1962], Chap. 7.

Translated by P. J. Shepherd