# Superradiance in layers of excited classical and quantum oscillators

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We present a theoretical study of two-directional superradiance of extended layers of quantum and nonresonant classical oscillators. We find the spatial structures and growth rates of the "hot" eigenmodes of the system which are produced in the linear stage of the superradiance process. The structure of the excited fields becomes complex (stochastization) in the nonlinear stage. We determine the time-dependence of the power of the radiation. We show that the behavior of the macroscopic characteristics (field, polarization, inversion) of the superradiance processes is similar in the quantum and the classical oscillator systems. We give estimates of the power of cyclotron superradiance.

## 1. INTRODUCTION

Great interest has recently been shown in studies of stimulated radiation in spatially bounded active media under conditions where such a medium is completely responsible for the formation of the structure of the emitted field. Amongst such studies we can refer to research on the channeling of radiation in free electron lasers,[1-5] and also to papers about the Dicke superradiance effect.[6] In its original version Dicke superradiance was studied for samples consisting of inverted quantum oscillators.[6-8] It was shown in a number of papers[9-15] that one can observe similar effects also in ensembles of excited classical oscillators with a lifetime which is infinite (on the emission time scale). The aim of the present paper is to consider this analogy using the example of stimulated emission of an extended layer of excited quantum and classical oscillators.[1]

## 2. SUPERRADIANCE OF A LAYER OF QUANTUM OSCILLATORS

We assume that the oscillators form a layer, unbounded in the $x$ and $y$ directions and with a width $b$ in the $z$ direction. The emission of plane TEM waves takes place in the $\pm z$ directions. There are no external electrodynamic systems. We start with an investigation of the superradiance of excited two-level quantum oscillators. In the semiclassical approximation we can use as the initial set of equations the set consisting of the wave equation for the emitted field and the equations for the average polarization $\mathbf{P}$ and the population difference $\Delta N = N_2 - N_1$ per unit volume of the active medium (we assume $\mathbf{P}\uparrow\uparrow\mathbf{d}\uparrow\uparrow\mathbf{x}_0$):[8]

$$\frac{\partial^2\mathbf{E}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=\frac{4\pi}{c^2}\frac{\partial^2\mathbf{P}}{\partial t^2}, \tag{1}$$

$$\frac{\partial^2\mathbf{P}}{\partial t^2}+2T_2^{-1}\frac{\partial\mathbf{P}}{\partial t}+(\omega_0^2+T_2^{-2})\mathbf{P}=-\frac{2d^2\omega_0\Delta N}{\hbar}\mathbf{E}, \tag{2}$$

$$\frac{\partial\Delta N}{\partial t}=-T_1^{-1}\Delta N+\frac{2}{\hbar\omega_0}\mathbf{E}\frac{\partial\mathbf{P}}{\partial t}. \tag{3}$$

Here $\omega_0$ is the transition frequency, $T_{1,2}$ are the longitudinal and transverse relaxation times, $d$ is the dipole moment of the molecules, and $\Delta N_0$ is the initial inversion. In what follows we use the forced solution of the wave equation (1) which can be written in the form

$$\mathbf{E}=-\frac{2\pi}{c}\int_{z-ct}^{z+ct}\frac{\partial\mathbf{P}}{\partial t}(z',t-|z-z'|/c)\,dz'. \tag{4}$$

It is natural to assume that the emission takes place in a spectral range concentrated near the transition frequency $\omega_0$. This enables us to write the emitted field and the polarization of the medium in the form

$$\mathbf{E}=\mathrm{Re}\,[\mathbf{x}_0A(z,t)e^{i\omega_0t}],\quad \mathbf{P}=\mathrm{Re}\,[\mathbf{x}_0P(z,t)e^{i\omega_0t}]. \tag{5}$$

Here $A(z,t)$ and $P(z,t)$ are slowly varying amplitudes for which we get, after averaging Eqs. (2)–(4),

$$\frac{\partial P}{\partial t}+T_2^{-1}P=id^2\,\Delta NA/\hbar, \tag{6}$$

$$\frac{\partial\Delta N}{\partial t}=-T_1^{-1}\,\Delta N+\mathrm{Im}\,(A^*P)/\hbar, \tag{7}$$

$$A=-\frac{2\pi i\omega_0}{c}\int_{z-ct}^{z+ct}P(z',t-|z-z'|/c)\exp(-i\omega_0|z-z'|/c)\,dz'. \tag{8}$$

In what follows we consider the case when the relaxation times are long on the time scale for the evolution of the superradiance instability:

$$T_{1,2}\gg 2\pi/\omega_c, \tag{9}$$

where $\omega_c=(8\pi d^2\omega_0\Delta N_0/\hbar)^{1/2}$ is the cooperative frequency. In this case we can neglect the relaxation processes and reduce the set of Eqs. (6) to (8) to the form

$$\frac{\partial \hat{P}}{\partial\tau}=iIan, \tag{10}$$

$$\frac{\partial n}{\partial\tau}=I\cdot\mathrm{Im}\,(a^*\hat{P}), \tag{11}$$

$$a=-i\int_{Z-\tau}^{Z+\tau}f(Z')\hat{P}(Z',\tau-|Z-Z'|)\exp(-i|Z-Z'|)\,dZ', \tag{12}$$

where we have $\tau=\omega_0t$, $Z=(\omega_0/c)z$, $\hat{P}=P/d\Delta N_0$, $a=A/d\Delta N_0 2\pi$, $n=\Delta N/\Delta N_0$, $I=\omega_c^2/4\omega_0^2$, and $f(Z)$ is a


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function describing the density distribution of the oscillators. The initial conditions for Eqs. (10) to (12) can be written in the form

$$\hat{P}|_{\tau=0}=\hat{P}_0, \quad n|_{\tau=0}=1. \tag{13}$$

We first consider superradiance of a layer which is thin on the scale of the wavelength of the layer: $f(Z) = B\delta(Z)$, where we have $B = (\omega_0/c)b$ and $\delta(Z)$ is a delta-function. For such a layer we have from (12)

$$a=-iB\hat{P}. \tag{14}$$

Taking into account the integral of Eqs. (10) and (11),

$$|\hat{P}|^2+n^2=|\hat{P}_0|^2+1, \tag{15}$$

and using (14) we easily find a solution of these equations. For $|\hat{P}_0| \ll 1$ we have (compare Ref. 8)

$$n=-\mathrm{th}[\Gamma(\tau-\tau_d)], \tag{16}$$

$$\hat{P}=|a|/B=\mathrm{ch}^{-2}[\Gamma(\tau-\tau_d)], \tag{17}$$

where $\tau_d = (1/2\Gamma)\ln(4/|\hat{P}_0|^2)$ is the delay time, and $\Gamma = IB$ is the instability growth rate. According to (16) and (17) the emission has the shape of a pulse; the maximum of its power is reached for $\tau = \tau_d$. After the passage of the pulse the population difference has changed its sign: $n(\tau\to\infty)\to -1$.

We start the study of the emission from an extended layer, $f(Z) = 1, Z\in[-B/2, B/2]$ with an analysis of the linear stage. Assuming that the population inversion is given: $n = 1$, we linearize the set of Eqs. (10) to (12). Writing the solution of the linearized set in the form $a = \tilde{a}(Z)e^{i\Omega\tau}$, $\hat{P} = \tilde{P}(Z)e^{i\Omega\tau}$, we obtain a characteristic equation which determines the frequency $\Omega$ and the spatial structure of the eigenmodes:

$$e^{2ik_iB} = \left(\frac{k_e+k_i}{k_e-k_i}\right)^2, \tag{18}$$

where $k_e = -(1+\Omega)$, $k_i = (k_e^2 - 2Ik_e/\Omega)^{1/2}$ are the wavenumbers outside and inside the layer, respectively. In Fig. 1 we show the growth rates $\Gamma = |\mathrm{Im}\Omega|$ of the symmetric $S_q$ ($q = 1,2,...$) and antisymmetric $A_q$ modes as functions of the width of the layer, found through a numerical solution of Eq. (18). It is clear that as the width of the layer increases there is a successive increase in the number of the mode corresponding to the maximum growth rate. In Fig. 2 we show the spatial structures of the first four modes.

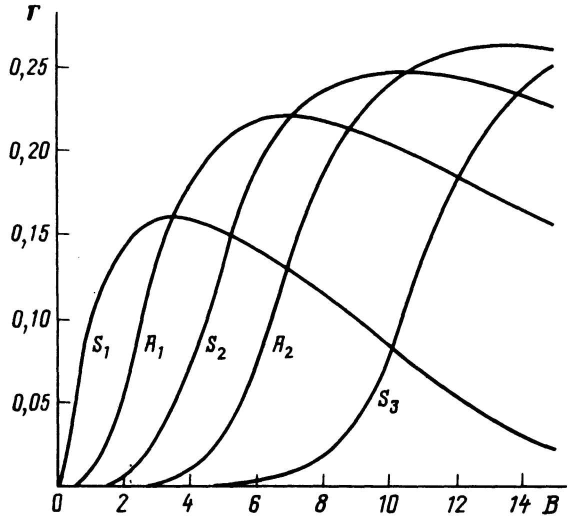



FIG. 1. The growth rates of the symmetric $S_q$ and antisymmetric $A_q$ modes as functions of the width of a layer of quantum oscillators; $I = 0.1$.

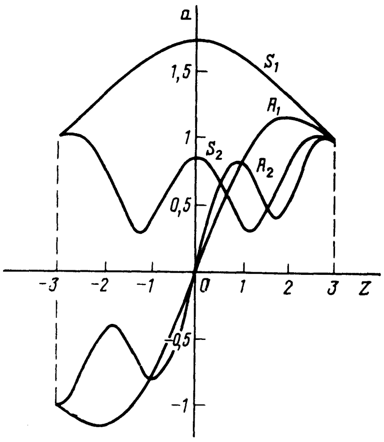



FIG. 2. Spatial structures of the modes in a layer of quantum oscillators; $I = 0.1$; $B = 6$.

Figures 3 and 4 illustrate the results of a numerical simulation of the nonlinear stage of an extended layer of quantum oscillators, using Eqs. (10) to (12). We show in Fig. 3 the time-dependence of the emission power $W = |a|^2|_{Z=\pm B/2}$. In contrast to the single-pulse superradiance regime, which occurs in the case of a thin layer [see Eq. (17)], one observes for an extended layer a number of additional power peaks after the main pulse has passed. The length of the train of pulses which are being emitted in a random sequence increases as the thickness of the layer increases. We show in Fig. 4 the evolution of the amplitude of the field and of the population difference along the layer. In the initial linear region, $\tau < 15$, we see the formation of the spatial structure of the main symmetric mode which is the same as the one shown in Fig. 2. Afterwards in the nonlinear interaction stage the structure of the field becomes complex and ultimately has become stochastic. In contrast to a thin layer the evolution of the amplitude of the polarization and of the population difference is oscillatory in nature. However, as in the case of a thin layer, asymptotically we have $n\to -1$, $\hat{P}\to 0$ for $\tau\to\infty$.

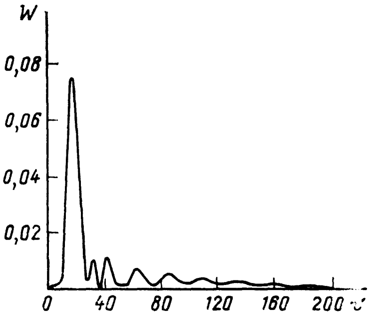
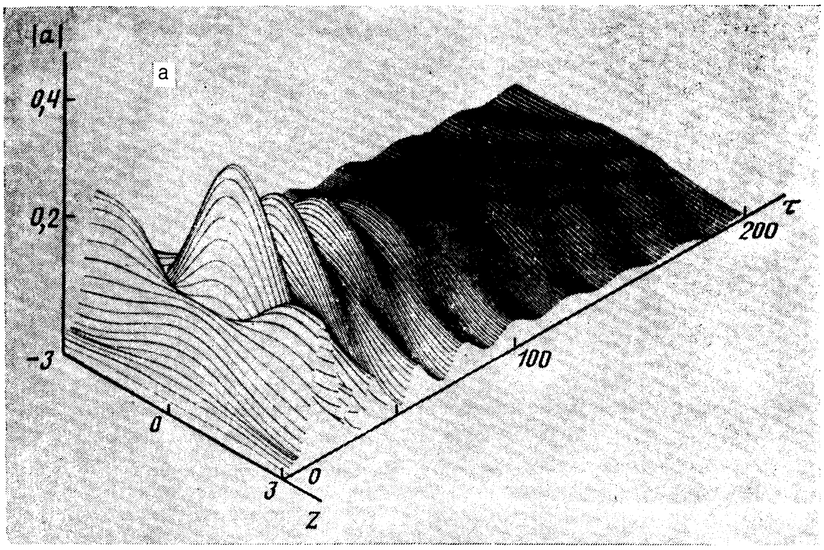



FIG. 3. Time-dependence of the radiation power of a layer of quantum oscillators; $I = 0.1$; $B = 6$.



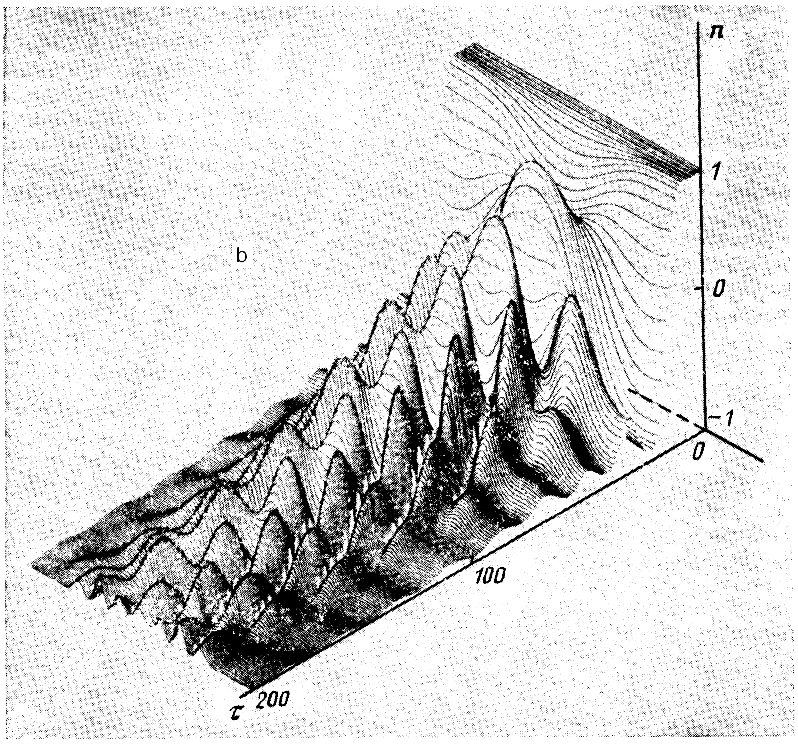




FIG. 4. Evolution of the distributions along a layer of quantum oscillators (a) of the electric field amplitude and (b) of the population difference; $I = 0.1$; $B = 6$.

## 3. SUPERRADIANCE OF A LAYER OF CLASSICAL OSCILLATORS

For the study of superradiance of a layer of excited classical oscillators we assume for the sake of definiteness that such oscillators are electrons gyrating in a uniform magnetic field: $\mathbf{H} = H_0 \mathbf{z}_0$.[2)] Let the electrons have the same initial transverse momentum, $p_{\perp 0} = m\gamma v_{\perp 0}$, and (apart from small fluctuations) be uniformly distributed in the phases of the cyclotron rotation. We assume that the electrons have no translational velocity.

The layer considered will emit circularly polarized waves in the $\pm z$ directions. The motion of the particles will be described by the equations

$$\frac{\partial p_+}{\partial t} - i\omega_H p_+ = -eE_+(z,t), \tag{19}$$

where we have $p_+ = p_x + ip_y$, $E_+ = E_x + iE_y$, $\omega_H = eH_0/mc\gamma$ is the relativistic gyrofrequency, and $\gamma = (1 + |p_+|^2/m^2c^2)^{1/2}$ is the relativistic mass factor.

For the radiation field we have an equation analogous to (4):

$$E_+ = -\frac{2\pi}{c} \int_{z-ct}^{z+ct} j_+(z', t - |z-z'|/c)\, dz', \tag{20}$$

where $j_+ = -e\rho_0 \langle v_+ \rangle$ is the electron current density, we have $v_+ = v_x + iv_y$, $\rho_0$ is the unperturbed electron density in the layer, and $\langle ... \rangle$ indicates averaging over the initial phases of the cyclotron rotation. In what follows we assume that the electrons are weakly relativistic, $\gamma \approx 1 + |p_+|^2/2m^2c^2$, and that the emission is concentrated near the nonrelativistic gyrofrequency $\omega_{H_0} = eH_0/mc$. Correspondingly writing $E_+$ and $p_+$ in the form

$$E_+ = A(z,t)\exp(i\omega_{H_0}t), \quad p_+ = p\exp(i\omega_{H_0}t),$$

we reduce the set of Eqs. (19) and (20) to the form[3)] [compare (10) to (12)]

$$\frac{\partial \hat{p}}{\partial \tau} + i\mu|\hat{p}|^2\hat{p} = -a, \tag{21}$$

$$\hat{p}|_{\tau=0} = \exp[i(\theta_0 + r\cos\theta_0)], \quad \theta_0 \in [0, 2\pi],$$

$$a = I\int_{Z-\tau}^{Z+\tau} f(Z')\langle \hat{p}(Z', \tau - |Z-Z'|)\rangle \exp(-i|Z-Z'|)\,dZ'. \tag{22}$$

We have used here the following dimensionless variables: $\tau = \omega_{H_0}t$, $Z = (\omega_{H_0}/c)z$, $\hat{p} = p/p_{\perp 0}$, $a = eA/m\omega_{H_0}v_{\perp 0}$, $I = \omega_p^2/2\omega_{H_0}^2$, and $\omega_p = (4\pi e^2\rho_0/m)^{1/2}$ is the plasma frequency, $\mu = v_{\perp 0}^2/2c^2$ is the desynchronization parameter, and $r \ll 1$ is a parameter characterizing the initial modulation of the electrons with respect to the phases of the cyclotron rotation.

In the limiting case of a layer which is thin compared with the wavelength of the radiation, $f(Z) = B\delta(Z)$, from (22) we get $a = IB\langle \hat{p}\rangle$. This relation together with the equations of motion (21) describes the superradiance of a thin layer which has been studied earlier in Refs. 12 to 14. We note merely that in contrast to the quantum oscillators the superradiance process of a layer of classical oscillators has a complex many-peak nature even in the case of a thin layer. Below we shall pay most attention to the study of the emission of extended layers with a uniform particle density distribution along the layer: $f(Z) = \text{const}$.

In the small signal approximation, $a \to 0$, we can linearize the equations of motion and we can reduce the set of Eqs. (21) and (22) to the form

$$\frac{\partial^2\langle p\rangle}{\partial \tau^2} = -\frac{\partial a}{\partial \tau} + i\mu a, \tag{23}$$

$$a = I\int_{Z-\tau}^{Z+\tau} f(Z')\langle p(Z', \tau - |Z-Z'|)\rangle \exp[-i(1-\mu)|Z-Z'|]\,dZ'. \tag{24}$$

Writing the solution of (23) and (24) in the form $a = \tilde{a}(Z)e^{i\Omega t}$, $\langle \hat{p}\rangle = \langle \tilde{p}(Z)\rangle e^{i\Omega t}$, we find a characteristic equation which has the same form as (18), where the normalized wavenumbers outside and inside the layer are given by the relations

$$k_e = \mu - \Omega - 1, \quad k_i = (k_e^2 + 2\alpha k_e)^{1/2}; \tag{25}$$

here we have written $\alpha = I(\Omega - \mu)/\Omega^2$.

In the case of a thin layer, $B \ll 1$, we find from (18) and (25)

$$\Omega^2 - iIB\Omega + i\mu IB = 0. \tag{26}$$

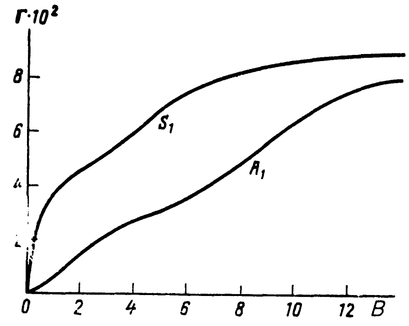



FIG. 5. The growth rates of the first symmetric ($S_1$) and the first antisymmetric ($A_1$) modes as functions of the width of a layer of classical oscillators; $I = 0.1$; $\mu = 0.1$.

For a small layer density, $I \ll 1$, we find, for the instability growth rate, neglecting the second term on the left-hand side of (26) (responsible for cyclotron absorption)

$$\Gamma = |\operatorname{Im}\Omega| = \left(\frac{IB\mu}{2}\right)^{1/2}. \tag{27}$$

We show in Fig. 5 the growth rates of the symmetric and the antisymmetric modes as functions of the layer width for arbitrary layer thicknesses. In the present case the first (main) symmetric mode $S_1$ has the maximum growth rate [the growth rate is for $B \ll 1$ determined by Eq. (27)]. However, as the width of the layer increases the growth rates of the other modes approach the growth rate of the main mode. Figure 6 illustrates the spatial structure of the first symmetric $S_1$ and the first antisymmetric $A_1$ modes.

We show in Fig. 7 the time-dependence of the radiation power and of the total electron efficiency,

$$\eta = 1 - \frac{1}{B}\int_{-B/2}^{B/2} \langle |\hat{p}|^2\rangle\,dZ$$

for different layer thicknesses. A comparison of Figs. 7a and 7b shows that the peak power and the superradiance pulse length increase as the layer thickness increases.

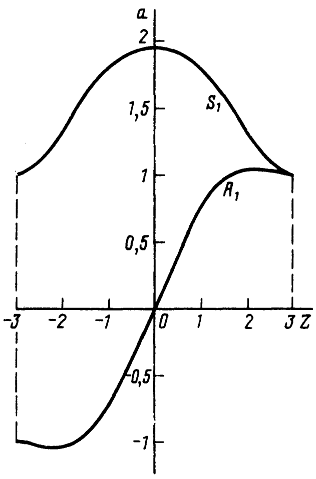



FIG. 6. Spatial structures of the modes in a layer of classical oscillators; $I = 0.1$; $\mu = 0.1$; $B = 6$.

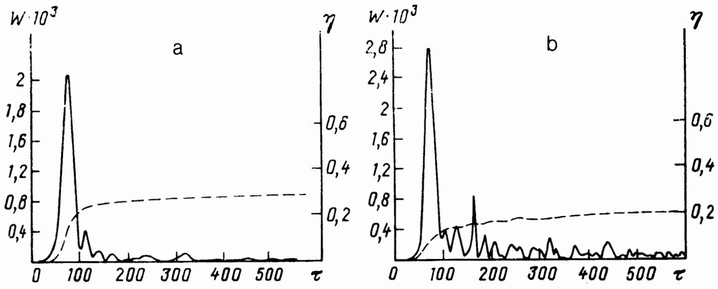



FIG. 7. Time-dependence of the radiation power (full-drawn line) and the electron efficiency (dashed line); $B = 6$ (a) and $B = 12$ (b); $I = 0.1$; $\mu = 0.1$.

We show in Fig. 8a the process of the formation of the spatial structure of the main symmetric mode in the initial linear stage of the emission for $\tau < 60$ and the complication (randomization) of that structure in the nonlinear stage for $\tau > 60$. Figure 8b characterizes the change in the mean square transverse electron momentum $\langle |\hat{p}^2(Z,\tau)| \rangle$ along the layer, i.e., the amount by which the inversion is reduced, which on average is larger at the edges of the layer than in the center. On the whole this behavior is completely analogous to the corresponding behavior for quantum oscillator layers (compare Figs. 3 and 4).

In conclusion we give a numerical estimate of the peak power of the cyclotron superradiance and the pulse length. Let the magnetic field strength be $H_0 = 100$ kOe, the radiation frequency be $\omega = 1.9 \times 10^{12}$ s$^{-1}$ (the wavelength be $\lambda \approx 1$ mm), the electron density be $\rho_0 = 2 \times 10^{14}$ cm$^{-3}$

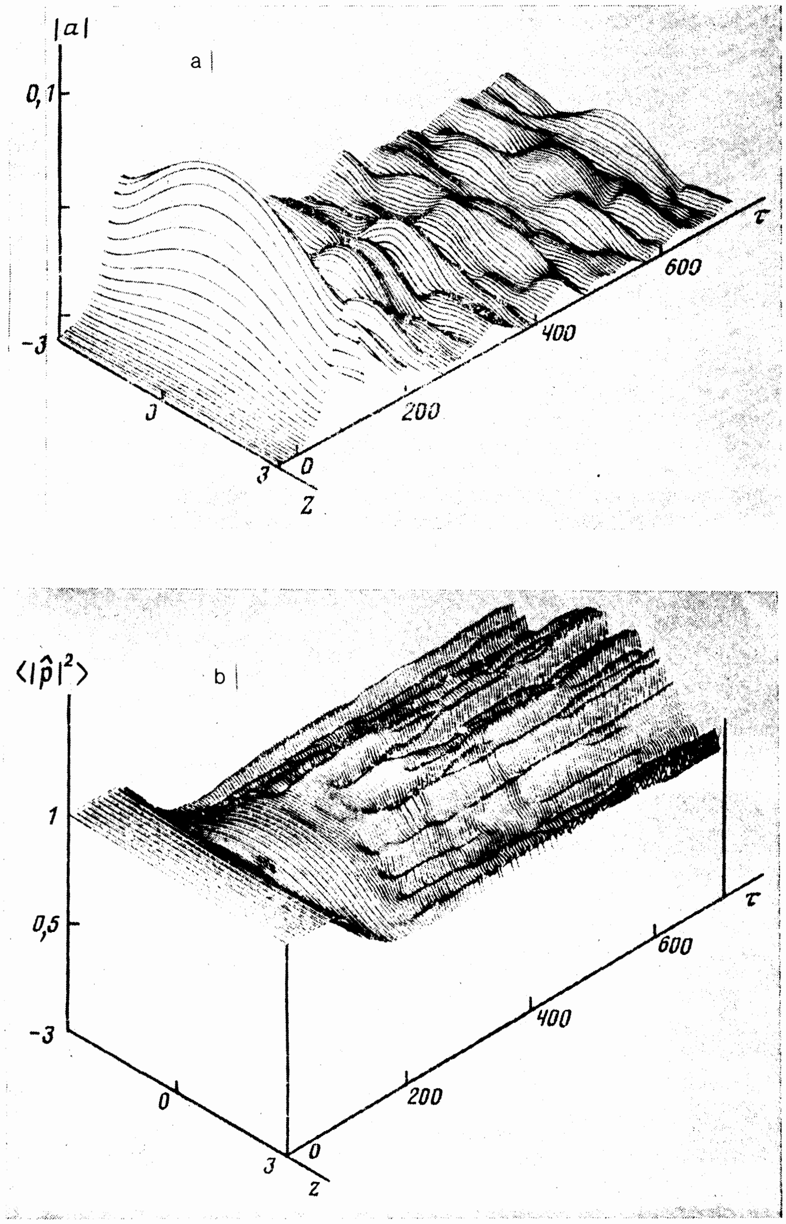



FIG. 8. Evolution of the distributions along a layer of classical oscillators (a) of the electric field amplitude and (b) of the mean square transverse electron momentum; $I = 0.1$; $\mu = 0.1$; $B = 6$.

($\omega_p = 8\times 10^{11}$ s$^{-1}$), and the electron rotational velocity be $v_\perp = 0.45c$. These system parameters correspond to the values $I = 0.1$, $\mu = 0.1$. From Fig. 7 we get for the reduced radiation peak power $W = 2.8\times 10^{-3}$ for a layer width $b \approx 2\lambda$ ($B = 12$). If we restore the dimensions, the power emitted from one square centimeter of the layer surface is 1.4 GW. The length of the pulse one $e$-fold down from the peak power is of the order of $1.5\times 10^{-11}$ cm$^{-1}$. We note that the superradiance power can be appreciably increased and the frequency be shifted to the short-wave range of the spectrum if we give the electron layer a translational velocity close to the light velocity.[13]

[1)] We note for comparison that if in an FEL the electron-oscillator layer plays the rôle of an active waveguide along which the radiation can be channeled and amplified, superradiance effects develop for wave propagation transverse to the layer, when this layer plays the role of an active resonator in which self-excitation of waves takes place.

[2)] The superradiance instability was studied in Refs. 9, 11, and 14 for layers formed when electrons move in the field of an undulator or when a strong electromagnetic pump wave acts on them.

[3)] In this form Eqs. (21) and (22) have a universal character and describe superradiance in oscillator layers of various physical nature, among them also acoustic oscillators (compare Ref. 15).


[1] C. M. Tang and P. Sprangle, *Physics of Quantum Electronics, Vol. 9, Free Electron Generators of Coherent Radiation*, Addison-Wesley, 1982, p. 627.

[2] E. T. Sharlemann, A. M. Sessler, and J. S. Wurtele, Phys. Rev. Lett. **54**, 1925 (1985).

[3] G. T. Moore, Opt. Commun. **52**, 46 (1985).

[4] N. S. Ginzburg and N. F. Kovalev, Pis'ma Zh. Tekh. Fiz. **13**, 274 (1987) [Sov. Tech. Phys. Lett. **13**, 112 (1987)].

[5] N. S. Ginzburg and A. S. Sergeev, Pis'ma Zh. Tekh. Fiz. **14**, 1844 (1988) [Sov. Tech. Phys. Lett. **14**, 800 (1988)].

[6] R. H. Dicke, Phys. Rev. **93**, 99 (1954).

[7] A. V. Andreev, V. I. Emel'yanov, and Yu. A. Il'inskiĭ, Usp. Fiz. Nauk **131**, 653 (1980) [Sov. Phys. Usp. **23**, 493 (1980)].

[8] V. V. Zheleznyakov, V. V. Kocharovskiĭ, and Vl. V. Kocharovskiĭ, Usp. Fiz. Nauk **159**, 193 (1989) [Sov. Phys. Usp. **32**, 835 (1989)].

[9] R. H. Bonifacio, C. Maroli, and N. Piovella, Opt. Commun. **68**, 369 (1988).

[10] V. V. Zheleznyakov, V. V. Kocharovskiĭ, and Vl. V. Kocharovskiĭ, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **29**, 1095 (1986). [Radiophys. Qu. Electron. **29**, 830 (1987)].

[11] N. S. Ginzburg, Pis'ma Zh. Tekh. Fiz. **14**, 440 (1988) [Sov. Tech. Phys. Lett. **14**, 197 (1988)].

[12] Yu. A. Il'inskiĭ and N. S. Maslova, Zh. Eksp. Teor. Fiz. **94**, No. 1, 171 (1988) [Sov. Phys. JETP **67**, 96 (1988)].

[13] N. S. Ginzburg and I. V. Zotova, Pis'ma Zh. Tekh. Fiz. **15**, (14), 83 (1989) [Sov. Tech. Phys. Lett. **15**, 573 (1989)].

[14] N. S. Ginzburg and Yu. V. Novozhilova, Pis'ma Zh. Tekh. Fiz. **15**, (19), 90 (1989) [Sov. Tech. Phys. Lett. **15**, 771 (1989)].

[15] Yu. A. Kobelev, L. A. Ostrovskiĭ, and I. A. Soustova, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **29**, 1129 (1986) [Radiophys. Qu. Electron. **29**, 857 (1987)].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