Density of phonon–fracton states of disordered solids in the vicinity of percolation phase transitions

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The behavior of the effective elastic moduli and of the density of phonon–fracton states near percolation phase transitions in elastic isotropic solids is investigated by the field renormalization-group method.

The development of a theory of phase transitions in disordered materials is still one of the central problems in solid-state physics. The model of a percolation phase transition plays the same role among the models put forward to account for phase transitions in disordered media as does the Ising model for second-order phase transitions in ideal crystals. In addition to the clear picture of the processes occurring in the course of a percolation phase transition, a scaling theory has been developed and various techniques have been used to calculate the critical exponents describing the thermodynamics of a medium in the vicinity of the percolation threshold.

An enormous amount of work has been done on the behavior of the effective transport coefficients of a percolation medium, including the thermal conductivity, diffusion coefficients, tunnel conductivity, and elastic moduli (see, for example, Refs. 1–13). However, the attention has been concentrated on the dynamic properties of "limiting" problems such as composition-induced insulator–metal and metal–superconductor phase transitions, statistics of random walk of "an ant or a termite in a maze" or calculation of the effective elastic moduli of harmonic lattices with randomly broken bonds. In all such cases the ratio of the values of the parameters of different phases is zero. The progress in the description of the transport coefficients of such media has depended largely on simple scaling relationships following from their fractal (self-similar geometry) within certain spatial scales.[4,10,14] However, such fractal objects can be modeled quite readily numerically or physically, so that the values of the critical exponents describing the behavior of a number of transport properties of "limiting media" near the percolation threshold are known quite accurately.[5,15,16]

Analytic calculations of the critical exponents have required much more intensive effort and have been proved to be technically more complicated than in the case of the thermodynamics of second-order phase transitions characterized by thermal fluctuations. The existence of scaling relationships makes it possible to reduce the problem of determinating the composition dependence of the coefficient describing the stiffness of spin waves in an isotropic Heisenberg magnetic material and the effective diffusion coefficient for a random walk of the "ants in a maze" type to calculation of the exponent t of the electrical conductivity of a random percolation network. Calculations of the exponent, carried out using the field theory with a nontrivial effective Hamiltonian requiring series of delicate limiting transitions in a number of parameters, were carried out to first order in $\varepsilon = 6 - d$ (d is the dimensionality of space and $d_c = 6$ is the upper critical dimensionality of the theory) and were reported in Refs. 17–19 (see also Ref. 20).

However, it is important to note that for the majority of phase transitions that occur in solids with large-scale inhomogeneities the ratios of the diffusion coefficients, electrical conductivity, elastic moduli, and densities of various phase states do not differ greatly from unity. If the framework of a phenomenological scaling approach to the description of the behavior of an effective characteristic of a two-phase medium near a percolation threshold is adopted, it is convenient to introduce a parameter h ($0 \leqslant h \leqslant 1$) as the ratio of the corresponding coefficients of different phases.[8] The parameter h plays the role of a dimensionless external field and the value $h = 0$ corresponds to the above-mentioned "limiting" problems in two possible ways when the value of the coefficient for one of the two phases is zero or infinity. If the solutions for both versions are known, we can construct scaling asymptotes of the effective characteristics of media with $0 < h \ll 1$.

We shall adopt a field-theoretic approach in a study of acoustic properties of disordered solids undergoing percolation phase transitions characterized by $h \approx 1$. Among these transitions we shall concentrate on the case with the simplest type of striction interaction when the solution of a stochastic vector differential equation of motion describing the behavior of an elastic medium in the critical region can be reduced to a scalar equation.

The results of our calculations by the field renormalization group method confirmed the existence of the scaling relationships between the critical exponents and also the conclusion on the nature of short- and long-wavelength vibrations near the percolation threshold, which follow from phenomenological considerations of the scaling theory. The values of the upper critical dimensionality and of the critical exponents of the problem will be shown to differ from the values applicable to percolation phase transitions characterized by $h \ll 1$.

PERTURBATION THEORY AND REGION OF STRONG FLUCTUATIONS IN A PERCOLATION ELASTIC MEDIUM

We shall assume that we can describe a phase transition in a solid with large-scale inhomogeneities (due to, for example, the presence of extended clusters of impurities or intrinsic point defects) simply by introducing a Gaussian random field of local values of the phase transition temperature $T_c(\mathbf{r})$ whose correlation scale is identical with a characteris-

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tic dimension $R_0 \gg r_c$ (r_c is the radius of thermal fluctuations). We assume that the local phase transition is of the first order, as is true of the majority of structural and ferroelectric transitions. Since we are mainly interested in a fairly narrow range of temperatures $\Delta T \sim \Delta T_d \ll \overline{T}_c$ [ΔT_d is the variance of the field $T_c(\mathbf{r})$ and $\overline{T}_c$ is its average value] near the percolation threshold, it is convenient to approximate the temperature dependences of the components of the order parameter by "step" functions[1)]

$$P_i(T) = P_i^* \theta(T_c(\mathbf{r}) - T), \quad \theta(a<0)=0, \quad \theta(a>0)=1.$$

Then, the local values of the elastic moduli and of the density occurring in the equation of motion of an inhomogeneous linear elastic medium

$$\rho(\mathbf{r}) \delta_{ij} \ddot{u}_j - \partial_k \{c_{ikjm} \partial_m u_j\} = \hat{L}_{ij} u_j = 0, \qquad (1)$$
$$\partial_k = \partial / \partial r_k$$

can be expressed in terms of an indicator function $\theta(\mathbf{r})$ of the new phase:

$$\rho = \rho_0 [1 + \varepsilon_{ii}^* \theta(\mathbf{r})], \quad c_{ijkm}(\mathbf{r}) = c_{ijkm} + \Delta c_{ijkm} \cdot \theta(\mathbf{r}),$$

where ε_{ii}^* and Δc_{ijkm} are discontinuities of the dilatation and of the elastic moduli that accompany a local phase transition. We show below that anomalous acoustic properties are due to an increase in the dimensions of a typical cluster near the percolation threshold, so that random fields $\rho(\mathbf{r})$ and $c_{ijkm}(\mathbf{r})$ can be regarded as static. The dynamic tensor of the effective elastic moduli $\tilde{c}_{ikjm}(\omega, \tau_0)$, where $\tau_0 = (x - x_c)/x_c$, x is the fraction of the new phase, and x_c is the critical value of this fraction, and also the density of states $N(\omega, \tau_0)$ can be determined provided we calculate the coordinate and time Fourier transforms of the averaged—over the field configurations $\theta(\mathbf{r})$—retarded Green's function $\langle G_{ik}(\omega, \tau_0)\rangle$ of Eq. (1):

$$\hat{L}_{ij} G_{jk} = \delta_{ik} \delta(\mathbf{r}) \delta(t), \quad G_{ik}(t<0) = 0.$$

The correlation corrections to the zeroth approximation of this Fourier transform have a relatively simple tensor structure only in the case of elastic isotropic phases and a quadratic striction of the type

$$H_{int} = q \int P^2(\mathbf{r}) \varepsilon_{ii}(\mathbf{r}) d^d r.$$

Below we consider precisely these phase transitions, but they are selected not only in order to simplify the matrix notation, but also because of the possibility of a radical change in the whole spatial structure of a heterophase state of a material, compared with a percolation medium, when these conditions are not satisfied. In fact, in the latter case the shape, dimensions, and spatial positions of the nuclei of a new phase are governed not only by the local values of the random field $T_c(\mathbf{r})$, but also by the interaction between these nuclei via long-range elastic fields.

In the case of crystals with sufficiently few defects a heterophase structure of the percolation type, which appears at the beginning of the phase transition process ($x \ll 1$), may be transformed—because of the elastic interaction between the nuclei—in a certain range of intermediate values of the fraction of the new phase near $x \approx 1/2$ over distances of mesoscopically large scale $l \sim (L r_c)^{1/2}$ (L is the size of a sample) into a regular superstructure[2)] (Refs. 21 and 22). Such a phase transition has been observed experimentally for some crystals (see, for example, Refs. 23 and 24), but we shall consider an alternative variant of a percolation-type phase transition when the spatially ordered heterophase structure does not appear in an elastic isotropic medium with the dilatation striction in the range $x \ll 1$ (up to $x = x_c$).

We determine the range of validity of perturbation theory on approach to the percolation threshold by calculating $\langle G_{ik}\rangle$, iterating Eq. (1) subject to the fact that the pair correlation functions $\langle \Delta\rho(\mathbf{r}) \Delta\rho(\mathbf{r}')\rangle$, $\langle \Delta K(\mathbf{r}) \Delta K(\mathbf{r}')\rangle$, and $\langle \Delta\rho(\mathbf{r}) \Delta K(\mathbf{r}')\rangle$ are proportional to the Green's function in the continuum percolation theory:

$$C(\mathbf{r}-\mathbf{r}') = [\langle \theta(\mathbf{r}) \theta(\mathbf{r}')\rangle - \langle\theta\rangle^2] \propto (R_0/r) \exp[-r/R],$$

where the correlation radius is $R = R_0 |(x - x_c)/x_c|^{-\nu}$. In the limit $x \to x_c$ the perturbation-theory iteration series for $\langle G\rangle$ corresponds to the following expansion for the dynamic effective bulk modulus:

$$\tilde{K}(\omega, \tau_0) = \overline{K}\{1 + \alpha_{im} \langle \Delta a_i \Delta a_m\rangle (kR_0) \ln(kR)$$
$$+ \beta_{ijmn} \langle \Delta a_i \Delta a_j\rangle \langle \Delta a_m \Delta a_n\rangle (kR_0)^2 (kR) \ln(kR) + \ldots\}, \quad (2)$$

where $\overline{K} = K_0 + xK^*$ and $\langle \Delta a_m \Delta a_n\rangle$ are averages of the form

$$\langle (\Delta K/K)^2\rangle = (\Delta K^*/\overline{K})^2 x(1-x),$$

α_{ij} and β_{ijmn} are complex numerical coefficients, and k is the average wave number. It follows from Eq. (2) that near the percolation threshold the effective dimensionless perturbation theory parameters are

$$g_{ij} \approx \langle \Delta a_i \Delta a_j\rangle (kR_0)(kR).$$

The diagonal elements of g_{ii} can be given a clear physical interpretation: $g_{ii} \approx R/l_i$, where l_i is the mean free path of a phonon calculated in the approximation of single scattering by spatial fluctuations of the density or of the elastic modulus.

The range of validity of perturbation theory is limited by the values $R < \min(l_i) \equiv R_0/\mathrm{Gi}$, where Gi is a dimensionless parameter which in our problem plays a role similar to the Ginzburg–Levanyuk parameter for a second-order phase transition. Clearly, in an analysis of acoustic properties of a percolation medium in the critical region when $\tau_0 < \mathrm{Gi}$ we have to carry out an effective summation of the series (2). Before doing this, we show that in the isotropic model an investigation of the solutions of the vector equation (1) in the range $R > R_0/\mathrm{Gi}$ can be replaced by an analysis of the solutions of scalar equations with a fluctuating density.

We first consider the scattering of an acoustic wave employing a simpler model of an elastic isotropic medium in which the density ρ_0 and the shear modulus μ_0 are constant, and only the bulk modulus $K(\mathbf{r}) = K_0 + \Delta K \cdot \theta(\mathbf{r})$ fluctuates (this is known as a Hill medium). In this case Eq. (1) becomes

$$\rho_0 \ddot{u}_i - [\partial_i (K(\mathbf{r}) + 4\mu_0/3) \partial_j + \mu_0 (\delta_{ij} \Delta - \partial_i \partial_j)] u_j = 0. \qquad (3)$$

We represent the field $u_i(\mathbf{r}, t)$ in the form of a sum of the potential and solenoidal fields:

$$\mathbf{u} = \mathbf{v} + \mathbf{w}, \quad \mathrm{curl}\ \mathbf{v} = \mathrm{div}\ \mathbf{w} = 0.$$

Applying the curl and div operations to Eq. (3), we can show that the components of the field w_i satisfy the following equations:

$$\rho_0 \ddot{w}_i - \mu_0 \Delta w_i = 0, \tag{4}$$

i.e., the transverse waves are not scattered. The field of a longitudinal wave is described by

$$\rho_0 \ddot{v}_i - \partial_i [K(\mathbf{r}) + 4\mu_0/3]\, \partial_j v_j = 0. \tag{5}$$

Introducing the scalar potential $\Phi(\mathbf{r}, t)$ of the field $v_i = \partial_i \Phi$, we obtain the following equation describing it:

$$\rho_0 \ddot{\Phi} - [K(\mathbf{r}) + 4\mu_0/3] \Delta \Phi = 0. \tag{6}$$

Therefore, the problem of propagation of sound in a two-phase Hill medium reduces to a solution of the scalar equations (4) and (6) for any fraction of the second phase $0 \leqslant x \leqslant 1$. We now turn back to the problem of acoustic fields in a solid that undergoes a percolation phase transition, for which not only the modulus $K(\mathbf{r})$, but also the density

$$\rho(\mathbf{r}) = \rho_0 [1 + \varepsilon_{ii}^* \theta(\mathbf{r})]$$

exhibits fluctuations. We can readily demonstrate that in the critical region near the percolation threshold $(R > R_0/\mathrm{Gi})$ the equations describing the propagation of an acoustic wave are obtained by a simple replacement of ρ_0 with $\rho(\mathbf{r})$ in Eqs. (4)–(6). In fact, singular corrections to the effective values of the elastic moduli appear when we evaluate the relevant integrals originating from narrow $(\sim 1/R)$ intervals of the momenta in the vicinity of poles of the bare phonon Green's functions. On the other hand, terms of the $\partial_m u \partial_n \theta$, $u \partial_m \partial_n \theta$,... type, which appear when the ∂_i operation is applied to Eq. (1) for media with a fluctuating density $\rho(\mathbf{r})$, fail to make singular contributions in the critical region near the percolation threshold. It should be mentioned that it is precisely because of their pole origin that the corrections due to the fluctuations $\rho(\mathbf{r})$ and $K(\mathbf{r})$ are equally singular [compared with the expansion described by Eq. (2)].

RENORMALIZATION PROCEDURE AND SOLUTION OF THE RENORMALIZATION GROUP EQUATIONS IN THE CRITICAL REGION

It therefore follows that in the critical region we have to solve the scalar equations (4) and (6), replacing ρ_0 with $\rho(\mathbf{r})$. We first establish a simple relationship between the average Green's functions of these equations in the case of the Gaussian statistics of the field $\theta(\mathbf{r})$. We introduce into the relevant equation a monochromatic source $I \exp(i\omega t)$. Then the amplitude of a longitudinal wave $\tilde{u}$ in the critical region obeys

$$[\rho_0 \omega^2 (1 + \Delta_\rho \theta) + K_1 (1 + \Delta_K \theta) \Delta] \tilde{u} = -I, \tag{7}$$
$$K_1 = K_0 + {}^4/_3 \mu_0,$$

where $\Delta_\rho = \Delta \varepsilon_{ii}^*$ and $\Delta_K = \Delta K^*/K_1$. After multiplying by $(1 + \Delta_K \theta)^{-1}$ and substituting $\theta = x + \varphi$, Eq. (7) becomes

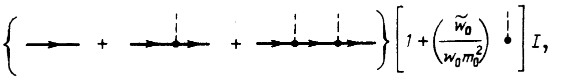
$$\hat{L}\tilde{u} \equiv -[m_0^2 + \Delta + w_0 \varphi] \tilde{u} = [1 + (\tilde{w}_0/m_0^2) \varphi] I(\mathbf{r}), \tag{8}$$

where

$$m_0^2 = \rho_0 \omega^2 K_1^{-1} [1 + x(\Delta_\rho - \Delta_K)(1 + \Delta_K)^{-1}],$$
$$w_0 = \rho_0 \omega^2 K_1^{-1} (\Delta_\rho - \Delta_K)(1 + \Delta_K)^{-1}, \quad \tilde{w}_0 = -\Delta_K w_0 (1 + \Delta_K)^{-1}.$$

Expanding the operator L^{-1} in powers of $w_0 \varphi$, we obtain the solution of Eq. (8) in the form

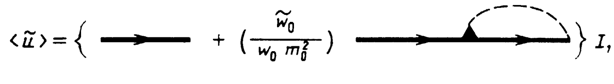
$$\tilde{u}(\mathbf{r}) = \{\hat{D}_0 + \hat{D}_0 w_0 \varphi \hat{D}_0 + \hat{D}_0 w_0 \varphi \hat{D}_0 w_0 \varphi \hat{D}_0 + \ldots\} \cdot [1 + (\tilde{w}_0/m_0^2) \varphi] I$$



(9)

where the operator $\hat{D}_0 = -(m_0^2 + \Delta)^{-1}$ is represented by the solid lines, the field φ by the dashed lines, and the factor w_0 by a dot.

Averaging Eq. (9) over fluctuations of the Gaussian field $\varphi(\mathbf{r})$, we obtain



(10)

where the thick line is the average Green's function $\hat{D} = \langle L^{-1} \rangle$ of Eq. (8) without a fluctuation correction to I, whereas the triangle represents a full three-legged vertex; the dashed curve is the Green's function in the continuum percolation theory. Using the "skeletal" representation of the Dyson equation, we find that the Fourier transform of the field $\langle \tilde{u}(\mathbf{p}) \rangle \equiv u(\mathbf{p})$ is described by

$$u(\mathbf{p}) = D(\mathbf{p}) [1 + \Sigma(\mathbf{p}) \tilde{w}_0 (w_0 m_0^2)^{-1}] I(\mathbf{p}) \equiv G(\mathbf{p}) I(\mathbf{p}),$$
$$\Sigma = D_0^{-1} - D^{-1}. \tag{11}$$

The statistics of percolation clusters is known to be described by a non-Gaussian effective Hamiltonian.[25] However, in an analysis of the scalar stochastic equation [of the form of Eq. (4) with a random density $\rho(\mathbf{r})$] in our previous investigation[26] we found that all the qualitative features of the behavior of the Green's function $D(\omega, \tau_0)$ of this equation, averaged over the percolation field, are already included in the Gaussian approximation. Non-Gaussian graphs simply result in some changes (of the order of the Fisher exponent of the percolation theory η) in the critical exponents of the relevant asymptotes, but this complicates greatly the mathematical treatment. Since the numerical values of the exponents are approximate in any variant of these calculations, we describe the percolation medium by a Gaussian random field $\varphi(\mathbf{r})$ whose correlation function is identical with the percolation-theory Green's function $C(\mathbf{r} - \mathbf{r}')$.

It follows from Eq. (11) that we can find the Green's function of Eq. (7) by calculating the Green's function D of the equation

$$-[m_0^2 + \Delta + w_0 \varphi] u = I. \tag{12}$$

The problem of calculation of $D(\mathbf{r} - \mathbf{r}')$ can be represented in the Hamiltonian form by writing down D in the standard manner

$$D(\mathbf{r} - \mathbf{r}') = c \left\langle \int D u_k D u_k^+ u_1(\mathbf{r}) u_1^+(\mathbf{r}') \exp(-H) \right\rangle_\varphi,$$
$$H(u, \varphi) = -\int d^d r \left\{ \sum_{n=1}^{N} u_n^+ [m_0^2 + \Delta + w_0 \varphi] u_n \right\}. \tag{13}$$

Here, c is the normalization constant and $N \to 0$. In view of the Gaussian nature of the field φ, the effective Hamiltonian needed to calculate D is obviously of the form

$$\bar{H}=H(u,\varphi)+\int d^d r\{{}^1/_2(\tau\varphi^2+(\partial_i\varphi)^2)\},\quad \tau=\tau_0^{2v}. \qquad (14)$$

Before renormalizing the Hamiltonian (14), we first demonstrate that, because of the absence of loops of φ ($N\to 0$), the part of the Hamiltonian which depends directly on this field is not renormalized. When considering the remaining part of the Hamiltonian (14) derived within the framework of dimensional regularization and the scheme of minimal residues (see, for example, Refs. 27 and 28), we note that the only nontrivial singular graph is the first correction to the 1-irreducible pair Green function of the field u, which diverges logarithmically for $d=4$. Consequently, the renormalization constant of the field $u(\mathbf{r})$ is trivial: $Z_u=1$.

A renormalized form of perturbation theory in a ($d=4-2\varepsilon$)-dimensional space is obtained using a Hamiltonian which differs from Eq. (14) because w_0 and m_0^2 are replaced with the renormalized values of wM^ε and m^2:

$$w=w_0M^{-\varepsilon},\quad m^2=m_0^2+\Delta m^2.$$

The renormalization mass M is an additional parameter with the dimensions of momentum. The counterterm in the renormalization of the mass Δm^2 is determined by a simple pole with respect to $1/\varepsilon$ for a single-loop graph of the self-energy part Σ:

$$\Delta m^2=w^2/(16\pi^2\varepsilon). \qquad (15)$$

The existence of the renormalization-group equations follows from the invariance of any quantity that depends on the set of unrenormalized parameters $e_0=\{\tau,\ m_0^2,\ w_0\}$ when M is altered:

$$\hat{R}A(e_0)=D_MA|_{e_0}=[D_M+(D_Mw)\partial_w+(D_Mm)\partial_m]A|_e=0, \qquad (16)$$

where $e=\{\tau,\ m^2,\ w,\ M\}$. Using the explicit form of Eq. (15), we obtain an expression for the operation $\hat{R}$ in terms of the renormalized variables:

$$\hat{R}=D_M-\varepsilon w\partial_w-2c_0w^2\partial/\partial m^2. \qquad (17)$$

In Eq. (17), we have $D_M=M\partial_M$, $\partial_s=\partial/\partial s$, $c_0=(16\pi^2)^{-1}$. Substituting the variables $v=c_0w^2/m^2$ in Eq. (17), we can represent the renormalization group operator $\hat{R}$ in the form

$$\hat{R}=D_M+\beta_v\partial_v-vD_m,\qquad \beta_v=2v(v-\varepsilon). \qquad (18)$$

We can see that the β_v function of Gell-Mann and Low has a nontrivial fixed point $v_*=\varepsilon$, which is infrared-stable:

$$(\partial\beta_v/\partial v)|_{v=v_*}=2\varepsilon>0,$$

whereas the charge v plays the role of the anomalous dimensionality γ_m of the mass m.

Using the existence of a stable fixed point $v_*=\varepsilon$, we can calculate the average Green's function $D(\mathbf{p})$ in the vicinity of a pole $p^2=m_*^2$ in the form of an unrenormalized perturbation-theory series in v_*, we can then determine the density of states $N(\omega,\ \tau)$, as well as the effective moduli $\tilde{K}$ and $\tilde{\mu}$. First of all, we shall find the position of a pole, governed by the condition $D^{-1}(p^2=m_*^2)=0$, and in the zeroth approximation we have $m_*^2=m^2$. Since m_*^2 (like D) is a renormalization group invariant, it satisfies the following equation:

$$\hat{R}\ m_*^2=(\beta_v\partial_v-d_mD_y-2D_z)m_*^2=0, \qquad (19)$$

where $y=m^2/M^2$, $z=\tau/M^2$, $d_m=2(1+v)$. The general solution of Eq. (19) in the critical range can be written in the form

$$m_*^2=m^2y^{-\alpha}P(zy^{-\beta}),\qquad \alpha=v_*(1+v_*)^{-1},\quad \beta=(1+v_*)^{-1}. \qquad (20)$$

Calculating m_*^2 in the form of a series in v_*, we obtain asymptotic forms of the function $P(s)$:

$$P(s\to 0)=P_1\exp(i\pi\varkappa\alpha),\quad \varkappa=\text{sign}\ \omega,\qquad P(s\to\infty)=Q_1s^{-\alpha/\beta}, \qquad (21)$$

where P_1 and Q_1 are real positive constants. The function $Z=(p^2-m_*^2)D(\mathbf{p})$ also satisfies the renormalization group equation $\hat{R}Z=0$, the general solution of which is

$$Z\left(\frac{m_*^2}{p^2},y,z,v\right)=Z\left(\frac{m_*^2}{p^2},\tilde{y},\tilde{z},\tilde{v}\right), \qquad (22)$$

where $\tilde{y}$, $\tilde{z}$, and $\tilde{v}$ are the first integrals described by the equations

$$\frac{dt}{t}=\frac{d\tilde{v}}{\beta_v(\tilde{v})}=-\frac{d\tilde{y}}{\tilde{y}d_m(\tilde{v})}=-\frac{d\tilde{z}}{2\tilde{z}},\quad \tilde{b}(t=1)=b. \qquad (23)$$

In the critical range we have $Z=Z(m_*^2/p^2,\ zy^{-\beta})$, but for $z=0$, we obtain

$$Z\approx A[p^2/(p^2-m_*^2)]^\delta,$$

where $\delta\approx\varepsilon$, while for $zy^{-\beta}\gg 1$ we find that $Z\approx B$, where A and B are real positive constants.

The condition $zy^{-\beta}\sim 1$, which determines the change in the asymptotic forms of the functions P and Z, implies the existence of a characteristic frequency

$$\omega_*\propto[\max(\Delta_\rho,\ \Delta_h)]^{-1}|(x-x_c)/x_c|^{v(1+\varepsilon)}. \qquad (24)$$

For $\omega\gg\omega_*$, the asymptotic expression for the effective elastic moduli $\tilde{K}$ and $\tilde{\mu}$ is determined by the fixed point $v_*=\varepsilon$ and is given by

$$\tilde{K}(\omega)\propto\tilde{\mu}(\omega)\propto\omega^{2\alpha}e^{-i\pi\varkappa\alpha}. \qquad (25)$$

This asymptotic form is reached by the motion of an invariant charge $\tilde{v}(t)$ to the fixed point v_* along a path

$$\tilde{v}(t)=vv_*/[v_*t+v(1-t)],\qquad t=z^{1/2}\sim|(x-x_c)/x_c|^v\to 0 \qquad (26)$$

at a fixed frequency.

The anomalous dispersion law of elastic waves and the appearance of a large imaginary part in the case of the moduli $\tilde{K}(\omega)$ and $\tilde{\mu}(\omega)$ in Eq. (25) corresponds to a change in the nature of elastic vibrations of the solid at $\omega>\omega_*$ from a phonon to a fraction, when short-wavelength vibrations become localized because of self-similarity of a heterophase structure on the scale of $r<R$. For $\omega<\omega_*$, the renormalization group equations admit the possibility that in the vicinity of the fixed point $v_*=\varepsilon$ the solutions for the effective moduli depend on the proximity to the percolation threshold in accordance with the power law:

$$\tilde{K}(\tau) \propto \tilde{\mu}(\tau) \propto |(x-x_c)/x_c|^{2\nu\varepsilon}. \quad (27)$$

However, in the case of percolation phase transitions characterized by small abrupt changes $\Delta\rho^*/\rho_0$ and $\Delta K^*/K_0$ ($h \approx 1$), which are under consideration here, the critical asymptotic form of Eq. (27) is not reached, because the motion of an invariant charge along the appropriate path $t = y^{1/d_m}$ does not result in the attainment of the fixed point $v_* = \varepsilon$. When the effective moduli $\tilde{K}$ and $\tilde{\mu}$ are calculated for the range $\omega \ll \omega_*$, it is sufficient to use unrenormalized perturbation theory. In general, the reason for the phonon nature of long-wavelength vibrations in the range $x < x_c$ is the existence of a translation symmetry of a percolation medium over distances $r > R$. At the percolation threshold, when $\omega_* \to 0$, the geometry of a heterophase structure becomes self-similar for all the physically attainable scales and the fracton nature of localized vibrations changes completely to the phonon nature.

The density of states $N(\omega, \tau)$ for long- and short-wavelength asymptotes can be calculated using the standard expression

$$N(\omega, \tau) = -(2\omega/\pi)\,\mathrm{Im}\int G_{ii}(\mathbf{p})\,d^d p. \quad (28)$$

If $\omega > \omega_*$, the asymptote G_{ii} is given by the dependence

$$\tilde{G}_{ii} = G + (d-1)D/d,$$
$$G(\mathbf{p}) = D(\mathbf{p})\left[1 + \frac{\tilde{w}_0}{w_0 m_0^2}(D_0^{-1} - D^{-1})\right], \quad (29)$$
$$D(\mathbf{p}) = A(p^2 - m_*^2)^{-1}[p^2/(p^2 - m_*^2)]^{\delta}.$$

Substituting Eq. (29) into Eq. (28), we find that the density of the fracton states is

$$N_{fr}(\omega \gg \omega_*) \propto \omega^{d(1-\alpha)-1} = \omega^{d_s - 1}, \quad (30)$$

where the spectral dimensionality exponent d_s is

$$d_s = d(1+\gamma_m^*)^{-1} = d(1+\varepsilon)^{-1}. \quad (31)$$

For $\omega \ll \omega_*$, phonons are retained in the system and the frequency dependence of $N_{\mathrm{ph}}(\omega, \tau)$ has its standard form

$$N_{ph} \propto a(\tau)\,\omega^{d-1}. \quad (32)$$

It should be noted that if the fixed point v_* had been attainable in the limit $\omega \to 0$, the function $a(\tau)$ would have exhibited a power-law dependence on ω_*: $a(\tau) \propto \omega_*^{d_s - d}$. However, as pointed out above, in the case of a phase transition characterized by $h \approx 1$ this asymptotic form is not realized and the dependence of the function $a(\tau)$ can be calculated using conventional perturbation theory.

DISCUSSION OF RESULTS

The results obtained in an analysis of acoustic properties are easily applied, because of the obvious analogy of the equations, to other transport characteristics, such as the effective diffusion coefficient or the stiffness of spin waves near the points where percolation phase transitions with $h \approx 1$ takes place. For example, the critical exponent of the effective diffusion coefficient is $\vartheta = 2\gamma_m^*$. Moreover, it follows from Eq. (31), describing the exponent of the spectral dimensionality d_s, that if we adopt the field-theoretic approach described above, we automatically satisfy the Alexander–Orbach scaling relationships applicable to fractal systems:[14]

$$d_s = d_f(1+\vartheta/2)^{-1}, \quad (33)$$

where in the case of a two-phase medium we have to replace—in contrast to the "limiting" problems—the fractal dimensionality d_f with the spatial dimensionality d (Ref. 10). Then, the "stronger" Alexander–Orbach hypothesis $d_s = 4/3$ cannot be confirmed analytically, irrespective of the dimensionality of space, as already demonstrated earlier for the "limiting" problems.[19,20]

It follows from the above considerations that the characteristic frequency ω_* corresponds to the localization threshold in the problem of motion of an electron in a random field. In an analysis of acoustic properties of such a system at $\omega \approx \omega_*$, as in the solution of the problem of the Anderson transition, we have to calculate the average two-particle Green function for a random field with an Ornstein–Zernike correlation function. We note that the problem of phonon localization in a medium with a Gaussian δ-correlated random field had been investigated earlier[29] for a $(2+\varepsilon)$-dimensional space by a method applied earlier to a similar electron problem.

An interesting aspect is the possibility of describing both the "limiting" problems with $h = 0$ and the percolation phase transitions with $h \approx 1$ by a single field theory. The development of such a theory would make it possible to calculate the effective characteristics of a composite percolation medium with arbitrary values of h and τ, and not only the asymptotic behavior of these characteristics in the cases when $\tau \ll 1$, $h \ll 1$ or $\tau \ll 1$, $h \approx 1$. Note that the qualitative difference between the nature of the effective Hamiltonians used to calculate the configurational averages in the problems with $h \approx 1$ and $h = 0$ is not surprising, because exactly the same situation occurs in the description of magnetic phase transitions in crystals containing paramagnetic impurities at various temperatures and impurity concentrations.

Fracton effects due to the scattering of acoustic vibrations by heterophase fluctuations near the percolation phase transitions in disordered crystals are less easy to observe experimentally using acoustic methods than by the method of low-angle scattering of light because the acoustic wavelengths are fairly long.[30] For example, when the characteristic scale of an inhomogeneity of a random field is $R_0 \sim 10^{-4}$ cm and the relative jumps in the elastic moduli due to a local phase transition amount to $\Delta_K \sim 0.1$, the critical frequency becomes $\omega_* \sim 5 \cdot 10^9$ Hz in the temperature interval $|T - T_c| \approx \Delta T_d$. Therefore, the phonon localization effects should manifest themselves not in ultrasonic experiments, but in an analysis of the spectra observed in the Brillouin scattering of light. Moreover, these effects should result in an anomalous behavior of the thermal diffusivity in the vicinity of a transition point if the inelastic scattering of phonons is sufficiently weak.

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[1)] This approximation is not of any fundamental importance, but it gives rise to simplification of some of the expressions. It is justified quantitatively if $(\partial P/\partial T)|_{T=T_c} \ll P^*/\Delta T_d$.

[2)] In the final stage of the transition (when $1-x \ll 1$) this structure loses its elastic stability and the crystal passes through a "chaotic" state.

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