# Nonlinear growth of domains of tempered incommensurate and related systems

E. B. Kolomeiskii

*A. V. Shubnikov Institute of Crystallography, USSR Academy of Sciences, Moscow*

(Submitted 20 June 1990; resubmitted 27 September 1990)
Zh. Eksp. Teor. Fiz. **99**, 562–567 (February 1991)

The time evolution is considered of an incommensurate structure, a film of wetting liquid, and the Arbikosov vortex lattice transformed discontinuously at the points of transitions into the incommensurate phase, the wetting state and to the boundary of the mixed and Meissner states respectively. It is shown that in the absence of inhomogeneities at zero temperature (and in three dimensions at any temperature) the size of the incommensurate phase domain, the thickness of the liquid film, and the period of the vortex lattice grow logarithmically. At finite temperature in the two-dimensional case the soliton density decreases as $t^{-1}$, the film thickness grows as $t^{1/4}$, and the vortex density decreases as $\exp(-\text{const}\cdot t^{1/2})$. In the presence of inhomogeneities at zero temperature relaxation does not occur in general. For $T\neq 0$ equilibrium is achieved via thermally activated jumps of the corresponding surfaces over the potential barriers created by the inhomogeneities. The structure periods and the film thickness grow in this case according to a nontrivial logarithmic law. In three dimensions the size of the incommensurate phase domain and the film thickness increase according to a universal law $(T\ \ln t)^{1/2}$ irrespective (to logarithmic accuracy) of the nature of defects and of their concentration.

For many years incommensurate structures and related systems have been the object of ongoing attention of the researchers. At present the equilibrium properties of these systems in the vicinity of the transition to the incommensurate phase may be regarded as well understood (see, e.g., Refs. 1–3). As far as nonequilibrium phenomena are concerned, they have not yet been properly studied.

The aim of this paper is the solution of a kinetic problem having a simple experimental realization: in the case of the incommensurate phase the latter is discontinuously transferred to the point of transition (or to its vicinity) into the commensurate phase. Since zero density of solitons (domain walls) now corresponds to the equilibrium, relaxation to a new equilibrium position begins. Below we will find an asymptotic time law according to which the soliton density vanishes as a function of temperature, inhomogeneities and space dimensionality. As specific objects of study, we consider the relaxation of a one-dimensional incommensurate structure (a lattice of domain walls), the related growth of a film of wetting liquid, and the time evolution of an Abrikosov vortex lattice in type II superconductors. To the author's knowledge, a problem of this sort has been considered only in two papers. In Ref. 4 the law governing the growth of a film of wetting liquid in a system without defects has been found; this law has been confirmed in a numerical experiment.[5] The result of Ref. 4 will be obtained below by means of a simpler natural technique.

**1.** Let us recall first how the commensurate—incommensurate phase transition is described.[1] In the vicinity of the transition point the specific free energy of a soliton system has the form

$$g(n)=n(\mu_c-\mu)+nV(n), \tag{1}$$

where $n$ is the soliton density, $\mu$ is an external parameter describing the proximity to the transition point $\mu_c$ (temperature, pressure, chemical potential, field, etc.), and $V(n)$ is the energy of the intersoliton interaction. If at large distances ($n\rightarrow 0$) the domain walls repel each other [$V(n)>0$] and the self-energy of a single soliton (the expression in the parantheses) is positive, the limit $n=0$ (the commensurate phase) corresponds to the minimum of (1). As soon as the self-energy becomes negative, creation of a large number of solitons becomes advantageous. However the latter process is inhibited by the intersoliton repulsion. As a result, for $\mu>\mu_c$ the soliton density increases continuously from zero by a law depending on $V(n)$ as $n\rightarrow 0$. The specific form of $V(n)$ is determined by several factors. At zero temperature and in the absence of inhomogeneities, $V(n)$ is an exponential function decreasing on a scale of the order of the width $\lambda$ of a domain wall:[1]

$$V(n)=C\exp(-1/\lambda n), \quad n\rightarrow 0.$$

When the temperature is finite and/or in the presence of inhomogeneities an extra long-range term is added to the exponential repulsion. The origin of this term can be understood from the following considerations. In the presence of a source of fluctuations (temperature, inhomogeneities) a soliton is strongly displaced in the transverse direction (becomes rough). Therefore, however far apart the neighboring walls may be, they are sure to "collide" (draw together at a distance of order $\lambda$) as the size of the system increases in the direction perpendicular to the modulation. After the first collision further collisions will occur, so that the notion of a mean free path may be introduced. All that means that the radius of effective interaction due to collision is infinite. This interaction is easily estimated as the density of elastic energy of collisions.[1] Since the maximum transverse displacement of a soliton is described by a quantity of the order of the intersoliton distance $l=n^{-1}$, the required estimate has the form

$$\varepsilon\,[n^{-1}/L(n)]^2,$$

where $\varepsilon$ is the soliton stiffness and $L(n)$ is the mean free path. The function $L(n)$ may be easily found, if the relation between the characteristic transverse displacement $w$ of the wall and the length $L$ is known:[1-3]

0038-5646/91/020314-04$03.00

$$w \approx AL^{\zeta}.$$

Here $A$ and $\zeta$ depend on the source of fluctuations. Substituting the mean intersoliton distance $n^{-1}$ for $w$, we obtain

$$L(n) \approx 1/(An)^{1/\zeta}.$$

Substituting $L(n)$ into the estimate of the elastic energy, we obtain for the total potential

$$V(n) = Bn^{\tau} + C\exp(-1/\lambda n), \qquad (2)$$
$$B = \varepsilon A^{2/\zeta}, \quad \tau = 2(1-\zeta)/\zeta.$$

Assume now that the system with a given soliton density determined by the minimum of Eqs. (1), (2) is out of equilibrium, so that $n = 0$ ($\mu = \mu_c$) corresponds to a new equilibrium state. The driving force of the initial relaxation is

$$F = -\frac{\partial g(n, \mu=\mu_c)}{\partial n} = -\frac{\partial}{\partial n}[nV(n)]. \qquad (3)$$

Consider first the case, when inhomogeneities are absent. Then it is natural to assume that the time evolution $n(t)$ is given by a simple relaxation equation (the point denotes time differentiation)

$$\dot{n} = -k\frac{\partial g(n, \mu=\mu_c)}{\partial n} = -k\frac{\partial}{\partial n}[nV(n)], \qquad (4)$$

where $k$ is the kinetic coefficient. The asymptotic form of $n(t)$ as $n \to 0$ is determined by the first long-range term in Eq. (2). Hence

$$\dot{n} = -k(\tau+1)Bn^{\tau},$$
$$n(t) \approx 1/[(\tau^2-1)Bkt]^{1/(\tau-1)}. \qquad (5)$$

This dependence is also valid when there is a long range repulsion between solitons whose own exponent $\tau$ is smaller than the quantity $2(1-\zeta)/\zeta(2)$. As seen from (5), the case $\tau = 1$ (which occurs in a superconductor; see below) is a special one and should be examined more thoroughly. Using the known expression for $A$ and $\zeta$ (Ref. 1),

$$A = (T/\varepsilon)^{1/2}, \quad \zeta = (3-d)/2,$$

and the expressions (2) for $B$ and $\tau$, we find that in the two-dimensional case, $d = 2$, the soliton density vanishes as

$$n(t) \approx \varepsilon/T^2kt.$$

In three dimensions we have $\zeta = 0$, hence $\tau = \infty$, which corresponds to a logarithmic dependence. The same result is obtained at zero temperature irrespective of the space dimensionality. One can easily prove this substituting the second term in Eq. (2) into the relaxation equation. Thus, at zero temperature (in three dimensions, at any temperature) the domain wall density vanishes logarithmically; in two dimensions, at finite temperature, it vanishes at $t^{-1}$. These results are dynamic analogs of the well-known ones related to the effect of thermal fluctuations on equilibrium properties of incommensurate phases.[1]

If at the moment of "tempering" of the incommensurate phase the condition $\mu = \mu_c$ is satisfied only approximately, the laws found above will have other forms at very late times. For $\mu_c - \mu < 0$, i.e. at equilibrium, we have a finite density $n_{\infty}$, at very late times ordinary exponential relaxation to this value will occur. The crossover time can be easily estimated by comparing both terms in (1). For example, for $d = 2$, $T = 0$, the law $n \propto 1/t$ will be replaced by the law $n - n_{\infty} \propto \exp(-\mathrm{const}\cdot t)$ at $t$ of order $[\varepsilon/T^2k^2(\mu-\mu_c)]^{1/2}$. For $\mu_c - \mu > 0$, i.e., when the tempering into the commensurate phase has occured, at very late times linear relaxation takes place:

$$\dot{n} = -k(\mu_c - \mu),$$
$$n(t) = \mathrm{const} - k(\mu_c - \mu)t,$$

and the process of achieving equilibrium takes a finite time. The time at which transition to the linear dependence occurs is estimated in the same manner. Similar relaxation behavior, which is observed when the equality $\mu = \mu_c$ probably does not hold, will also take place in the cases considered below. In what follows, we will always assume, for simplicity, that $\mu = \mu_c$.

2. Consider now the growth of a film of wetting liquid. The expression for specific free energy in the vicinity of the wetting transition point recalls Eq. (1) of Ref. 1:

$$g(l) = l(\mu_c - \mu) + V(1/l).$$

Here $l$ is the film thickness. The first term is the local excess free energy of the film layer in comparison with the energy of the neighboring phase of the same thickness, and the second one is the energy of interaction between the film surface and the substrate. The equilibrium film width obtained by mimimization of $g(l)$ diverges as $\mu \to \mu_c$, which corresponds to the wetting transition. If the system under consideration has been discontinuously transferred to the wetting transition point ($\mu = \mu_c$), the relaxation to equilibrium is described by an equation analogous to Eq. (4):

$$\dot{l} = -k\frac{\partial g(l, \mu=\mu_c)}{\partial l} = -k\frac{\partial}{\partial l}\left[V\left(\frac{1}{l}\right)\right]. \qquad (6)$$

As above, we find that in the presence of long range forces [the first term in Eq. (2)], the film grows according to the law

$$l(t) \approx [\tau(\tau+2)kBt]^{1/(\tau+2)}.$$

When the long range interaction is related to "collisions" between the film surface and the substrate, substituting expression (2) for $\tau$, we obtain the result of Ref. 4:

$$l(t) \propto t^{\zeta/2}.$$

Thus, in two dimensions we have $l(t) \propto t^{1/4}$, and in three dimensions (and, irrespective of the space dimensionality, at zero temperature) the width of the wetting film increases logarithmically.

3. In the cases considered above the relaxation occurs according to a power (logarithmic) law. More exotic time dependence is realized in a type II superconductor tempered from the mixed state to the boundary between the latter and the Meissner phase. The expression for the specific free energy as the function of the vortex density at fields close to the lower critical field $H_{c1}$ has the form[6]

$$g(n) = \frac{\Phi_0}{4\pi}(H_{c1} - H)n + n\left(\frac{3\Phi_0^2}{8\pi^2\lambda^2}\right)\mathrm{K}_0\left(\frac{1}{\lambda n^{1/2}}\right) + \frac{2\pi T^2 n^2}{\varepsilon\ln(1/n\lambda^2)}. \qquad (7)$$

Here $\Phi_0$ is the magnetic flux quantum, $\lambda$ is the London penetration depth of magnetic field, $K_0(x)$ is a modified Bessel function of the second kind and $\varepsilon$ is the linear energy of a vortex. The last term arises due to collisions between vortices, and the logarithmic factor reflects the fact that vortices can go around one another. This naturally decreases the efficiency of collisions in comparison with the soliton interaction in the two-dimensional incommensurate phase. If the external field decreases rapidly to $H = H_c$, the time evolution $n(t)$ again obeys the equation

$$\dot{n} = -k \frac{\partial g(n, H = H_{c1})}{\partial n},$$

and the asymptotic form $n(t)$ at $t \to \infty$ is determined by the last term in (7):

$$\dot{n} = -\frac{4\pi T^2 nk}{\varepsilon \ln(1/n\lambda^2)}.$$

Hence

$$n\lambda^2 \approx \exp[-(8\pi T^2 kt/\varepsilon)^{1/2}].$$

At zero temperature logarithmic relaxation determined by direct interaction of vortices [the second term in (7)] takes place.

**4.** Up to now we considered only systems without defects. On the face of it, the above consideration holds also for systems with defects, the only difference being that the parameters $A$ and $\tau$ given by Eq. (2) must be expressed through other well known formulae depending on the nature and concentration of the defects.[1-3] However, this point of view is erroneous. In fact, a dynamic description based upon a relaxation equation of the form (4) or (6) implies that application of infinitesimal force to the system brings about motion at a velocity proportional to this force, the proprotionality factor being the kinetic coefficient.

In an inhomogeneous system the situation is different: at zero temperature motion does not occur at all if the applied force is smaller than a critical one. At finite temperature the velocity depends exponentially on the applied force, so that the notion of the kinetic coefficient cannot be introduced.[7] Therefore relaxation in the presence of defects is governed by metastability effects. The driving force for the growth is given, as before, by the right-hand side of the kinetic equations (3), (4), (6) (without the kinetic coefficient $k$). However, for the motion to begin, the driving force should not be smaller than the dry friction force (the so-called pinning force $F_{\text{pin}}$ Refs. 2, 3).

Consider first the relaxation of an incommensurate structure and growth of a wetting film. In the limit of large $l = 1/n$ the driving force per surface (per one soliton) is given by the formula, which follows from Eq. (6),

$$F_d \approx \varepsilon A^{2/\zeta} / l^{\tau+1}. \tag{8}$$

The expression for the pinning force per surface is given in Ref. 2:

$$F_{pin} \approx \varepsilon A \left\{ \max\left[ \left(\frac{a}{A}\right)^{1/\zeta}, \left(\frac{T \ln \omega t}{\varepsilon A^2}\right)^{1/(d-3+2\zeta)} \right] \right\}^{\zeta-2}. \tag{9}$$

Here $a$ is the radius of interaction between a defect and a domain wall (a film boundary) and $\omega$ is a microscopic hopping frequency. At zero temperature the largest pinning force arises from the smallest distances (of order $a$) in the direction perpendicular to the surface.[2] Therefore, if when tempering takes place the equilibrium intersoliton distance (or the film width) is larger than $a$, then the driving force of the growth (8) is smaller than the dry friction force (9). As a result, the system remains in a frozen state for an arbitrarily long time. Let us explain the meaning of the time dependence of $F_{\text{pin}}$ [see (9)] at finite temperature.[2] It is natural to assume that for a surface area of size $L$ with characteristic transverse displacement $w(L)$ metastable states are separated by a distance of order $w(L)$ and by energy barriers of height

$$E_{pin} \approx \varepsilon L^{d-1} w^2 / L^2 \approx \varepsilon A^2 L^{d-3+2\zeta}.$$

These barriers create pinning forces

$$F_{pin}(L) \approx \frac{1}{L^{d-1}} \left| \frac{dE_{pin}}{dw} \right| \approx \frac{\varepsilon w}{L^2} \approx \varepsilon A L^{\zeta-2},$$

which increase with decreasing $L$. Therefore the largest pinning force corresponds to a scale $L_{\text{min}}$ in the wall plane such that $a \approx A L_{\text{min}}^{\zeta}$. However, at finite temperature small scale barriers are overcome by thermally activated jumps.[2,7] Thus, pinning at these barriers is ineffective. According to the Arrenius law, the barrier $E_{\text{pin}}(L)$ is overcome in a time

$$t(L) \approx \omega^{-1} \exp \frac{E_{pin}(L)}{T} \approx \omega^{-1} \exp \frac{\varepsilon A^2 L^{d-3+2\zeta}}{T}.$$

Therefore in a time $t$ barriers at distances

$$L < L(t) \approx [T \ln \omega t / \varepsilon A^2]^{1/(d-3+2\zeta)}$$

will be overcome. Hence at $T \neq 0$ we must substitute $\max[L_{\text{min}}, L(t)]$ into the expression for $F_{\text{pin}}(L)$, whence the expression (9). However, in this case also, the motion in these systems will begin not immediately, but only when the magnitude of the dry friction force decreases to the magnitude of the driving force (8). When mechanical equilibrium is established, i.e., for

$$F_d = F_{pin}, \tag{10}$$

the domains (or the film thickness) begin to grow. The condition (10) analogous to the critical state condition in type II superconductors will hold at all later times. Substituting (8) and (9) into (10) and using the definition (2) for $\tau$, we finally find the time dependence $l(t)$:

$$l(t) \approx A \left( \frac{T \ln \omega t}{\varepsilon A^2} \right)^{\zeta/(d-3+2\zeta)}. \tag{11}$$

It is remarkable that in the most interesting case, $d = 3$, the period of the incommensurate structure (or the film thickness) increases according to the universal law

$$l(t) \approx (T \ln \omega t / \varepsilon)^{1/2}$$

irrespective of the nature of the defects or of their concentration. Since the methods used have logarithmic accuracy, the universality should be understood with this proviso.

In type II superconductors the period of the vortex lattice diverges according to the law obtained from (11) at $d = 2$:

$$l(t) \approx A \left( \frac{T \ln \omega t}{\varepsilon A^2} \right)^{\zeta/(2\zeta-1)}$$

Using the exact value $\zeta = 0.6$ for the case of an uncorrelated potential,[3] we obtain

$$l(t) \propto (T \ln \omega t)^3.$$

Note, in conclusion, that logarithmic behavior takes place even in the absence of inhomogeneities (always at zero temperature, and, in three dimensions, at any temperature). The fundamental distinction of logarithmic growth in inhomogeneous medium lies in its strong temperature dependence, reflecting thermally activated motion of surfaces.

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Translated by E. Khmelnitskii