δ-expansion of the fluctuation spectrum of a nonlinear system subjected to the action of noise

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Based on a new perturbation theory involving expansion in powers of the nonlinearity of the theory (for a system with a power-law nonlinearity), we calculate the fluctuation spectrum of a nonlinear system to second order. For an anharmonic oscillator in the strong-damping limit we show that as the power of the nonlinearity and the intensity of the external noise change a reconstruction of the spectrum is possible due to a sign change in the nonlinear contribution to the total spectrum.

In Refs. 1 and 2 a new analytic technique was proposed for studying nonlinear field theory, which was later generalized to supersymmetric models[3] and stochastic quantization. Reference 3 also provided the stimulus for the investigation we describe here, in which we apply this new perturbation theory to problems involving the dynamics of nonlinear systems in the presence of noise. In Ref. 3 the authors constructed expansions not in the interaction constant but rather in the nonlinear exponent δ of the theory (for a power-law nonlinearity, i.e., $x^{2\delta+1}$). Therefore we should expect nontrivial dependences on the physical parameters which ultimately will allow us to identify features of the theory which could not be studied using previous methods of investigation.

In this paper we obtain an expression for the fluctuation spectrum (i.e., the Fourier transform of the autocorrelation function) to second order in δ for the example of a one-dimensional system with strong damping. This example shows that a reconstruction of the spectrum is possible. We note that after the necessary regularization (see Refs. 5 and 6) our expressions for the spectrum can be used for any nonlinear system with a power-law nonlinearity in a space of arbitrary dimension. The Appendices contain a discussion of how to average expressions containing the logarithm of a random function, using a different method from the one given in Refs. 1 and 2, along with some technical details.

Consider the following equation:

$$\hat{K}x=-\gamma x(x^{2\delta}-1)+f(t), \tag{1}$$

where $\hat{K}$ is an arbitrary linear differential operator in which we have included a term γx for convenience; $f(t)$ is a Gaussian random process with correlation function

$$\langle f(t_1)f(t_2)\rangle=B(t_2-t_1). \tag{2}$$

Assume that x is a dimensionless quantity, so that the dimensions of γ coincide with the dimensions of $\hat{K}$ (we have in mind dimensions of frequency). For example, let $\hat{K}$ be the operator for a damped harmonic oscillator. Then by using a dimensionless action we obtain the dimensions of the following quantities: $[t] = -1$, $[x] = -1/2$, $[\gamma] = 2+\delta$. If we introduce a dimensional parameter ν with the dimensions of frequency and make the replacement $\nu^{1/2}x\to x$, retaining a dimensionality of 2 for γ, we obtain an expression analogous to (1).

Following Refs. 1 and 2, we write x in the form

$$x=\sum_{n=0}^{\infty}\delta^n x_n. \tag{3}$$

Expanding x on the right side of (1), we obtain the following system of equations:

$$\hat{K}x_0=f(t), \tag{4}$$

$$\hat{K}x_1=-\gamma x_0\ln x_0^2, \tag{5}$$

$$\hat{K}x_2=-\gamma[2x_1+x_1\ln x_0^2+{}^1/_2x_0(\ln x_0^2)^2]. \tag{6}$$

The fluctuation spectrum is determined as follows:

$$F(\Omega)=\int_{-\infty}^{+\infty}\langle x(t+\tau)x(t)\rangle e^{-i\Omega\tau}\,d\tau. \tag{7}$$

Accordingly, to second order in δ the autocorrelation function has the form

$$\langle x(1)x(2)\rangle=\langle x_0(1)x_0(2)\rangle+\delta[\langle x_0(1)x_1(2)\rangle+(1\leftrightarrow 2)]$$
$$+\delta^2[\langle x_0(1)x_2(2)\rangle+\langle x_0(2)x_2(1)\rangle+\langle x_1(1)x_1(2)\rangle], \tag{8}$$

where $x(1)\equiv x(t_1)$.

To zero order in δ, taking into account that

$$x_0(1)=G(1,2)f(2)\equiv\int_{-\infty}^{\infty}G(t_1-t_2)f(t_2)dt_2, \tag{9}$$

where $G=\hat{K}^{-1}$ is the Green's function, it is not difficult to obtain

$$F_0(\Omega)=|G(\Omega)|^2B(\Omega), \tag{10}$$

where $G(\Omega)$ and $B(\Omega)$ are the Fourier transforms of the Green's function and the noise correlator, respectively.

To first order in δ it is necessary to calculate the average

$$\langle x_0(1)x_1(2)\rangle=-\gamma G(2,3)\langle x_0(1)x_0(3)\ln x_0^2(0)\rangle. \tag{11}$$

For this we write

$$\ln x_0^2=\frac{d}{dk}(x_0^2)^k|_{k=0},$$

we then assume that k is an integer, and carry out the average by evaluating the derivative at zero (a different derivation is given in Appendix 1). As a result, we find the following first-order fluctuation spectrum:

$$F_1(\Omega)=-2\delta\gamma L\,\mathrm{Re}\,G(-\Omega)F_0(\Omega), \tag{12}$$

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where

$$L=\ln 2I+\psi\left(\frac{3}{2}\right),\qquad I=\langle x_0^2\rangle=\int\frac{d\Omega}{2\pi}F_0(\Omega),$$

$$\psi(k)=\frac{d}{dk}\ln\Gamma(k).$$

We can use analogous methods to perform the second-order calculations in δ (see Appendix 2). In this case, the expression for the spectrum has the form

$$F(\Omega)=F_0(\Omega)+(-\gamma)\operatorname{Re}G(-\Omega)F_0(\Omega)[(\delta L+1)^2-1+\delta^2\psi'(^3/_2)$$
$$-4\gamma\delta^2A]+2\delta^2\gamma^2F_0(\Omega)\operatorname{Re}G(-\Omega)$$
$$\times\int d\tau\, G(\tau)e^{i\Omega\tau}\left[L^2-\int_0^1 ds\frac{1}{s(1-s)^{1/2}}\right.$$
$$\left.\times\ln\left(1-s\frac{F_0^2(\tau)}{I^2}\right)\right]+\delta^2\gamma^2|G(\Omega)|^2\int d\tau e^{-i\Omega\tau}F_0(\tau)$$
$$\times\left[L^2-\int_0^1 ds\frac{(1-s)^{1/2}}{s}\ln\left(1-s\frac{F_0^2(\tau)}{I^2}\right)\right],\tag{13}$$

where

$$F_0(\tau)=\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}F_0(\Omega)e^{i\Omega\tau},\qquad \psi'(k)=\frac{d}{dk}\psi(k),$$

$$A=\int_{-\infty}^{\infty}d\tau\, G(\tau)\frac{F_0(\tau)}{I}\left[L-\frac{F_0^2(\tau)}{I^2}\int_0^1 ds(1-s)^{1/2}\right.$$
$$\left.\times\left(1-s\frac{F_0^2(\tau)}{I^2}\right)^{-1}\right].$$

It is not difficult to see that the frequency dependence of the second term in (13) coincides with the expression obtained by perturbing with respect to γ. However, in our case it is now possible for the sign of the coefficient in the square brackets to change. To first order in δ, the coefficient becomes negative when $L<0$; naturally, the inclusion of second-order terms in δ provides necessary bounds on this. Certainly all this agrees with the usual perturbation theory in γ, where the contribution of this term is proportional to $(-\gamma)$. Let

$$\hat{K}=\frac{d^2}{dt^2}+\omega^2+2\Gamma\frac{d}{dt},$$

where $\omega^2=\omega_0{}^2+\gamma$ is the renormalized frequency, so that $F_0(\Omega)$ depends on γ. Expanding (13) to first order in γ, we see that the coefficient of the second term is equal to $(\delta L+1)^2+1+\delta^2\psi'(3/2)$, i.e., it is positive, so that the contribution is determined by $(-\gamma)$, which agrees with the usual perturbation theory.

In order to obtain the necessary bound for δ, let us evaluate the integral A in (13). For this we go to the strong-damping limit, where the second derivative with respect to time in $\hat{K}$ can be neglected. In this case

$$G(\tau)=\exp\left(-\frac{\omega^2}{2\Gamma}\tau\right)\theta(\tau),$$
$$F_0(\tau)=\frac{D}{2\Gamma\omega^2}\exp\left(-\frac{\omega^2}{2\Gamma}|\tau|\right),\tag{14}$$

where $\theta(\tau)$ is a step function and $B(t_1-t_2)=2D\delta(t_1-t_2)$ (we have made the replacement $\gamma\to\gamma/2\Gamma$). Substituting (14) into the expression for A, we obtain (see Appendix 3)

$$A=\frac{\Gamma}{\omega^2}\left[L-1+\frac{1}{2}\psi'\left(\frac{3}{2}\right)\right].\tag{15}$$

Using (15) in (13), we find that for $L<0$ and $\gamma\lesssim 1.63\omega^2$

$$\delta<-\frac{2L}{L^2-2\beta L+2\beta+\psi'(^3/_2)(1-\beta)}=f(L),\tag{16}$$

where $\beta=\gamma/\omega^2$. For $L>0$ we have $f(L)<0$; at the point $L=0$ the function $f(L)$ changes sign, then goes through a maximum for

$$L_{max}=-[2\beta+\psi'(^3/_2)(1-\beta)]^{1/2}.$$

As β increases, the value $\delta_{\max}=f(L_{\max})$ decreases, e.g., from $\delta_{\max}=0.89$ for $\beta=0.1$ to $\delta_{\max}=0.58$ for $\beta=0.5$. It is obvious that for each δ there is a value of L in this interval determined by the equation $f(L)=\delta$. Thus, as δ and L vary a restructuring of the spectrum is possible, associated with the change in sign of the nonlinear contribution to the total spectrum [the second term in (13)].

Let us consider a specific example. Let $\delta=0.5$, and assume a nonlinearity in (1) of the form $-\gamma x|x|$. Let us take $\beta=0.5$; then for $0.09<D/\Gamma\omega^2<0.54$ we observe a restructuring of the spectrum for this system, which takes place for fixed damping parameters of the anharmonic oscillator as the intensity of external noise varies.

Finally, we must keep in mind that for small frequencies or near resonance the contribution of the third and fourth terms to (14) can be quite significant, so that it is necessary to consider the high-frequency region in order to identify effects associated with the reconstruction of the spectrum.

We note that changing the sign of the nonlinear contribution to the fluctuation spectrum of the system leads to a change in the direction of transfer of energy due to the nonlinear interaction. For example, the behavior of a plasma with a linear inhomogeneity in the Lagrange variables is described by Eq. (1) with $\delta=0.5$ (see Ref. 7), so that for certain noise intensities we may expect the formation of an energy-containing region for small frequencies. We can hope that a similar picture will also be correct [when we introduce spatial variations into (13), i.e., $\Omega\tau\to\omega t-qx$, $d\tau\to dtdx$, and $\Omega\to(\omega,q)$], in which case the analogous restructuring of the spectrum can play the role of a mechanism for pumping down the large-scale structure of the noise.

APPENDIX 1

In this section we show how to obtain the average of the logarithm of a random function without representing the logarithm as the derivative of a power-law function evaluated at zero. For this we will use the integral formula:

$$\ln x_0{}^2=\int_{-\infty}^{\infty}\frac{d\beta}{\pi^{1/2}}\exp(-\beta^2)\int_0^{\infty}\frac{d\alpha}{\alpha}[\exp(-\alpha)-\exp(2i\alpha^{1/2}\beta x_0)].\tag{1.1}$$

We note that we could bypass the integration with respect to β; however, because our application involves a system in which a deterministic source is present along with the random force, we find that the linear dependence on x_0 in the exponent has its advantages. Substituting (1.1) into (11) and using the relation

$$\langle x_0(1)x_0(2)\exp(2i\alpha^{1/2}\beta x_0(2)\rangle$$
$$=\langle x_0(1)x_0(2)\rangle(1-4\alpha\beta^2 I)\exp(-2\alpha\beta^2 I), \qquad (1.2)$$

after integrating with respect to β we obtain

$$\langle x_0(1)x_0(2)\ln x_0^2(2)\rangle$$
$$=\int_0^\infty \frac{d\alpha}{\alpha}\left[e^{-\alpha}-\frac{1}{(1+2\alpha I)^{1/2}}\right]\langle x_0(1)x_0(2)\rangle. \qquad (1.3)$$

We now use the transformation

$$\frac{1}{(1+2\alpha I)^{1/2}}=\frac{1}{\Gamma(^3/_2)}\int_0^\infty dy\, y^{1/2}\exp(-y-2\alpha y I). \qquad (1.4)$$

and substitute (1.4) into (1.3); then after integrating with respect to α we obtain

$$\langle x_0(1)x_0(2)\ln x_0^2(2)\rangle=\langle x_0(1)x_0(2)\rangle\int_0^\infty\frac{dy}{\Gamma(^3/_2)}y^{1/2}e^{-y}\ln 2yI$$
$$=\langle x_0(1)x_0(2)\rangle L.$$

APPENDIX 2

We illustrate the second-order calculation in δ with the following example:

$$\langle x_1(1)x_1(2)\rangle$$
$$=\gamma^2 G(1,3)G(2,4)\langle x_0(3)\ln x_0^2(3)x_0(4)\ln x_0^2(4)\rangle. \qquad (2.1)$$

Following Refs. 1 and 2, we write

$$\langle x_0(3)\ln x_0^2(3)x_0(4)\ln x_0^2(4)\rangle$$
$$=\frac{d}{dk_1}\frac{d}{dk_2}\langle x_0^{2k_1+1}(3)x_0^{2k_2+1}(4)\rangle\Big|_{k=0}$$
$$=\frac{d}{dk_1}\frac{d}{dk_2}\sum_{n=0}^{\min(k_1k_2)}C_{2n+1}^{2k_1+1}C_{2n+1}^{2k_2+1}(2k_1-2n-1)!!\,(2k_2-2n-1)!!$$
$$\times(2n+1)!I^{k_1+k_2-2n}\langle x_0(3)x_0(4)\rangle^{2n+1}\Big|_{k=0}. \qquad (2.2)$$

By differentiation we obtain

$$\langle x_0(3)x_0(4)\rangle\left[L^2+\frac{\pi^{1/2}}{2}\int_0^{a^2}dz\sum_{n=0}^\infty\frac{\Gamma(n+1)}{\Gamma(n+^5/_2)}z^n\right], \qquad (2.3)$$

where $a=I^{-1}\langle x_0(3)x_0(4)\rangle$. We now use the definition of the hypergeometric function and its integral representation; as a result we find

$$\langle x_1(1)x_1(2)\rangle$$
$$=\gamma^2 G(1,3)G(2,4)\left[L^2-\int_0^1 ds\frac{(1-s)^{1/2}}{s}\ln(1-sa^2)\right]aI. \qquad (2.4)$$

Note once more the necessity for the following partition:

$$\langle x_0(1)x_0^{2k_1+1}(2)x_0^{2k_2}(3)\rangle$$
$$=(2k_1+1)\langle x_0(1)x_0(2)\rangle\langle x_0^{2k_1}(2)x_0^{2k_2}(3)\rangle$$
$$+2k_2\langle x_0(1)x_0(3)\rangle\langle x_0^{2k_1+1}(2)x_0^{2k_2-1}(3)\rangle.$$

This term arises in taking the average

$$\langle x_0(1)x_0(2)\ln x_0^2(2)\ln x_0^2(3)\rangle.$$

APPENDIX 3

In the course of the calculations leading up to (15) it becomes necessary to sum the following series:

$$R=\frac{\pi^{1/2}}{2}\sum_{n=0}^\infty\frac{\Gamma(n+1)}{(n+2)\Gamma(n+^5/_2)}. \qquad (3.1)$$

Using the hypergeometric function, it is easy to obtain

$$R=\int_0^1 dz\,z\int_0^1 ds(1-s)^{1/2}(1-sz)^{-1}=-\lim_{\mu\to0}\left\{\int_0^1 s^{\mu-1}(1-s)^{1/2}ds\right.$$
$$\left.+\frac{d}{d\nu}\int_0^1 ds[s^{\mu-2}(1-s)^{\nu+3/2}+s^{\mu-1}(1-s)^{\nu+1/2}]\Big|_{\nu=0}\right\} \qquad (3.2)$$

where we have used

$$\ln(1-s)=\frac{d}{d\nu}(1-s)^\nu\Big|_{\nu=0}.$$

Let us calculate the integral

$$\int_0^1 ds\, s^{\mu-1}(1-s)^{1/2}=\frac{\Gamma(\mu)\Gamma(^3/_2)}{\Gamma(\mu+^3/_2)}=-\psi(^3/_2)-\gamma_E+\frac{1}{\mu}+O(\mu),$$
$$\frac{d}{d\nu}\int_0^1 ds(1-s)^{\nu+1/2}s^{\mu-1}=-\psi'(^3/_2)+O(\mu),$$
$$\frac{d}{d\nu}\int_0^1 ds(1-s)^{\nu+3/2}s^{\mu-2}$$
$$=-\frac{1}{\mu}-1+\gamma_E+\psi(^3/_2)+^3/_2\psi'(^3/_2)+O(\mu),$$

where γ_E is the Euler-Mascheroni constant. Assembling all the integrals we obtain

$$R=1-^1/_2\psi'(^3/_2).$$

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