Quasiclassical theory of radiation emission from high-energy particles in an external field and the problem of boundary conditions

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The question of radiation emission by relativistic particles in an external field is studied in the quasiclassical approximation. In describing this process special attention is devoted to the problem of boundary conditions. It is shown that the cross section for emission taking into account recoil is determined by the classical trajectories, of which there can be several for prescribed boundary conditions, of particles in the external field. The possibility of the existence of several trajectories leads to a unique radiation interference pattern which is analogous to the interference pattern arising in the case of rainbow scattering of particles.

1. INTRODUCTION

Radiation emission from a charged relativistic particle (electron or positron) has been studied on the basis of classical and quantum electrodynamics in a large number of papers (see Refs. 1–9 and the work cited there). In classical electrodynamics the trajectory of a particle was prescribed through the initial values of the coordinates and the momentum, and the intensity of the radiation was determined. In the quantum theory the momentum of a particle is usually prescribed before and after emission of radiation. But, in principle, this can also be done in the classical formulation of the problem. In this case, however, we can obtain not one but several classical trajectories corresponding to the same values of the initial and final momenta. Such a situation occurs, for example, in the process of rainbow scattering of particles.[10,11] In this case two trajectories correspond to the same boundary conditions and the scattering cross section in the classical theory is assumed to be equal to the sum of the scattering cross sections corresponding to these two trajectories. From the viewpoint of quantum mechanics, however, this result is not always correct, since the scattering amplitudes associated with both trajectories can interfere. This becomes obvious when the scattering problem is studied in the quasiclassical approximation.

Indeed, in this approximation we employ both the quantum-mechanical description and the concept of a classical trajectory. As a result the scattering cross section is determined by the squared modulus of the sum of the quantum scattering amplitudes, each of which is determined by the corresponding classical trajectory.[10,11]

An analogous situation can also occur in the study of the process of radiation emission from a relativistic particle in an external field in the case if the final state of the radiating particle is fixed. We can obtain, in this case, several trajectories corresponding to the same boundary conditions (the momenta of the particle before and after emission are prescribed). In the classical description we can find the intensity of the radiation corresponding to each trajectory, and if the effective radiation is studied (radiation from a flux of particles), then these intensities will add. In so doing, however, as in the case of rainbow scattering, the most important property of the radiation—the possibility of interference of radiation from different trajectories—is lost. We cannot describe this phenomenon on the basis of classical electrodynamics, since it presupposes the possibility that the particle moves simultaneously along two trajectories (we emphasize that in finding the effective radiation we are actually dealing with not one particle, but a flux of particles).

Thus in studying emission of radiation we must take into account the possibility that the particle moves along several trajectories. This can be done if the quasiclassical approximation in quantum mechanics is employed. The particle in this case, naturally, is assumed to be fast. The quasiclassical approximation implies that if several classical trajectories exist corresponding to one and the same boundary conditions, radiation interference is possible. The present paper is devoted to this problem.

We first note that the radiation process for relativistic particles has been studied in the quasiclassical approximation in a number of papers (see Ref. 9 and the work cited there), but the formulation of the initial and boundary conditions for the trajectory was not actually studied there. For this reason, how the particle trajectory in an external field should be determined and the procedure that should be used for averaging the emission probability over these trajectories remain unclear. Naturally, in this case there is no possibility of the existence of several trajectories with the same boundary conditions and therefore there is no interference of radiation accompanying the motion of a particle simultaneously along several trajectories.

Reference 12 is also devoted to the application of the quasiclassical approximation to the problem of radiation emission. In Ref. 12 the role of higher-order approximations in the Planck constant $\hbar$ in radiation emission at low frequencies was studied using coherent-trajectory states. In this work, a narrow wave packet of the radiating particle was taken as the initial state. This eliminated the possibility of the existence of several trajectories corresponding to one and the same boundary conditions.

Our work in Refs. 13 and 14 is also devoted to this problem. In these papers, however, the effect of recoil accompanying emission is neglected.

In the present paper we study the effect of recoil and give a more rigorous analysis of the problem. We study in detail the case of the eikonal approximation and clarify how the laws of conservation of energy and momentum in the radiation process arise in the problem.

Our approach is conceptually close to the method used by Fock[15] in studying the relation between the unitary transformations in quantum mechanics and canonical transformations in classical mechanics. We relate the emission cross section to the classical trajectories of a particle in an external field which satisfy prescribed boundary conditions. We spe-

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cially investigate the case when the emission cross section factors and the elastic scattering cross section is separated out; in addition, the latter takes into account the possibility of the existence of rainbow scattering. It is shown that the interference pattern accompanying emission is related with the interference pattern arising in the case of elastic rainbow scattering.

2. EMISSION PROBABILITY IN THE QUASICLASSICAL APPROXIMATION

The probability of the emission of a photon by a charged particle in an external field in a transition of the particle from the initial state with momentum $\mathbf{p}_i$ into the final state with momentum $\mathbf{p}_f$ is determined in quantum electrodynamics by the following formula:[3,4]

$$dW_{fi}=\frac{e^2\,d^3k}{4\pi^2\hbar\omega}\frac{d^3p_f}{(2\pi\hbar)^3}|M_{fi}|^2, \qquad (2.1)$$

where ω and $\mathbf{k}$ are the frequency and wave vector of the emitted photon and M_{fi} is the matrix element of the transition,

$$M_{fi}=2\int dt\,d^3r\varphi_f^*(\mathbf{r},t)\mathbf{e}\hat{\mathbf{P}}\varphi_i(\mathbf{r},t)\exp[i(\omega t-\mathbf{kr})]. \qquad (2.2)$$

Here $\mathbf{e}$ is the polarization vector of the photon, $\hat{\mathbf{P}}$ is the generalized momentum operator, and φ_i and φ_f are the wave functions of the particle in the external field under study.

For simplicity we shall neglect the spin of the particle, since the essence of the problem is not related to the existence of spin.

In the quasiclassical approximation, which we shall employ, the wave function of the particle in an external field is related, as is well known,[16,17] to the classical action of the particle in this field $S(\mathbf{r},\mathbf{p},t)$ by

$$\varphi(\mathbf{r},t)=\frac{1}{\{2[\varepsilon-U(r)]\}^{1/2}}\left|\frac{\partial^2 S}{\partial\mathbf{r}\,\partial\mathbf{p}}\right|^{1/2}\exp\left(\frac{i}{\hbar}S\right), \qquad (2.3)$$

where $\mathbf{p}$ is the momentum of the particle before or after scattering (see Sec. 3) and $|\partial^2S/\partial\mathbf{r}\partial\,\mathbf{p}|$ is the determinant of the matrix $(\partial^2S/\partial r_i\partial j_k)$.

We shall show below (Secs. 3 and 4) that in the quasiclassical approximation the emission probability (2.1) can be expressed with the help of the following formula in terms of the scattering matrix S_{pp_i} of the particle in an external field:

$$dW_{fi}=\frac{e^2\,d^3k}{4\pi^2\hbar\omega}\frac{d^3p}{(2\pi\hbar)^3}\frac{\varepsilon_i}{\varepsilon_f}\left|\Phi\left\{i\hbar\frac{\partial}{\partial\mathbf{p}}\right\}S_{pp_i}\right|^2\bigg|_{\mathbf{p}=\mathbf{p}_f+\hbar\mathbf{k}}, \qquad (2.4)$$

where $\varepsilon_f=\varepsilon_i-\hbar\omega$ is the energy of the particle after emission,

$$\Phi\{\mathbf{x}\}=\int dt\,\mathbf{ev}(t)\exp\left\{i\frac{\varepsilon_i}{\varepsilon_f}[\omega t-\mathbf{kr}(t)]\right\}, \qquad (2.5)$$

$\mathbf{r}(t)=\mathbf{r}(t,\mathbf{x},\mathbf{p}')$ is the trajectory of the particle in an external field and is determined by the final value of the momentum $\mathbf{p}'=\mathbf{p}_f+\hbar\mathbf{k}$ and the coordinates $\mathbf{x}$.

If in the final state only the direction of motion of the charged particle and the photon momentum $\hbar\mathbf{k}$ are fixed, then the formula (2.4) must be integrated over the momentum $(p_z=p_f+\hbar k)_z$ (the z-axis is oriented along the momentum $\mathbf{p}_i$). We do not present here the result of such integration (see Sec. 3), and we confine our attention only to the result pertaining to the eikonal approximation. In this approximation, which is valid for fast particles, the scattering matrix with accuracy up to terms proportional to p^{-1} has the form

$$S_{pp_i}\approx\int d\tau\,d^2\rho\exp\left\{\frac{i}{\hbar}[(\varepsilon_p-\varepsilon_i)\tau-(\mathbf{p}-\mathbf{p}_i)\mathbf{r}+\chi_0(\boldsymbol{\rho})]\right\}, \qquad (2.6)$$

where $\boldsymbol{\rho}$ is the impact parameter,

$$\chi_0=-\frac{1}{v}\int_{-\infty}^{\infty}dz\,U(\boldsymbol{\rho},z),$$

and $U(\boldsymbol{\rho},z)$ is the potential of the external field (it is assumed that the vector potential of the external field is equal to zero). Starting from the formula (2.4) we can arrive at the following expression for the differential cross section for emission (the relation between the differential cross section and the emission probability is given in Ref. 3):

$$d\sigma_{fi}=\frac{e^2\,d^3k}{4\pi^2\hbar\omega}\frac{\varepsilon_i}{\varepsilon_f}\frac{d^2q_\perp}{(2\pi\hbar)^3}|A(\mathbf{q},\mathbf{k})|^2, \qquad (2.7)$$

where $\mathbf{q}=\mathbf{p}_i-\mathbf{p}$, $q_\perp$ are the components of $\mathbf{q}$ that are orthogonal to $\mathbf{p}_i$, and

$$A(\mathbf{q},\mathbf{k})=\int d^2\rho\,\Phi\{\boldsymbol{\rho}\}\exp\left\{\frac{i}{\hbar}[\mathbf{q}\boldsymbol{\rho}+\chi_0(\boldsymbol{\rho})]\right\} \qquad (2.8)$$

is the emission amplitude.

In the quasiclassical approximation $|\chi_0|\gg\hbar$, so that the integral over $\boldsymbol{\rho}$ in Eq. (2.8) can be calculated by the stationary-phase method. The stationary point $\boldsymbol{\rho}^*$ is determined from the condition

$$\mathbf{q}=-\frac{\partial}{\partial\boldsymbol{\rho}}\chi_0(\boldsymbol{\rho}). \qquad (2.9)$$

There can be several stationary points, so that in the general case for $|\chi_0|\gg\hbar$ the emission cross section (2.8) has the form

$$d\sigma_{fi}=\frac{e^2\,d^3k}{4\pi^2\hbar\omega}\frac{\varepsilon_i}{\varepsilon_f}d^2q_\perp\left|\sum_n\Phi\{\boldsymbol{\rho}_n^*\}\left|\frac{\partial\boldsymbol{\rho}_n^*}{\partial\mathbf{q}}\right|^{1/2}\exp\left(\frac{i}{\hbar}F_n\right)\right|^2, \qquad (2.10)$$

where $F_n=\mathbf{q}\boldsymbol{\rho}_n^*+\chi_0(\boldsymbol{\rho}_n^*)$ is the phase of scattering, corresponding to the stationary point $\boldsymbol{\rho}_n^*$, and the summation over n is performed over different stationary points.

If there is only one stationary point, then we arrive at the formula

$$d\sigma_{fi}=\frac{e^2\,d^3k}{4\pi^2\hbar\omega}\frac{\varepsilon_i}{\varepsilon_f}d^2\rho^*|\Phi\{\boldsymbol{\rho}^*\}|^2. \qquad (2.11)$$

We can see that in the quasiclassical approximation the emission cross section is determined by the classical trajectory of the particle in the external field neglecting recoil. The trajectory is determined, in this case, by the boundary conditions, i.e., the values of the particle momenta before and after interaction. With given boundary conditions there can be more than one trajectory. This situation is manifested in the fact that the matrix element of the emission probability contains summation over different trajectories with the same boundary conditions. The existence of different trajectories is manifested in turn in the fact that the scattering phase $\mathbf{q}\boldsymbol{\rho}+\chi_0(\boldsymbol{\rho})$ has several stationary points [they are determined from the condition (2.9)]. The summation in Eq. (2.10) is performed over these stationary points.

If in the expression for the emission amplitude (2.8) the function Φ is replaced by unity, then we obtain the elastic scattering amplitude in the eikonal approximation

$$a(\mathbf{q})=\int d^2\rho \exp\left\{\frac{i}{\hbar}[\mathbf{q}\rho+\chi_0(\rho)]\right\}. \qquad (2.12)$$

In the quasiclassical approximation this amplitude is expressed in terms of the same classical trajectories which were discussed above. The summation over different trajectories, i.e., different stationary points of the phase $\mathbf{q}\rho+\chi_0(\rho)$, leads to the well-known phenomenon of rainbow scattering.[10,11] This phenomenon consists of the fact that when a particle is elastically scattered in a field $U(\mathbf{r})$ the scattering amplitudes corresponding to different stationary points of the integral (2.12) interfere.

On transferring from elastic scattering to radiation emission, i.e., when $\Phi\neq 1$, there arises a more complicated interference pattern than in the case of rainbow scattering. As before the emission amplitudes interfere analogously to the scattering amplitudes in the case of elastic rainbcw scattering. But, there also arises interference between electromagnetic waves emitted by a particle moving along classical trajectories. This interference is caused both by emission from the particle from different sections of one and the same trajectory and by emission from different sections of different trajectories.

We note that the formulas obtained take into account recoil accompanying emission ($\varepsilon_i\neq\varepsilon_f$).

In the formulas presented above the final state of the emitting particle is fixed. If the formula (2.7) is integrated over the transferred momentum $\mathbf{q}$, then we arrive at the formula

$$d\sigma=\frac{e^2 d^3k}{4\pi^2\hbar\omega}\frac{\varepsilon_i}{\varepsilon_f}\int d^2\rho|\Phi\{\rho\}|^2. \qquad (2.13)$$

This differs from the corresponding formula of Ref. 9 in that it takes into account the boundary conditions that determine the trajectory of the particle and it contains a procedure for averaging over them (the presence of the integral over ρ).

3. RELATION BETWEEN THE EMISSION MATRIX ELEMENT AND THE CLASSICAL TRAJECTORIES

Our problem now is to derive the formula (2.4) for the probability of emission from a fast charged particle in an external field in the quasiclassical approximation. For this it is first necessary to calculate the emission matrix element with quasiclassical wave functions. These wave functions, however, have different asymptotic forms. The wave function $\varphi_i(\mathbf{r},t)$ corresponding to the initial state is a plane wave with momentum $\mathbf{p}_i$ in the limit $t\to-\infty$ and the wave function $\varphi_f(\mathbf{r},t)$ corresponding to the final state of the emitting particle is a plane wave with momentum $\mathbf{p}_f$ in the limit $t\to+\infty$. Similarly to the wave functions, the actions of the particle S_i and S_f, appearing in the quasiclassical wave functions of the initial and final states, also have different asymptotic behavior in the limits $t\to\pm\infty$. As a result when the quasiclassical wave functions are substituted directly into the matrix element it will contain an exponential that oscillates rapidly as $\hbar\to 0$:

$$\exp\left\{\frac{i}{\hbar}[S_f(\mathbf{r},\mathbf{p}_f,t)-S_i(\mathbf{r},\mathbf{p}_i,t)]\right\}.$$

It becomes very difficult to calculate exactly the matrix element in this case, so that there arises the question of how to avoid this difficulty. We shall show that this can be achieved if the wave function of the initial state is expanded in the wave functions which have the asymptotic form of the wave function of the final state and after this expansion is substituted into the matrix element only the terms corresponding to momenta close to the momentum of the final state remain.

This expansion has the form

$$\varphi_i(r,t)=\int\frac{d^3p}{(2\pi\hbar)^3}\varphi_p(\mathbf{r},t)S_{pp_i}, \qquad (3.1)$$

where $\varphi_p(\mathbf{r},t)$ is the wave function of the particle in the external field, and in the limit $t\to+\infty$ it has the asymptotic form

$$\varphi_p(\mathbf{r},t)\to\frac{1}{(2\varepsilon_p)^{1/2}}\exp\left[\frac{i}{\hbar}(\mathbf{pr}-\varepsilon_p t)\right],$$

and S_{pp_i} is the scattering matrix of the particle in the external field. In a stationary field $U(\vec{r})$ the scattering matrix has the form[18]

$$S_{pp_i}=(2\pi\hbar)^3\delta(\mathbf{p}-\mathbf{p}_i)-\frac{i}{\hbar}(2\varepsilon_i)^{1/2}\int d\tau\, d^3r \times\exp\left[\frac{i}{\hbar}(\varepsilon\tau-\mathbf{pr})\right]U(\mathbf{r})\varphi_i(\mathbf{r},\mathbf{t}). \qquad (3.2)$$

Substituting the expansion (3.1) into Eq. (2.2) gives the following expression for the matrix element in the quasiclassical approximation

$$M_{fi}=\int dt\, d^3r\frac{d^3p}{(2\pi\hbar)^3}\left|\frac{\partial^2 S_f}{\partial\mathbf{r}\,\partial\mathbf{p}_f}\right|^{1/2}\left|\frac{\partial^2 S_p}{\partial\mathbf{r}\,\partial\mathbf{p}}\right|^{1/2} \times e\mathbf{v}(\mathbf{r})\exp[i(\omega t-\mathbf{kr})]\exp\left[\frac{i}{\hbar}(S_p-S_f)\right]S_{pp_i}, \qquad (3.3)$$

where $\mathbf{v}(\mathbf{r})=(\nabla S_p-\mathbf{A})/(\varepsilon-U)$. [We employed here the fact that as $\hbar\to 0$ the operator $\hat{\mathbf{P}}$ acts only on the exponential part of the wave function $\varphi_i(\mathbf{r},t)$.]

We note that the integration over $\mathbf{p}$ in this expression is performed under the condition that $\varepsilon_p=\varepsilon_i$. This is because the scattering matrix S is proportional to the δ-function $\delta(\varepsilon_p-\varepsilon_i)$ expressing the law of conservation of energy in the scattering process.

The matrix element (3.3) contains the factor

$$\exp\left\{\frac{i}{\hbar}[S_p(\mathbf{r},t)-S_f(\mathbf{r},t)]\right\}, \qquad (3.4)$$

and in addition S_p and S_f have the same asymptotic form in the limit $t\to+\infty$. It is clear that if $\mathbf{p}$ differs strongly from $\mathbf{p}_f$, then this factor will oscillate rapidly, so that such values of $\mathbf{p}$ will make a small contribution to the matrix element.

There now arises the question of how to determine the region of values of $\mathbf{p}$ making the main contribution to the matrix element. The situation is simplest in the case when recoil can be neglected, i.e., when $\varepsilon_f\approx\varepsilon_p=\varepsilon_i$. In this case the values of $\mathbf{p}$ close to $\mathbf{p}_f$ will make the main contribution to the matrix element, so that the quantity (S_p-S_f) can be expanded in a series in powers $(\mathbf{p}-\mathbf{p}_f)$. Retaining in the expansion the first nonvanishing term we obtain

$$S_p(\mathbf{r},t)-S_f(\mathbf{r},t)=(\mathbf{p}-\mathbf{p}_f)\mathbf{x}, \tag{3.5}$$

where

$$\mathbf{x}=\frac{\partial}{\partial \mathbf{p}_f}S_f(\mathbf{r},t). \tag{3.6}$$

The quantity $\mathbf{x}$ is, according to Hamiltonian theory,[17,19] the final point of the particle trajectory in the external field. The relation (3.6) determines the trajectory of the particle $\mathbf{r}(t)=\mathbf{r}(t,\mathbf{x},\mathbf{p}_f)$.

Since $\mathbf{p}$ is close to $\mathbf{p}_f$, in the preexponential factor the determinant $|\partial^2 S/\partial \mathbf{r}\partial\mathbf{p}|$ can be assumed to be equal to $|\partial^2 S_f/\partial \mathbf{r}\partial\mathbf{p}_f|$. Using the relation

$$d^3r\left|\frac{\partial^2 S_f}{\partial\mathbf{r}\,\partial\mathbf{p}_f}\right|=d^3x \tag{3.7}$$

we arrive at the following expression for the matrix element:

$$M_{fi}=(2\pi\hbar)^{-3}\int d^3p\,d^3x\,dt$$

$$\times\exp\left[\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}_f)\mathbf{x}\right]e\mathbf{v}(t)\exp[i(\omega t-\mathbf{kr}(t))]S_{pp_i}. \tag{3.8}$$

The procedure employed here is analogous to the method used by Fock[15] to establish the relation between unitary transformations in quantum mechanics and canonical transformations in classical mechanics. For this reason, our method for calculating the matrix element of the emission probability can be called the method of canonical transformations in the quantum theory of radiation.

Introducing the function Φ determined by the formula (2.5) and setting $\varepsilon_i=\varepsilon_f$ in Φ (neglecting recoil), we write the radiation matrix element in the form

$$M_{fi}=\int\frac{d^3p}{(2\pi\hbar)^3}S_{pp_i}\Phi\left\{-i\hbar\frac{\partial}{\partial\mathbf{p}}\right\}\int d^3x\exp\left[\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}_f)\mathbf{x}\right]. \tag{3.9}$$

The integral over x appearing here is a δ-function $\delta(\mathbf{p}-\mathbf{p}_f)$, expressing the law of conservation of momentum in the emission process neglecting recoil, so that

$$M_{fi}=\int d^3p\,S_{pp_i}\Phi\left\{-i\hbar\frac{\partial}{\partial\mathbf{p}}\right\}\delta(\mathbf{p}-\mathbf{p}_f). \tag{3.10}$$

Next, transferring the action of the operator $(-\hbar\partial/\partial\mathbf{p})$, appearing in Φ, from the δ-function to the S-matrix with the help of integration by parts, we obtain

$$M_{fi}=\Phi\left\{i\hbar\frac{\partial}{\partial\mathbf{p}}\right\}S_{pp_i}|_{\mathbf{p}=\mathbf{p}_f}. \tag{3.11}$$

Since the operator Φ is determined by the trajectory of the particle in the external field, this formula relates the emission matrix element to the trajectory and the scattering matrix S. This is the basic formula of the theory being developed.

We shall now determine the action of the operator Φ on the scattering matrix (3.2). Since the operator $i\hbar\partial/\partial\mathbf{p}$ acts only on the functions $\exp(i\varepsilon\tau/\hbar)$ and $\exp(i\mathbf{pr}/\hbar)$ in the expression for S, we obtain

$$\Phi\left\{i\hbar\frac{\partial}{\partial\mathbf{p}}\right\}S_{pp_i}=\int d^2\rho'\,dz'\left[\Phi\{\rho',z\}\exp\left[\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}_i)\mathbf{r}'\right]\right.$$

$$-\frac{i}{\hbar}(2\varepsilon_i)^{1/2}\int d\tau\,\Phi\{\rho',z'-v\tau\}$$

$$\left.\times\exp\left\{\frac{i}{\hbar}[(\varepsilon-\varepsilon_i)\tau-\mathbf{pr}'\right\}U(\mathbf{r}')\varphi_i(\mathbf{r}')\right], \tag{3.12}$$

where the z' axis is oriented along $\mathbf{p}$ and ρ' denotes two coordinates orthogonal to $\mathbf{p}$.

In what follows we shall be interested in the emission probability integrated over p_{fz}, where the z-axis is oriented along the direction of the initial momentum. It is determined, neglecting recoil, by the following formula

$$dW_{fi}=\frac{e^2d^3k}{4\pi^2\hbar\omega}\frac{d^2q_\perp}{(2\pi\hbar)^2}\int\frac{dP_{fz}}{2\pi\hbar}$$

$$\times\left[\Phi\left\{i\hbar\frac{\partial}{\partial\mathbf{p}}\right\}S_{pp_i}\right]\left[\Phi^*\left\{i\hbar\frac{\partial}{\partial\mathbf{p}}\right\}S^*_{pp_i}\right], \tag{3.13}$$

where $\mathbf{q}$ is the component of the momentum $\mathbf{p}_f$ that is orthogonal to $\mathbf{p}_i$ and $\mathbf{p}\approx\mathbf{p}_f$.

We now transform the emission probability so that it contains the δ-function expressing the law of conservation of energy. To this end, performing in the emission probability (3.13) the integration by parts over p_{fz}, we transfer the operator $i\hbar\partial/\partial p_{fz}$ appearing in Φ from the factor S_{pp_i} to the function $(\Phi^*S^*_{pp_i})$. Using the relation (3.12), we obtain finally

$$dW_{fi}=\frac{e^2d^3k}{4\pi^2\hbar\omega}\frac{d^2q}{(2\pi\hbar)^2}\int dP_{fz}\delta(\varepsilon_f-\varepsilon_i)\int d\tau|A_\tau(\mathbf{q},\mathbf{k})|^2, \tag{3.14}$$

where

$$A_\tau=\int d^2\rho\,\Phi\{\rho,v\tau\}\left[\exp\left(\frac{i}{\hbar}\mathbf{q}\rho\right)\right.$$

$$\left.-\frac{i}{\hbar}(2\varepsilon_i)^{1/2}\int dz\exp\left(-\frac{i}{\hbar}\mathbf{pr}\right)U(\mathbf{r})\varphi_i(\mathbf{r})\right]. \tag{3.15}$$

We note that if the function Φ in the integrand for A_τ is replaced by unity, then we obtain the amplitude of elastic scattering of a particle in an external field. For $\Phi\neq1$ the quantity A_τ can be called the emission amplitude.

The general formula (3.15) for the emission amplitude, obtained in the quasiclassical approximation, greatly simplifies in the eikonal approximation. In this case the formula (3.15) transforms into the previously derived formula (2.8).

4. TAKING RECOIL INTO ACCOUNT

The formula (3.14) for the emission probability neglects recoil. In order to take recoil into account we shall turn to the general formula (3.3) for the matrix element. The action S_f appearing in this formula refers to an energy ε_f, which is not equal to the initial energy of the particle $(\varepsilon_f=\varepsilon_i-\hbar\omega)$. In calculating the matrix element neglecting recoil, we took into account the fact that values of $\mathbf{p}$ close to $\mathbf{p}_f$ make the main contribution to the integral over $\mathbf{p}$. The value of the energy $\varepsilon_p=\varepsilon_i$ was equal to the value of the final energy of the particle ε_f. When recoil is taken into account this relation is not satisfied, but the relation $\varepsilon_f=\varepsilon_i-\hbar\omega$ is satisfied. For this reason, it makes no sense to expand directly the difference of the actions $S_p - S_f$ in powers of $\mathbf{p} - \mathbf{p}_f$, and obviously the action S_f must be transformed so that the energy $\varepsilon' = \sqrt{\mathbf{p}'^2 + m^2}$ of the transformed action $S_{p'}$ would be equal to ε_p. In this case we shall also expand the action S_p in powers of $\mathbf{p} - \mathbf{p}'$, but now $\mathbf{p}'$ is not the final momentum of the particle $\mathbf{p}_f$ but rather the momentum $\mathbf{p}_f + \hbar\mathbf{k}$.

Thus we transform from the action S_f to the action S according to the formula

$$S_f = S + \hbar(\omega t - \mathbf{kr}). \tag{4.1}$$

Then S satisfies the equation

$$(\partial_t S + U + \hbar\omega)^2 = (\nabla S - \mathbf{A} - \hbar\mathbf{k})^2 + m^2 \tag{4.2}$$

and in the limit $t \to +\infty$ has the asymptotic form

$$S \to -(\varepsilon_f + \hbar\omega)t + (\mathbf{p}_f + \hbar\mathbf{k})\mathbf{r}. \tag{4.3}$$

We shall seek the solution of Eq. (4.2) in the form

$$S = S_{p'}(\mathbf{r}) - \varepsilon' t + \chi(t), \tag{4.4}$$

where the function $S_{p'}(\mathbf{r})$ satisfies the equation

$$(\varepsilon_{p'} - U)^2 = (\nabla S_{p'} - \mathbf{A})^2 + m^2 \tag{4.5}$$

and the asymptotic condition, in the limit $t \to +\infty$,

$$S_{p'} \to (\mathbf{p}_f + \hbar\mathbf{k})\mathbf{r}.$$

Then we arrive at the following equation for the function $\chi(t)$:

$$\partial_t\chi = -\hbar\frac{\varepsilon'}{\varepsilon' - \hbar\omega}\left(\omega - \mathbf{k}\frac{\nabla S_{p'} - \mathbf{A}}{\varepsilon' - U}\right) + \frac{1}{2(\varepsilon' - \hbar\omega)}(\partial_t\chi)^2. \tag{4.6}$$

Dropping the term proportional to $(\varepsilon' - \hbar\omega)^{-1}$ in Eq. (4.6), we arrive at the equation

$$\partial_t\chi = -\hbar\frac{\varepsilon'}{\varepsilon' - \hbar\omega}(\omega - \mathbf{kv}), \tag{4.7}$$

whose solution has the form

$$\chi(t) = \hbar\frac{\varepsilon'}{\varepsilon' - \hbar\omega}[\omega t - \mathbf{kr}(t)]. \tag{4.8}$$

Thus the emission matrix element will now contain the exponential factor

$$S_p - S_f + \hbar(\omega t - \mathbf{kr}) = S_p - S_{p'} - \chi(t).$$

We shall expand this expression in a series in powers of $\mathbf{p} - \mathbf{p}'$:

$$S_p - S_{p'} - \chi(t) \approx (\mathbf{p} - \mathbf{p}')\frac{\partial S_{p'}}{\partial \mathbf{p}'} - \chi(t), \tag{4.9}$$

after which we shall proceed in the same manner as in the derivation of the formula (3.14) for the emission probability neglecting recoil. But first we must transform in addition the preexponential factor of the quasiclassical wave function

$$\varphi_f = g^{1/2}\exp\left(\frac{i}{\hbar}S_f\right).$$

The function g in the quasiclassical approximation satisfies the equation

$$\partial_t[(\partial_t S_f + U)g] - \nabla[(\nabla S_f - \mathbf{A})g] = 0. \tag{4.10}$$

Using the relations (4.1) and (4.4) we arrive at the following equation for the function g:

$$(\varepsilon' - U)\partial_t g + \nabla[(\nabla S_{p'} - \mathbf{A})g] = \omega\partial_t g + \mathbf{k}\nabla g + \partial_t(\partial_t\chi g).$$

We shall seek the solution of the last equation in the form

$$g = cfu,$$

where f is determined by the equation

$$(\varepsilon' - U)\partial_t f + \nabla[(\nabla S_{p'} - \mathbf{A})f] = 0, \tag{4.11}$$

and c is a normalization factor.

Then we obtain the following equation for u:

$$(\varepsilon_{p'} - U)\partial_t u + (\nabla S_{p'} - \mathbf{A})\nabla u = f^{-1}[\omega\partial_t g + \mathbf{k}\nabla g + \partial_t(\partial_t\chi g)]. \tag{4.12}$$

It follows from Eq. (4.11) that

$$f = \frac{1}{\varepsilon_{p'} - U}\left|\frac{\partial^2 S_{p'}}{\partial\mathbf{r}\,\partial\mathbf{p}'}\right|. \tag{4.13}$$

Using this relation it is easy to verify that the function u is close to unity:

$$u \approx 1 + O(\varepsilon^{-2}).$$

Thus the function g with accuracy up to terms of the order of ε^{-2} is determined by the relation

$$g \approx \frac{c}{\varepsilon' - U}\left|\frac{\partial^2 S_{p'}}{\partial\mathbf{r}\,\partial\mathbf{p}'}\right|. \tag{4.14}$$

This relation differs from the corresponding preexponential factor of the wave function φ_f neglecting recoil only in the fact that in Eq. (4.14) the momentum $\mathbf{p}' = \mathbf{p}_f + \hbar\mathbf{k}$ appears instead of the final momentum of the particle $\mathbf{p}_f$. From the normalization condition for $\varphi_f(\mathbf{r})$ we find that $c = \varepsilon'/2\varepsilon_f$.

We note that with the same accuracy

$$\varepsilon' \approx \varepsilon_f + \hbar\omega. \tag{4.15}$$

Taking these facts into account and proceeding just as in the derivation of the formula for the emission probability neglecting recoil we arrive at the formula (2.4) for the emission probability taking recoil into account.

As is well known, in spinor electrodynamics the process of pair production by a photon in a fixed external field is closely related to the process of emission from an electron in this field taking recoil into account. Namely, having an expression for the emission probability, it is easy to obtain an expression for the probability of pair production by a photon in an external field, if the following substitutions are made in the matrix element of the emission process:

$$\varepsilon_f,\ \mathbf{p}_f \to \varepsilon_-,\ \mathbf{p}_-;\quad \omega,\ \mathbf{k} \to -\omega,\ -\mathbf{k};\quad \varepsilon_i,\ \mathbf{p}_i \to -\varepsilon_+,\ -\mathbf{p}_+, \tag{4.16}$$

where ε_-, $\mathbf{p}_-$ and ε_+, $\mathbf{p}_+$ are the energies and momenta of the electron and positron of the produced pair. In addition, in the emission probability the phase volume $d^3k\,d^3p_f$ must be replaced by $d^3p_+\,d^3p_-$. An analogous situation also occurs in scalar electrodynamics. For this reason, if the emission probability for a spinless particle neglecting the recoil is known, then it is easy to obtain the probability of production of a pair of spinless particles in a prescribed external field by a photon with momentum $\hbar k$ and energy $\hbar\omega$:

$$dW_{fi}=\frac{2\pi e^2}{\hbar\omega}\frac{d^3p_+}{(2\pi\hbar)^3}\frac{d^3p_-}{(2\pi\hbar)^3}|M_\pm|^2, \tag{4.17}$$

where $M_\pm$ is the matrix element of the pair production process.

In the quasiclassical approximation this general formula assumes the form

$$dW_\pm=\frac{e^2d^3p_+}{4\pi^2\hbar\omega}\frac{d^3p_-}{(2\pi\hbar)^3}\frac{\varepsilon_+}{\varepsilon_-}\left|\Phi_+\left\{i\hbar\frac{\partial}{\partial\mathbf{p}}\right\}S_{p,p_+}\right|^2\Big|_{\mathbf{p}=\hbar\mathbf{k}-\mathbf{p}_-}, \tag{4.18}$$

where S_{p,p_+} is the scattering matrix S of a positron scattered by an external field and

$$\Phi_+\{\mathbf{x}\}=\int_{-\infty}^{\infty}dt\,\mathbf{ev}_+(t)\exp\left\{i\frac{\varepsilon_+}{\varepsilon_-}[\omega t-\mathbf{kr}_+(t)]\right\}. \tag{4.19}$$

Here $\mathbf{r}_+(t)=\mathbf{r}_+(t,\mathbf{x},\mathbf{p}_+)$ is the trajectory of the positron in the external field and is determined by the final momentum $\mathbf{p}_+=\hbar\mathbf{k}-\mathbf{p}_-$ and the coordinates $\mathbf{x}$.

In the eikonal approximation the formula (4.18) leads to the following expression for the pair-production cross section

$$d\sigma_\pm=\frac{e^2d^3p_+}{4\pi^2\hbar\omega}\frac{\varepsilon_+}{\varepsilon_-}\frac{d^2q_\perp}{(2\pi\hbar)^2}|A_+(\mathbf{q},\mathbf{k})|^2, \tag{4.20}$$

where $\mathbf{q}_\perp=(\mathbf{p}_++\mathbf{p}_-)_\perp$ are the components of the momentum transferred to the pair which are orthogonal to the momentum of the photon $\hbar\mathbf{k}$ and

$$A_+(\mathbf{q},\mathbf{k})=\int d^2\rho\,\Phi_+\{\boldsymbol{\rho}\}\exp\left\{\frac{i}{\hbar}[\mathbf{q}\boldsymbol{\rho}+\chi_+(\boldsymbol{\rho})]\right\} \tag{4.21}$$

is the pair-production amplitude.

As is the case of the emission process, the integration over ρ in Eq. (4.21) can be performed by the stationary-phase method. If there are several points of stationary phase, then

$$d\sigma_\pm=\frac{e^2d^3p_+}{4\pi^2\hbar\omega}\frac{\varepsilon_+}{\varepsilon_-}d^2q_\perp\left|\sum_n\Phi_+\{\boldsymbol{\rho}_n^*\}\left|\frac{\partial\boldsymbol{\rho}_n^*}{\partial\mathbf{q}}\right|^{1/2}\exp\left(\frac{i}{\hbar}F_n^+\right)\right|^2, \tag{4.22}$$

where $F_n^*=\mathbf{q}\boldsymbol{\rho}_n^*+\chi_+(\boldsymbol{\rho}_n^*)$ is the scattering phase of the positron, corresponding to the stationary point $\boldsymbol{\rho}^*$.

This formula, as also the corresponding formula (2.10) for the emission cross section, describes the interference effect in pair production.

We can see that the cross section for pair production by a photon in an external field is determined by the classical trajectory of the positron. This cross section can also be expressed in terms of the classical trajectory of the electron. We call attention to the fact that in the formulas derived the trajectory of the positron or electron corresponds to the problem of scattering of a particle in an external field. In addition, it is implicitly assumed that no bound states are formed during scattering.

5. EMISSION PROBABILITY IN THE CASE OF RAINBOW SCATTERING

If the emission process occurs in a region whose length is much greater than the longitudinal dimensions of the range of the external forces acting on the particle, then, as is well known,[3,4] the cross section for elastic scattering of the particle can be removed from the emission cross section as a separate factor. This is also true in the quasiclassical approximation. In this case, the elastic-scattering cross section also takes into account the rainbow scattering effect.

Indeed, we now turn to the general formula (2.10) determining the emission cross section in the quasiclassical approximation. In the expression for the integral Φ appearing in this formula we perform the integration over time by parts:

$$\Phi=i\frac{\varepsilon_f}{\varepsilon_i}\int_{-\infty}^{\infty}dt\exp\left\{i\frac{\varepsilon_i}{\varepsilon_f}[\omega t-\mathbf{kr}(t)]\right\}\frac{d}{dt}\frac{\mathbf{ev}(t)}{\omega-\mathbf{kv}(t)}. \tag{5.1}$$

The time interval Δt in which the particle is exposed to the external field makes the main contribution to this integral. If in this time interval the exponent in the exponential in Eq. (5.1) is small compared with unity, i.e., if

$$\frac{\varepsilon_i}{\varepsilon_f}[\omega\Delta t-\mathbf{kr}(\Delta t)]\ll 1, \tag{5.2}$$

then the exponential in Eq. (5.1) can be replaced by unity and the function Φ assumes the form

$$\Phi\approx i\frac{\varepsilon_f}{\varepsilon_i}\left(\frac{\mathbf{ev}'}{\omega-\mathbf{kv}'}-\frac{\mathbf{ev}}{\omega-\mathbf{kv}}\right), \tag{5.3}$$

where $\mathbf{v}$ and $\mathbf{v}'$ are the velocity of the particle before and after scattering.

Substituting this expression for Φ into Eq. (2.9) we obtain the following expression for the emission cross section

$$d\sigma=dw\,d\sigma_{el}, \tag{5.4}$$

where $d\sigma_{el}$ is the differential cross section for elastic scattering

$$d\sigma_{el}=d^2q\left|\sum_n\left|\frac{\partial\boldsymbol{\rho}_n^*}{\partial\mathbf{q}}\right|^{1/2}\exp\left(\frac{i}{\hbar}F_n\right)\right|^2 \tag{5.5}$$

and dw is a factor that determines the emission probability when the velocity of the particle changes discontinuously from $\mathbf{v}$ to $\mathbf{v}'$,

$$dw=\frac{e^2d^3k}{4\pi^2\hbar\omega}\frac{\varepsilon_f}{\varepsilon_i}\left(\frac{\mathbf{ev}'}{\omega-\mathbf{kv}'}-\frac{\mathbf{ev}}{\omega-\mathbf{kv}}\right)^2. \tag{5.6}$$

Thus we can see that if the condition (5.2) is satisfied, then the scattering cross section factors. As regards the elastic scattering entering here, it in principle can take into account also the rainbow scattering of the particle. This happens if the deflection function of the particle in the external field is a two-valued function of the impact parameter.[10,11] As a result, rainbow scattering of the particle arises in the emission process. In other words, in the emission there arises an interference pattern owing to the existence of two classical trajectories with the same boundary conditions. This interference in the case of a long radiation formation length studied here is determined completely by the interference effect in elastic scattering.

The situation changes significantly if the radiation formation length is comparable to the longitudinal range of the external force acting on the particle. The interference pattern arising in this case in emission does not reduce only to the interference effect in elastic scattering; it is also determined by the interference effect in the radiation itself. The latter effect is caused both by interference for emission from different trajectories and by interference for emission from different sections of the same trajectory.

The interference pattern can become much more complicated, if there are more than two trajectories corresponding to the same boundary conditions. This situation can occur, for example, when relativistic particles pass through a crystal (see, in this connection, Refs. 13 and 20). This question, of course, requires a special investigation, since in this case, when a particle passes through a crystal, bound states, which are neglected in the present work, can arise.

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