Scaling of the conductance of a disordered conductor

A. M. Satanin

N. I. Lobachevskiĭ Nizhegorodskiĭ State University

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Exact scaling equations are derived for the elements of the S-matrix describing the scattering of electrons in a disordered conductor. Localization in a $2D$ system is analyzed in the weak-scattering case. A Fokker-Planck equation is derived for the parameters which determine the scaling of the conductance. This equation contains only one scaling parameter, the localization length l. The latter is determined by the correlation function D of the random potential and by the number of channels, N: $l \approx Nk^2/D$ for $N \gg 1$, where k is the wave vector. The growth exponents of the resistance and the conductivity are found and are expressed in terms of l.

1. FORMULATION OF THE PROBLEM

Several methods have recently been proposed for describing the localization of electrons in disordered conductors: localization field theories,[1,2] the random-ensemble method,[3-5] and approximate renormalization-group equations[6,7] (see the review by Lee and Ramakrishnan[8]). Conductance fluctuations in the weak-localization regime are of a universal nature. In the formulation of a scaling theory of localization, various assumptions are made regarding the scale of the splitting of the correlations. The number of effective scaling parameters and the limiting distribution of the conductance (or resistance) of a macroscopic sample are under debate.

The scaling properties of the conductance of a $1D$ disordered conductor are well understood.[9-14] In the $1D$ case it is possible to formulate a Cauchy problem for the parameters which determine how the conductance depends on the length of the conductor. A method based on the Fokker-Planck equation can be used to solve the Cauchy problem for equations with random coefficients.

It is important to generalize the method which has been worked out for $1D$ systems to the cases of $2D$ and $3D$ conductors. The conductance g of a conductor of finite dimensions can of course be expressed in terms of the elements of an S-matrix:

$$S=\begin{pmatrix} r & t' \\ t & r' \end{pmatrix}, \tag{1}$$

which relates the amplitudes of the incident and scattered waves. If there are a large number of open channels, g is given by the Landauer equation[15] (in units of $e^2/2\pi\hbar$)

$$g=\mathrm{Tr}(t^+t). \tag{2}$$

The scaling properties of g can be described by formulating a scaling equation for the matrix t.

In the present paper we use a method developed by Wigner and Eisenbud[16] in nuclear theory to derive exact scaling equations for the elements of the S-matrix. As in the $1D$ case, these equations constitute a Cauchy problem. To illustrate the use of these equations, we analyze the case of localization in the finite-dimensional approximation, i.e., the approximation of a finite number of channels. We construct a Fokker-Planck equation for the parameters which determine the scaling of the conductance. This equation contains only one scaling parameter: the localization length. We derive an exact expression for the localization length, which holds for an arbitrary number of open channels. We examine the scaling of the statistical moments of the conductance and the resistance.

2. SCALING EQUATION FOR THE S-MATRIX

We consider a conductor which occupies the interval $(0, L)$ along the x axis, while it is unbounded in the other dimensions (in the z direction in the case of a film; in the y and z directions in the case of a $3D$ layer). The relationship between the amplitudes of the incident and reflected waves is determined by the S-matrix in (1). To study the scaling of the conductance it is sufficient to write equations for r and t. The matrices r' and t' can be expressed in terms of r and t by using the conditions for unitarity and symmetry under time reversal.

For definiteness we consider a $3D$ layer. Inside this layer the wave function satisfies the equation

$$\frac{\partial^2\psi_{\mathbf{q}}(x)}{\partial x^2}=-k_q^2\psi_{\mathbf{q}}(x)+\int V_{\mathbf{q},\mathbf{q}'}(x)\psi_{\mathbf{q}'}(x)d\mathbf{q}', \tag{3}$$

where

$$k_q^2=(2mE/\hbar^2-\mathbf{q}^2)^{1/2},$$

$$\psi_{\mathbf{q}}(x)=\int d\boldsymbol{\rho}\exp(-i\mathbf{q}\boldsymbol{\rho})\psi(x,\boldsymbol{\rho}),\qquad \boldsymbol{\rho}=(y,z),$$

$$V_{\mathbf{q},\mathbf{q}'}(x)=\frac{2m}{\hbar^2}\int d\boldsymbol{\rho}\exp[-i\boldsymbol{\rho}(\mathbf{q}-\mathbf{q}')]V(x,\boldsymbol{\rho}),$$

and $V(x, \boldsymbol{\rho})$ is the impurity potential. We write the general solution of (3) in the form

$$\psi_{\mathbf{q}}(x)=\int d\mathbf{q}'\, c_{\mathbf{q}'}\hat{\psi}_{\mathbf{q},\mathbf{q}'}(x)$$

or, in matrix notation,

$$\psi_q=(\hat{\psi}c_q).$$

Outside this layer we have

$$\begin{aligned}\psi^R&=\int\frac{d\mathbf{q}}{k_q^{1/2}}\{A_{\mathbf{q}}^R\exp[-ik_q(x-L)]\\&\quad+B_{\mathbf{q}}^R\exp[ik_q(x-L)]\}\exp(i\mathbf{q}\boldsymbol{\rho}),\quad x>L.\\ \psi^L&=\int\frac{d\mathbf{q}}{k_q^{1/2}}B_{\mathbf{q}}^L\exp(-ik_qx+i\mathbf{q}\boldsymbol{\rho}),\qquad x<0.\end{aligned} \tag{4}$$

Introducing the operator

$$\hat{k}=\left(\frac{2mE}{\hbar^2}-\Delta_\rho\right)^{1/2},\qquad \Delta_\rho=\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},$$

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we write the matching condition at $x = L$:

$$\hat{k}^{-1/2}(I+r)A^R=\hat{\psi}(L)c,$$
$$-i\hat{k}^{1/2}(I-r)A^R=\frac{d\hat{\psi}}{dL}c. \tag{5}$$

At $x = 0$ it is sufficient to write

$$tA^R=\hat{\psi}(0)c. \tag{6}$$

Introducing the matrix

$$R=\frac{d\hat{\psi}}{dL}\hat{\psi}^{-1}, \tag{7}$$

we can put (5) and (6) in the following form, after we eliminate A^R:

$$-i\hat{k}^{1/2}(I-r)=R\hat{k}^{-1/2}(I+r), \tag{8}$$

$$t=\hat{\psi}(0)\hat{\psi}^{-1}(\hat{L})k^{-1/2}(I+r). \tag{9}$$

From (7) and (3) we find the following equations for R:

$$\dot{R}=-\hat{k}^2+V_L-R^2, \quad \dot{R}=dR/dL, \tag{10}$$

where V_L is the matrix V at $x = L$. Differentiating (8) and (9) with respect to L, and using (10), we find

$$\dot{r}=i(\hat{k}r+r\hat{k})+\frac{1}{2i}(I+r)\hat{k}^{-1/2}V_L\hat{k}^{-1/2}(I+r), \tag{11}$$

$$\dot{t}=it\hat{k}+\frac{1}{2i}t\hat{k}^{-1/2}V_L\hat{k}^{-1/2}(I+r). \tag{12}$$

Equations (11) and (12) describe the evolution of r and t as a function of the layer thickness L. The initial conditions are found at $L = 0$:

$$r(0)=0, \quad t(0)=I. \tag{13}$$

From the symmetry under time reversal we have

$$r_{q,q'}=r_{-q',-q}, \quad r'_{q,q'}=r'_{-q',-q},$$
$$t'_{-q',-q}=t_{q,q'}. \tag{14}$$

The unitarity condition gives us

$$r'=-tr^+(t^+)^{-1} \tag{15}$$

and the relations

$$r^+r+t^+t=I, \quad r'^+r'+t^*t^T=I. \tag{16}$$

These relations impose constraints on the elements of the S-matrix and can be used to express r' and t' in terms of r and t. Equations (11) and (12), along with (13)–(16), are scaling equations for the S-matrix and constitute a Cauchy problem. An equation like (11) for the reflection amplitude r was derived previously by Babkin *et al.*[17] by an embedding method.

3. FINITE-DIMENSIONAL APPROXIMATION IN A 2D SYSTEM

Attempts to analyze Eqs. (11) and (12) in the general case run into serious difficulties, since the Fokker-Planck equation is a functional-derivative equation. Let us consider a $2D$ system (a film) with a finite dimension L_z in the transverse direction ($0<z<L_z$). For a given E, equal to the Fermi energy, there is a finite number of open channels $n = 1, 2,..., N$, where

$$N=[(2mL_z^2E/\pi^2\hbar^2)^{1/2}].$$

In this event,

$$(\hat{k})_{nn'}=k_n\delta_{nn'}, \quad k_n^2=2mE/\hbar^2-E_n,$$
$$E_n=\pi^2\hbar^2n^2/2mL_z^2.$$

The conductance component coming from virtual transitions accompanying scattering into higher-lying bands with $E_n > E$ is small by virtue of the inequality

$$\hbar/|E-E_n|\tau\ll 1,$$

where τ is the collision time; interband transitions are taken into account here. This point can be verified by analyzing the Green's functions, which have poles for $E<E_n$ but not for $E>E_n$. The time τ is determined by the imaginary part of the eigenenergy of the average one-particle Green's function. Assuming that the scattering is weak, we ignore virtual transitions to high-lying levels. Under this assumption, the matrices r and t have $N\times N$ elements. The wave functions of the transverse motion, $\psi_n(z)$, are real if we impose the boundary conditions $\psi_n(0) = \psi_n(L_z) = 0$. In the basis $\psi_n(z)$, relations (14)–(16) become

$$r=r^T, \quad r'=(r')^T, \quad t'=t^T, \quad r'=-tr^+(t^+)^{-1}, \quad r^+r+t^+t=I.$$

We introduce the parametrization $r = v^T\Lambda v$, $t = u\Sigma v$, where Λ and Σ are real diagonal matrices, related by $\Lambda^2 + \Sigma^2 = I$, and v and u are unitary matrices: $v^+v = I$, $u^+u = I$. We set

$$\Lambda_n=\text{th}(\chi_n/2), \quad \Sigma_n=\text{ch}^{-1}(\chi_n/2),$$

then the conductance is given by

$$g=\sum_n\frac{2}{\text{ch}\,\chi_n+1}. \tag{17}$$

From the equations for r and t we have

$$\dot{\chi}_n=-{}^1/_2\,\text{Im}(v\bar{V}_Lv^T)_{nn}, \quad \bar{V}_L=\hat{k}^{-1/2}V_L\hat{k}^{-1/2}, \tag{18}$$

$$\dot{v}=iv\hat{k}+(2i)^{-1}Fv, \tag{19}$$

$$\dot{u}=(2i)^{-1}uM, \tag{20}$$

where the matrices F and M are given by

$$F_{nn'}=(v\bar{V}_Lv^+)_{nn'}+\text{cth}[(\chi_n+\chi_{n'})/2]\times\text{Re}(v\bar{V}_Lv^T)_{nn'}$$
$$-i\,\text{cth}[(\chi_n-\chi_{n'})/2]\times\text{Im}(v\bar{V}_Lv^T)_{nn'}(1-\delta_{nn'}), \tag{21}$$

$$M_{nn'}=-\frac{\text{Re}(v\bar{V}_Lv^T)_{nn'}}{\text{sh}[(\chi_n+\chi_{n'})/2]}+i\frac{\text{Im}(v\bar{V}_Lv^T)_{nn'}}{\text{sh}[(\chi_n-\chi_{n'})/2]}(1-\delta_{nn'}). \tag{22}$$

Equations (18)–(20) describe the dynamics of a system on a manifold which is parametrized by the angles $\{\chi_n\}$, $0<\chi_n<\infty$, and the matrices v and u ($v^+v = I$, $u^+u = I$). To construct a Fokker-Planck equation we need to make a further reduction, parametrizing each of the matrices v and u by sets of N^2 real variables. As can be seen from (17), (18), and (21), to calculate the conductance it is sufficient to analyze the equations for $\{\chi_n\}$ and v only. We write v as

$$v=h^+\Phi h, \quad h^+h=I,$$

where Φ is the diagonal matrix $\Phi_{nn'} = \delta_{nn'}\exp(i\varphi_{n/2})$, and

h depends on $N(N-1)$ real parameters. From (19) we have

$$\dot\varphi_n = 2\bar k_{nn} - \bar F_{nn}, \tag{23}$$

$$(\dot h h^+)_{nn'} = \frac{i\Phi_n \bar k_{nn'}}{\Phi_n - \Phi_{n'}} + \frac{1}{2i}\frac{\bar F_{nn'}\Phi_{n'}}{\Phi_n - \Phi_{n'}}, \qquad \Phi_n = \exp(i\varphi_n/z), \tag{24}$$

where $\bar k = h\hat k h^+$, $\bar F = hFh^+$. We denote by $\theta_1,\ldots,\theta_{N(N-1)}$ the angles which parametrize h, and we introduce

$$(e_\alpha)_{nn'} = \left(\frac{\partial h}{\partial \theta_\alpha} h^+\right)_{nn'}, \qquad \alpha = 1,\ldots, N(N-1). \tag{25}$$

All the matrices in (24) and (25) are expanded in a complete set of orthogonal $N \times N$ matrices τ_α, which are normalized by

$$\mathrm{Tr}(\tau_\alpha \tau_\beta) = 2\delta_{\alpha\beta}.$$

Equation (24) then becomes

$$e_{\alpha\beta}\dot\theta_\beta = ip_\alpha + \frac{1}{2i}\Pi_\alpha, \tag{26}$$

where

$$e_{\alpha\beta} = {}^1\!/_2 \mathrm{Tr}(\tau_\alpha e_\beta),$$

$$p_\alpha = \frac{1}{2}\mathrm{Tr}(\tau_\alpha \hat p), \qquad (\hat p)_{nn'} = \frac{\Phi_n \bar k_{nn'}}{\Phi_n - \Phi_{n'}},$$

$$\Pi_\alpha = \frac{1}{2}\mathrm{Tr}(\tau_\alpha \hat\Pi), \qquad (\hat\Pi)_{nn'} = \frac{\bar F_{nn'}\Phi_{n'}}{\Phi_n - \Phi_{n'}}.$$

From (26) we have

$$\dot\theta_\alpha = \bar p_\alpha + \bar\Pi_\alpha, \tag{27}$$

where

$$\bar p_\alpha = i(e^{-1})_{\alpha\beta} p_\beta,$$

$$\bar\Pi_\alpha = \frac{1}{2i}(e^{-1})_{\alpha\beta}\Pi_\beta.$$

Equations (18), (23), and (27) are thus stochastic equations in real variables. They give a complete description of the scaling properties of the conductance. A Fokker-Planck equation is derived for these equations in the following section of this paper.

In the case $N = 1$ we find

$$\dot\chi = -\frac{V_L}{k}\sin\varphi, \qquad \dot\theta = \frac{V_L\cos\varphi}{2k\,\mathrm{ch}^2(\chi/2)},$$

$$\dot\varphi = 2k - (V_L/k)(1 + \mathrm{cht}\,\chi\cos\varphi).$$

These equations are the well-known equations for the $1D$ case. They were analyzed in detail for various random processes in Ref. 14.

4. FOKKER-PLANCK EQUATION

In principle, we could work from Eqs. (18), (23), and (27) to find the exact distributions of the parameters $\{\chi_n\}$, $\{\varphi_n\}$, and $\{\theta_\alpha\}$ are thus determine the statistical characteristics of the conductance. However, the Fokker-Planck equation which results does not lend itself to a general analysis, even in the case $N = 1$. Simplifications are found in the case of weak scattering, in which the characteristic matrix elements $\hbar^2|(V_L)_{nn'}|/2m$ are small in comparison with the spacing of the transverse-quantization levels, $|E_n - E_{n'}|$.

We begin with a qualitative analysis and a geometric interpretation of the parameters $\{\chi_n\}$, $\{\varphi_n\}$, and $\{\theta_\alpha\}$. The angles χ_n and φ_n can be treated, as in the $1D$ case, as the coordinates of a "particle" on a hyperboloid of index n. The angles θ_α are responsible for the interaction of such particles. At $L = 0$, we have $\chi_n = 0$ for all n. As the size of the system increases, the scattering by the impurity field V_L gives rise to diffusion of the particles among the hyperboloids. It can be seen from (23) that the angles φ_n rotate with characteristic frequencies $\sim k_n$ as L varies. Correspondingly, we see from (27) that for θ_α these frequencies are $\sim|k_n - k_{n'}|$. When the size of the system, L, becomes substantially greater than k_n^{-1} and $|k_n - k_{n'}|^{-1}$, the distribution of $\{\varphi_n\}$ and $\{\theta_\alpha\}$ becomes uniform, because of the compactness of these variables. In the weak-scattering case, the variables $\{\varphi_n\}$ and $\{\theta_\alpha\}$ are thus fast, while the $\{\chi_n\}$ are slow. In the $1D$ case, this circumstance is utilized in the following way. The joint distribution function of the angles χ and φ can be expanded in eigenfunctions which realize a representation of the rotation group on a circle [isomorphic to the group $U(1)$]; then the coefficient of the zeroth harmonic can be distinguished. It is this coefficient which determines the distribution function of the slow variable χ. This procedure is effectively equivalent to taking an average of the Fokker-Planck equation over the angles φ, i.e., integrating over the rotation group. In the multichannel case, the distribution function must be expanded in the group $U(N)$, and the Fokker-Planck equation must be averaged over the group measure of the $U(N)$ group in order to distinguish the slow variables.

We consider a Gaussian random field $V(x,z)$ with the correlation function

$$\langle V(x,z)V(x',z')\rangle = 2D\delta(x-x')\delta(z-z').$$

In the basis $\psi_n(z)$, the correlation function for V_L can be written

$$\langle (V_L)_{nn'}(V_L)_{mm'}\rangle = 2D_{nn',mm'}\delta(L-L'), \tag{28}$$

where

$$D_{nn',mm'} = D\int_0^{L_z} dz\, \psi_n(z)\psi_{n'}(z)\psi_m(z)\psi_{m'}(z).$$

Writing the Fokker-Planck equation with the help of Eqs. (18), (23), and (27), and taking an average over the fast variables, we find

$$\dot P(\{\chi_n\},L) = \left(\sum D_{nn'}\frac{\partial^2}{\partial\chi_n\partial\chi_{n'}} - \sum\frac{\partial}{\partial\chi_n}f_n\right)P(\{\chi_n\},L), \tag{29}$$

where

$$D_{nn'} = \frac{1}{2}\sum\frac{D_{bcdf}}{(k_b k_c k_d k_f)^{1/2}}\langle\langle \mathrm{Im}(v_{nb}v_{nc})\,\mathrm{Im}(v_{n'd}v_{n'f})\rangle\rangle, \tag{30}$$

$$f_n = \sum_{n'}\mathrm{cth}\left(\frac{\chi_n+\chi_{n'}}{2}\right)R_{nn'} + \sum_{n'\neq n}\mathrm{cth}\left(\frac{\chi_n-\chi_{n'}}{2}\right)I_{nn'}. \tag{31}$$

$$R_{nn'} = \frac{1}{2}\sum\frac{D_{bcdf}}{(k_b k_c k_d k_f)^{1/2}}\left[\langle\langle \mathrm{Re}(v_{nd}v_{n'f})\,\mathrm{Re}(v_{n'c}v_{n'b})\rangle\rangle + \langle\langle \mathrm{Re}(v_{nd}v_{n'f})\,\mathrm{Re}(v_{n'c}v_{nb})\rangle\rangle\right], \tag{32}$$

$$I_{nn'}=\frac{1}{2}\sum\frac{D_{bcdf}}{(k_bk_ck_dk_f)^{1/2}}[\langle\langle\mathrm{Im}(v_{nd}v_{n'f})\,\mathrm{Im}(v_{nc}v_{n'b})\rangle\rangle$$
$$+\langle\langle\mathrm{Im}(v_{nd}v_{n'f})\,\mathrm{Im}(v_{n'c}v_{nb})\rangle\rangle]. \qquad (33)$$

The double angle brackets $\langle\langle...\rangle\rangle$ in (30)–(33) mean an average over the angles $\{\varphi_n\}$ and $\{\theta_\alpha\}$. Since $\{\varphi_n\}$ and $\{\theta_\alpha\}$ are parameters of the group $U(N)$, this average actually means an integration over the group measure of $U(N)$.

The group average can be carried out with the help of

$$\langle\langle v_{ab}v_{cd}v^*_{ef}v^*_{pq}\rangle\rangle=(N^2-1)^{-1}\times(\delta_{ae}\delta_{bf}\delta_{cp}\delta_{dq}$$
$$+\delta_{ap}\delta_{bq}\delta_{ce}\delta_{df})-N^{-1}\times(\delta_{ae}\delta_{bq}\delta_{df}\delta_{cp}+\delta_{ap}\delta_{ce}\delta_{bf}\delta_{dq}). \qquad (34)$$

Equation (34) can be proved by direct evaluation, through an expansion of the product of group matrices in irreducible representations[18] or through the use of the generating function. Using (34), we find from (30)–(33)

$$D_{nn'}=\delta_{nn'}/l,\quad R_{nn'}=I_{nn'}=(1+\delta_{nn'})/2l,$$
$$\frac{1}{l}=\frac{1}{N(N+1)}\sum_{n,n'}\frac{D_{nn',nn'}}{k_nk_{n'}}. \qquad (35)$$

The Fokker-Planck equation thus becomes

$$\dot P(\{\chi_n\},L)=\frac{1}{l}\Big(\sum\frac{\partial^2}{\partial\chi_n^2}-\sum\frac{\partial}{\partial\chi_n}f_n\Big)P(\{\chi_n\},L), \qquad (36)$$

where

$$f_n=\mathrm{cth}\,\chi_n+\frac{1}{2}\sum_{n\neq n'}\Big[\mathrm{cth}\Big(\frac{\chi_n+\chi_{n'}}{2}\Big)+\mathrm{cth}\Big(\frac{\chi_n-\chi_{n'}}{2}\Big)\Big]. \qquad (37)$$

The Fokker-Planck equation has a single scale l, the localization length. Using the explicit expression for the functions $\psi_n(z)$, we find from (35)

$$\frac{1}{l}=\frac{D}{2N(N+1)L_z}\Big(\sum\frac{1}{k_n^2}+2\sum\frac{1}{k_nk_{n'}}\Big). \qquad (38)$$

At large N, we easily find from this expression the result

$$l\approx Nk^2/D,\quad k^2=2mE/\hbar^2. \qquad (39)$$

5. ANALYSIS OF THE FOKKER-PLANCK EQUATION

In the $1D$ case ($N=1$), Eq. (36) is

$$\dot P(\chi,L)=\frac{1}{l}\Big(\frac{\partial^2}{\partial\chi^2}-\frac{\partial}{\partial\chi}\mathrm{cth}\,\chi\Big)P(\chi,L). \qquad (40)$$

This equation can be solved exactly.[9,11] For comparison with the multichannel case, we write the solution of (4) in the form

$$P(\chi,L)=(\mathrm{sh}\,\chi)^{1/2}\int_0^\infty ds\exp\Big[-\Big(\Lambda_s+\frac{1}{4}\Big)\frac{L}{l}\Big]C_s\Psi_s(\chi), \qquad (41)$$

where $\Psi_x(\chi)$ is the solution of the equation

$$H\Psi_s(\chi)=\Lambda_s\Psi_s(\chi),\quad H=-\frac{\partial^2}{\partial\chi^2}-\frac{1}{4\,\mathrm{sh}^2\chi}-\frac{1}{4},$$
$$\Psi_s=\frac{1}{2}\Big(\frac{\pi}{2}\mathrm{sh}\,\chi\Big)^{1/2}$$
$$\times\exp\Big[-\Big(is-\frac{1}{2}\Big)\chi\Big]F\Big(\frac{1}{2},-is-\frac{1}{4},1,-e^x\,\mathrm{sh}\,\chi\Big), \qquad (42)$$

$$s^2=\Lambda_s+{}^1/_4,$$

$F(\alpha,\beta,\gamma,z)$ is the hypergeometric function, and C_s is found from the initial condition $P(\chi,0)=\delta(\chi)$. Using (41), we can easily calculate the probability distribution for the resistance and the conductance.[9,11] In the $1D$ case, according to Landauer, we should consider the reflection of electrons from a disordered region. The resistance is then given by

$$\rho=g^{-1}=(\mathrm{ch}\chi-1)/2,$$

and the resistance distribution is

$$W(\rho,L)=\frac{\exp(-L/4l)}{(2\pi)^{1/2}(L/l)^{1/2}}\int_{2\rho+1}^\infty\frac{dx\,x\exp(-x^2L/4l)}{[\mathrm{ch}\,x-\mathrm{ch}(2\rho+1)]}.$$

Under the condition $L\gg l$, the distribution $\chi\simeq\ln\rho$ obeys a normal law. To make a comparison with the multichannel case, we follow Abrikosov,[10] introducing a different definition of the conductance, which holds for the case of a barrier with a small transmission:[7]

$$g=|t|^2=2/(\mathrm{ch}\chi+1).$$

The moments $\langle g^n\rangle$ have a finite value even for integers.[10] From (41) we find ($L\gg l$)

$$\langle g^n\rangle=B_{n,1}\exp\Big(-\frac{L}{4l}\Big),\quad B_{n,1}=\frac{\pi^{3/2}\Gamma(n-{}^1/_2)}{\Gamma^2(n)}\Big(\frac{l}{L}\Big)^{3/2}, \qquad (43)$$

where $\Gamma(x)$ is the gamma function. The scale of the variation in the moments is determined by the lower boundary of the spectrum of the operator H, which is 1/4, according to (42).

We can draw the following conclusions from an analysis of the $1D$ system. Under the condition $L\gg l$ the variables χ are Gaussian, and the decay index of the moments is related to the lower boundary of the spectrum of the operator H. Let us attempt to find the corresponding properties of the multichannel system.

For $N>1$, the Fokker-Planck equation can again be reduced to a Schrödinger equation with an imaginary time, if we use the substitution $P=\exp(-G)\Psi$, where the function G is determined by the solution of the equation

$$\frac{\partial G}{\partial\chi_n}=\frac{1}{2}f_n. \qquad (44)$$

If (44) is to have a solution, the following integrability condition must hold:

$$\frac{\partial f_n'}{\partial\chi_n}=\frac{\partial f_n}{\partial\chi_{n'}}. \qquad (45)$$

It is a simple matter to verify that this condition holds for the function f_n in (37). Solving (44), we find

$$G=\sum\ln\mathrm{sh}\,\chi_n$$
$$+\frac{1}{2}\sum_{n\neq n'}\Big[\ln\mathrm{sh}\Big(\frac{\chi_n+\chi_{n'}}{2}\Big)+\ln\mathrm{sh}\Big|\frac{\chi_n-\chi_{n'}}{2}\Big|\Big]. \qquad (46)$$

As in the single-channel case, we thus have

$$-l\dot\Psi=H\Psi,\quad H=-\sum\frac{\partial^2}{\partial\chi_n^2}+u(\{\chi_n\}),$$
$$u=\frac{1}{2}\sum\frac{\partial f_n}{\partial\chi_n}+\frac{1}{4}\sum f_n^2. \qquad (47)$$

The operator H can be written in the form

$$H=\sum_n Q_n{}^+Q_n,$$

where

$$Q_n=\frac{\partial}{\partial\chi_n}-\frac{1}{2}f_n,\qquad Q_n{}^+=-\frac{\partial}{\partial\chi_n}-\frac{1}{2}f_n.$$

It follows that H has positive definite eigenvalues. In the limit $\chi_n\to\infty$ ($n=1,...,N$), $u\to u_N$, we have

$$u\to u_N=1/24N(N+1)(2N+1). \qquad (48)$$

The continuous spectrum of H thus begins at u_N. We write the general solution of (36) in the form

$$P=\Bigl[\Bigl(\prod_n \operatorname{sh}\chi_n\Bigr)^{1/2}\prod_{n,n'}\operatorname{sh}\Bigl(\frac{\chi_n+\chi_{n'}}{2}\Bigr)\operatorname{sh}\Bigl|\frac{\chi_n-\chi_{n'}}{2}\Bigr|\Bigr]^{1/4}$$
$$\times\int\prod_\lambda ds_\lambda \exp\Bigl[-(\Lambda_{\{s\}}+u_N)\frac{L}{l}\Bigr]C_{\{s\}}\Psi_{\{s\}}(\{\chi\}), \qquad (49)$$

where $\Psi_{\{s\}}$ and $\Lambda_{\{s\}}$ are the eigenfunctions and eigenvalues of the equation

$$H\Psi_{\{s\}}=(\Lambda_{\{s\}}+u_N)\Psi_{\{s\}}. \qquad (50)$$

The Hamiltonian H in (47) describes a system of N interacting particles which are moving in the half-space $0<x_n<\infty$. If the spectrum of H of the system of particles has no bound states in the interval $(0, u_N)$, as is true in the case $N=1$, the asymptotic behavior of the moments of the conductance, $\langle g^n\rangle$, can be found from (49):

$$\langle g^n\rangle\approx B_{n,N}\exp(-u_N L/l), \qquad (51)$$

where $B_{n,N}$ are power functions of L.

This assumption can be verified only for $N=2$. Analysis of u in (47) shows that the potential of the interaction between particles contains, in addition to the nonuniform field (terms of the form $\sim 1/\sinh^2\chi_n$), some terms which describe two- and three-particle interactions. To trace the evolution of an initial distribution $P(\{\chi\},0)=\Pi\delta(\chi_n)$ over the entire range of the length ($L<l$ and $L>l$), we need to know the exact spectrum of H.

Certain conclusions regarding the limiting behavior of the resistance can be drawn by considering the system of stochastic equations to which Eq. (36) corresponds. We introduce Gaussian random forces with a correlation function

$$\langle\xi_n(L)\xi_{n'}(L')\rangle=2l^{-1}\delta_{nn'}\delta(L-L'). \qquad (52)$$

It is simple to verify that Eq. (36) corresponds to the system of stochastic equations

$$\dot\chi_n=l^{-1}f_n+\xi_n,\qquad n=1,\ldots,N, \qquad (53)$$

where ξ_n are defined in accordance with (52). In the $N=1$ case, we have

$$\dot\chi=l^{-1}\operatorname{cth}\chi+\xi$$

and the evolution of χ is as follows. In the region $L<l$, the average value of χ increases linearly with the length. If $L>l$, then the force $\coth\chi/l$ pushes the particle out of the region $\chi\lesssim 1$. Later, for $x\gg 1$, it becomes independent of χ. The mean value here is $\langle\chi\rangle=L/l$, and the measure of the dispersion is also proportional to the length. In typical realizations, χ increases in proportion to the length. Under the condition $L\gg l$, the logarithm of the resistance, $\chi\simeq\ln\rho$, obeys a normal law.

For $N\gg 1$, the evolution of the system is considerably more complex. If $L\gg l$, the particles which start from the origin, $\chi_n=0$, are moved by the forces $\sim\coth\chi_n/l$ into the region $\chi_n\gtrsim 1$. For $\chi_n>1$, they become ordered. By virtue of the mutual repulsion $\sim\coth[(\chi_n-\chi_{n'})/2]/l$, they diffuse under the influence of the random forces ξ_n. Their order of occurrence, determined in the region $\chi_n\lesssim 1$, cannot change. Let us assume that for $L>l$ an order

$$1<\chi_1<\chi_2<\ldots<\chi_n$$

has been established. For the mean value $\chi_s=\Sigma\chi_n/N$ we find the following exact expression from (53):

$$\langle\dot\chi_s\rangle=N/2l,$$

i.e., the center of the distribution, $\langle\chi_s\rangle=NL/2l$, shifts into the region $\chi\gg 1$ at $L\gg l$. The relative distance between particles also increases in proportion to the length of the system. Under the condition $\chi_n\gg 1$ we find from (53)

$$\langle\dot\chi_n\rangle\approx n/l,\qquad \langle\chi_n\rangle=nL/l.$$

The channel with the smallest value of n thus contributes to the conductance at $L\gg l$. The resistance of the overall system, which is determined primarily by this channel, as in the 1D case, will have a log-normal distribution. Similar arguments were offered by Dorokhov,[12] who studied localization in a system of weakly coupled conductors with 1D topology.

6. CONCLUSION

The scaling equations derived here are exact. They are a natural generalization of the method developed in Refs. 9–14 for 1D systems. In addition to the conductance, one can express the transmission coefficients, the delay times, the density of states of the open system, and other properties in terms of the S-matrix. Since the scaling equations constitute a Cauchy problem, one can calculate statistical characteristics from them by means of the sophisticated apparatus of the theory of Markov processes and the efficient numerical methods for solving the Cauchy problem.

The method developed in this paper differs from the approach taken in Refs. 3–5, where the random nature of the T-matrix was postulated, and the scale of the splitting of the correlations was not clear. The method of this paper is based on exact scattering characteristics in a random potential.

Let us consider the conclusions which follow from an analysis of localization in a 2D system. As was shown in Sec. 5, the Fokker-Planck equation for the parameters which determine the scaling properties of the conductance contain a single scaling parameter, the localization length l. An explicit expression for this length has been derived in terms of the correlation function of the scattering potential, and the dependence on the number of channels has been determined. The growth exponents of the parameters which determine the behavior of the conductance and the resistance, however, are characterized by a set of lengths which are expressed in terms of l. We have analyzed the structure of the phase space of the dynamic system to which the localization problem reduces in the multichannel case. We are thus able to find a qualitative picture of this phenomenon and to study its relationship with other many-particle problems.

A nontrivial behavior of the system arises in the limit $N \to \infty$. This limit can be taken at the outset, by treating the equations for r and t in the Fourier representation in the transverse coordinates. In this parametrization, it is possible to study the scaling equations for averages of the form $\langle (t^+)^n t^n \rangle$.

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