Propagation of a soliton and emission from it in a resonant nonlinear optical waveguide

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The motion of a soliton pulse in a resonant nonlinear single-mode optical waveguide and the emission by this pulse are studied for the case in which there is a deviation from that matching of the carrier frequency and the waveguide properties which would lead to a pure soliton propagation regime. This disruption is treated as a perturbation of an integrable system. A perturbation theory for solitons is modified to deal with the effect of the emission by the soliton on the medium. The properties of the perturbed soliton are calculated. The spectral power density of the emission is found. It reveals a very nonmonotonic dependence on the wave number.

1. INTRODUCTION

Nonlinear-optics phenomena can serve as an effective proving ground for developing and refining theoretical concepts of soliton physics and the corresponding experimental base. With a eye on the properties of optical media, we can distinguish two directions for research of this type. The first is aimed at optical solitons in a nonresonant medium, the classic example of which is a plane waveguide channel which arises during steady-state two-dimensional focusing in a medium with a Kerr nonlinearity.[1] Included in this category is the problem of the propagation of ultrashort pulses through an optical waveguide with a nonlinearity which is quadratic in the field.[2,3] The second direction is toward solitons in resonant media, which are responsible for self-induced transparency,[4–6] superfluorescence,[6–8] the passage of pulses through a nonlinear film,[9,10] and several other effects. The present status of self-induced transparency is reflected in the review by Maimistov *et al.*[11]

One should also bear in mind that in actual optical media both these nonlinearities are present to some extent. There is accordingly the fundamentally important question of whether a soliton can exist in a "composite" medium of this sort. An affirmative answer was given to this question in Refs. 12–15, where it was shown that a soliton arises when a plane light beam is scanned over the surface of a resonant Kerr medium. A soliton regime arises when a certain condition involving the parameters of this system is satisfied. Specifically, the scanning velocity must be correlated in a certain way with the parameters of the medium and of the emission. The physical meaning of the condition is that a balance is struck among such competing factors as the diffractive divergence, the self-focusing, and the evolution of the two-level subsystem. A soliton in a composite medium has characteristics of both a waveguide channel and a 2π pulse. This situation is described by the Maxwell-Bloch equations, but in this case, in contrast with Refs. 4–6, Maxwell's equation includes terms which stem from the diffractive divergence and the Kerr nonlinearity.

As was first shown in Ref. 16, the same equations can be used to describe the propagation of ultrashort pulses through a nonlinear single-mode optical waveguide containing resonant impurities. The impurities qualitatively change the evolution of the pulse as a soliton. It becomes necessary to match the frequency of the light wave with the parameters of the nonlinear medium. This matching condition is rather restrictive, but one might attempt to satisfy it by varying the frequency of the light wave over the inhomogeneously broadened line of the resonant transition.[16] A question which arises here is how a mismatch (which might arise, in particular, from errors in the determination of the characteristics of the medium) would affect the propagation of the pulse through the waveguide. Mathematically, a violation of this condition can be treated as a perturbation of an integrable system; if this perturbation is small, one can use perturbation theory for the solitons.

Our purpose in the present paper is to analyze this perturbed system. In first order we find corrections to the parameters and shape of the soliton. We also find the characteristics of the radiation emitted by the soliton as a result of the perturbation. We show that a mismatch does not affect the soliton amplitude, but it does cause a small and asymmetric distortion of the shape of the soliton. We might mention several points which reflect distinctive features of this study. First, in the spirit of the formulation of the problem we do not assume—in contrast with Lamb[4]—that the carrier frequency coincides with the central frequency of the atomic transition. Second, the perturbation theory for solitons in the form it was developed in Ref. 17 and used in Ref. 18 requires some modification. The reason is that the effect of the radiation emitted by the soliton on the medium must be taken into account; the effect is seen in the difference between the evolution of the scattering matrix here and the evolution in Ref. 17. Finally, the perturbation-theory method is based on the Riemann problem. That approach is technically more transparent. In particular, it allows us to avoid the appearance of integral equations in the calculations of the corrections to the solitons. The corrections to the shape of a soliton which arise when a light beam is scanned over the surface of a medium with a composite nonlinearity were recently calculated,[19] but the reaction of the medium to the emission by the soliton was ignored.

2. EQUATIONS OF THE MODEL

The propagation of an ultrashort pulse propagating through a nonlinear single-mode optical waveguide containing resonant impurities, which are modeled by two-level atoms, is described by the modified Maxwell-Bloch equations:[16]

$$iE_z+dE_{tt}+e|E|^2E+\langle\sigma\rangle=0,$$
$$\sigma_t=i\delta\sigma+ifEu,\quad u_t=2if(\sigma\bar{E}-\bar{\sigma}E). \tag{1}$$

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Here z and t are dimensionless coordinates in the moving system; $E = A/R_0$, where $A(z, t)$ is a slowly varying complex amplitude, and R_0 is the maximum value of this amplitude; and σ and u are elements of the density matrix of the two-level subsystem, which are associated with the induced polarization and the population inversion, respectively. A superior bar means complex conjugation. The angle brackets mean an average with the distribution function $g(\delta)$ over all normalized detunings of the transition frequency of an individual atom, ω_{12}, from the frequency of the carrier wave, ω, where $\delta = (\omega - \omega_{12})\Omega_0^{-1}$. We are not assuming that the carrier frequency coincides with the central frequency of the atomic transition, ω_0. Here $\Omega_0^2 = 2\pi N_0 \omega \tilde{d}^2 \hbar^{-1}$, where N_0 is the density of resonant atoms, and $\tilde{d}$ is the effective matrix element of the dipole transition between resonant states. The coefficients d and e in (1) are given by[16]

$$d = \frac{L_a}{L_d}, \quad e = \frac{L_a}{L_n}, \tag{2}$$

where L_a is the resonant-absorption length, L_d is the dispersion length, and L_n is the nonlinear length. These lengths are given by

$$L_a = \frac{c}{2\pi\Omega_0^2 g(0)}, \quad L_d = \frac{4\beta t_p^2}{|\partial^2 k^2(\omega)/\partial\omega^2|},$$
$$L_n = \frac{\beta c^2}{2\pi\omega^2 R_0^2 |\tilde{\chi}|}. \tag{3}$$

Here t_p is the pulse length, β is the propagation constant, $|\tilde{\chi}|$ is the effective nonlinear susceptibility, and the dimensionless constant $f = \tilde{d} R_0 t_p \hbar^{-1}$ characterizes the interaction of the light with the two-level subsystem. Equations (1) have soliton solutions if the parameters satisfy the condition[16]

$$e = 2f^2 d. \tag{4}$$

Below we will use some slightly different quantities: $\mathscr{E} = 2ifE$, $\lambda = -2f\sigma$, $N = -fu$, $2\alpha = -\delta$. We can then rewrite Eqs. (1) as

$$i\mathscr{E}_z + d\mathscr{E}_{tt} + (e/4f^2)|\mathscr{E}|^2\mathscr{E} - i\langle\lambda\rangle = 0,$$
$$\lambda_t + 2i\alpha\lambda = \mathscr{E}N, \quad N_t = -{}^1/_2(\lambda\bar{\mathscr{E}} + \bar{\lambda}\mathscr{E}). \tag{5}$$

We denote by Ω that frequency within the linewidth of the atomic transition at which condition (4) holds, and for which we have $d(\Omega) = d_0$ and $e(\Omega) = e_0$. We then have $e_0 = 2f^2 d_0$. Writing $d(\omega)$ as

$$d(\omega) = d_0 + (\partial d/\partial\omega)_{\omega=\Omega}\Delta\omega, \quad \Delta\omega = \omega - \Omega,$$

and writing a corresponding expression for $e(\omega)$, we find, using (2) and (3),

$$d(\omega) = (1 - \varepsilon p_1) d_0, \quad e(\omega) = (1 + \varepsilon p_2) e_0, \tag{6}$$

where

$$\varepsilon = \frac{\Delta\omega}{\Omega},$$

$$p_1 = 1 + \Omega\left[\frac{(k'/k)^2 - 2k''/k - k'''/k'}{k''/k' + k'/k}\right]_{\omega=\Omega}, \quad p_2 = \Omega\left(\frac{n_2'}{n_2}\right)_{\omega=\Omega}. \tag{7}$$

Here $k' = \partial k(\omega)/\partial\omega$, etc., and n_2 is the nonlinear component of the refractive index:

$$n(\omega, \mathscr{E}) = n_0(\omega) + n_2(\omega)|\mathscr{E}|^2.$$

Let us look at some estimates. We assume $\omega \sim 10^{15}$ cm^{-1}, $n_0 = 1.5$, $k = (\omega/c) n_0$, $\lambda \approx 1.5$ μm, $k'' = -2.5 \cdot 10^{-28}$ s^2/cm, and $k''' = 10^{-42}$ s^3/cm. We then find $|\varepsilon p_1| \sim 10^{-3}$–$10^{-4}$. With regard to p_2, estimates yield $|\varepsilon p_2| \sim 10^{-4}$–$10^{-6}$ depending on the particular mechanism for the optical anharmonicity.[20,21] These corrections are therefore small, justifying our use of a perturbation theory for solitons.

Using (6) and (7), we can thus write the Maxwell-Bloch equations (1) as follows:

$$i\mathscr{E}_z + d_0\mathscr{E}_{tt} + {}^1/_2 d_0|\mathscr{E}|^2\mathscr{E} - i\langle\lambda\rangle = \varepsilon d_0(p_1\mathscr{E}_{tt} - {}^1/_2 p_2|\mathscr{E}|^2\mathscr{E}),$$
$$\lambda_t + 2i\alpha\lambda = \mathscr{E}N, \quad N_t = -{}^1/_2(\lambda\bar{\mathscr{E}} + \bar{\lambda}\mathscr{E}). \tag{8}$$

The right-hand side of Maxwell's equation specifies the perturbation. For Eqs. (8) we have an "ε-curvature representation":

$$U_z - V_t + [U, V] = i(\varepsilon\hat{R} - \zeta_z\sigma_3), \quad \zeta \in \mathbf{C},$$

where the 2×2 matrices U, V, and $\hat{R}$ are

$$U = \begin{pmatrix} -i\zeta & \mathscr{E}/2 \\ -\bar{\mathscr{E}}/2 & i\zeta \end{pmatrix} \equiv -i\zeta\sigma_3 + Q(t),$$
$$V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} 0 & R \\ \bar{R} & 0 \end{pmatrix},$$
$$A = \frac{i}{2}\left[\frac{1}{2}d_0|\mathscr{E}|^2 - 4\zeta^2 d_0 + \frac{1}{2}\left\langle\frac{N}{\zeta-\alpha}\right\rangle\right],$$
$$B = \frac{i}{2}\left[d_0\mathscr{E}_t - 2i\zeta d_0\mathscr{E} - \frac{1}{2}\left\langle\frac{\lambda}{\zeta-\alpha}\right\rangle\right], \tag{9}$$
$$C = \frac{i}{2}\left[d_0\bar{\mathscr{E}}_t + 2i\zeta d_0\bar{\mathscr{E}} - \frac{1}{2}\left\langle\frac{\bar{\lambda}}{\zeta-\alpha}\right\rangle\right],$$
$$R = {}^1/_2 d_0({}^1/_2 p_2|\mathscr{E}|^2\mathscr{E} - p_1\mathscr{E}_{tt}).$$

In the case $\varepsilon = 0$, Eqs. (8) can be integrable by the inverse scattering method.[16]

3. PERTURBATION THEORY FOR THE MAXWELL-BLOCH EQUATIONS

Let us outline the procedure for finding solutions of the perturbed system (8) through the use of a Riemann problem. We denote by $T_\pm(t,\zeta)$ the matrix Jost solutions of the Zakharov-Shabat spectral problem:[1]

$$\Phi_t(t, \zeta) - U(t, \zeta)\Phi(t, \zeta) = 0,$$

where $|\mathscr{E}| \to 0$ as $|t| \to \infty$, with the asymptotic behavior

$$T_\pm(t, \zeta) \xrightarrow[t\to\pm\infty]{} J(\zeta t) = \exp(-i\zeta t\sigma_3).$$

The scattering matrix $S(\zeta)$ can be expressed in terms of the Jost solutions:

$$T_-(t, \zeta) = T_+(t, \zeta) S(\zeta), \quad S(\zeta) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad \det S = 1. \tag{10}$$

We denote by $T_\pm^{(i)}$, $i = 1, 2$, the columns of the matrices $T_\pm$. We introduce the new matrices

$$\theta(t, \zeta) = (T_-^{(1)} e^{i\zeta t}, T_+^{(2)} e^{-i\zeta t}), \quad \bar{\theta}(t, \zeta) = (T_+^{(1)} e^{i\zeta t}, T_-^{(2)} e^{-i\zeta t}).$$

The matrix θ is analytic in the upper ζ half-plane, and we have $\det\theta = a(\zeta)$. The matrix $\bar\theta$, in contrast, is analytic in the lower half-plane, and we have $\det\bar\theta = \bar a(\zeta)$. Asymptotically the matrix θ becomes triangular:

$$\theta(t,\zeta)\underset{t\to\pm\infty}{\longrightarrow} J(\zeta t)\theta_\pm(\zeta)J^{-1}(\zeta t),$$

where

$$\theta_+=\begin{pmatrix} a & 0\\ b & 1\end{pmatrix},\quad \theta_-=\begin{pmatrix} 1 & \bar b\\ 0 & a\end{pmatrix}$$

(corresponding expressions can be written for $\bar\theta$). The relationship with the scattering matrix is $\theta_+(\zeta)=S(\zeta)\theta_-(\zeta)$. We denote by ζ_j and $\bar\zeta_j$ the zeros of the determinants $a(\zeta)$ and $\bar a(\zeta)$, respectively ($j=1,\ldots,\tilde N$). The columns of the matrices θ and $\bar\theta$ then satisfy the proportionalities

$$\theta^{(1)}(t,\zeta_j)=\gamma_j(t)\theta^{(2)}(t,\zeta_j),$$
$$\gamma_j(t)=\gamma_j\exp(2i\zeta_j t),\quad \operatorname{Im}\zeta_j>0,\quad \gamma_j\in\mathbf{C},$$
$$\bar\theta^{(2)}(t,\bar\zeta_j)=-\bar\gamma_j(t)\bar\theta^{(1)}(t,\bar\zeta_j),$$
$$\bar\gamma_j(t)=\bar\gamma_j\exp(-2i\bar\zeta_j t),\quad \operatorname{Im}\bar\zeta_j<0,\quad \bar\gamma_j\in\mathbf{C}.$$

The set $b(\xi)$, $\xi=\operatorname{Re}\zeta$, ζ_j, $\bar\zeta_j$, γ_j, and $\bar\gamma_j$ constitutes the scattering data.

The relationship (10) between the Jost solutions becomes

$$\bar\theta^+(t,\xi)\theta(t,\xi)=G(t,\xi), \tag{11}$$

where the matrix $G(t,\xi)$ is

$$G(t,\xi)=J(\xi t)\begin{pmatrix} 1 & \bar b(\xi)\\ b(\xi) & 1\end{pmatrix}J^{-1}(\xi t)$$

and $\bar\theta^+$ is defined by $\bar\theta^+=\bar\theta^{-1}\det\bar\theta$. Expression (11) can be treated as a matrix Riemann problem with zeros,[22] i.e., a problem of the factorization of a nonsingular matrix $G(t,\xi)$ which is specified on the real axis, into a product of two matrices with the specified analytic properties. The variable t serves as a parameter here. The solution of the Riemann problem (11) can be written in the form

$$\theta(t,\zeta)=I-\sum_{j=1}^{\tilde N}(\bar\zeta_j-\zeta)^{-1}\frac{\bar\theta(t,\bar\zeta_j)}{\dot{\bar a}_j}\bar F_j(t) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\zeta}\bar\theta(t,\xi)\bar\rho(t,\xi),\ \operatorname{Im}\zeta>0,$$
$$\bar\theta(t,\zeta)=I-\sum_{j=1}^{\tilde N}(\zeta_j-\zeta)^{-1}\frac{\theta(t,\zeta_j)}{\dot a_j}F_j(t) - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\zeta}\theta(t,\xi)\rho^+(t,\xi),\ \operatorname{Im}\zeta<0. \tag{12}$$

Here

$$\dot a_j=\frac{d}{d\zeta}a(\zeta)_{\zeta=\zeta_j},\quad \rho(t,\xi)=\frac{1}{a(\xi)}[G(t,\xi)-I],$$
$$\bar\rho(t,\xi)=\frac{1}{\bar a(\xi)}[G(t,\xi)-I].$$

The matrices

$$F_j(t)=\begin{pmatrix} 0 & \gamma_j^{-1}(t)\\ \gamma_j(t) & 0\end{pmatrix},\quad \bar F_j(t)=-\begin{pmatrix} 0 & \bar\gamma_j(t)\\ \bar\gamma_j^{-1}(t) & 0\end{pmatrix}$$

explicitly incorporate the proportionality of the columns at the points ζ_j and $\bar\zeta_j$.

Let us formulate boundary conditions for the population inversion N and for the induced polarization λ. We will discuss "causal" solutions (in the terminology of Zakharov[7]). We then have $N\to -1$ and $\lambda\to 0$ as $t\to -\infty$. Since the Bloch part of system (8) is specified by first-order equations, the asymptotic behavior of λ and N as $t\to\infty$ arises as a consequence of the solution of the equations. Lamb[4] showed that N and λ can be expressed in terms of the Jost solutions of the spectral problem. In terms of the matrices θ and $\bar\theta$ we have

$$N=-{}^{tr}\theta^{(1)}(t,\alpha)\sigma_1\bar\theta^{(2)}(t,\alpha),$$
$$\lambda=-{}^{tr}\theta^{(1)}(t,\alpha)(1+\sigma_3)\bar\theta^{(2)}(t,\alpha)$$

(the superscript tr means transposition). As $t\to\infty$ we thus find

$$N\to -1+2b(\alpha)\bar b(\alpha),\quad \lambda\to 2a(\alpha)\bar b(\alpha)\exp(-2i\alpha t).$$

The equation for the z evolution of the scattering matrix is written (see the Appendix)

$$S_z-[V_-,S]-J^{-1}V_+JS=i\varepsilon\theta_+\left(\int_{-\infty}^{\infty}J^{-1}\theta^{-1}\hat R\theta J\,dt\right)\theta_-^{-1}, \tag{13}$$

where the term with V_+ incorporates the reaction of the medium to the emission by the soliton. Here

$$V_+=\begin{pmatrix} A_+ & B_+\\ C_+ & -A_+\end{pmatrix},\quad A_+=\frac{i}{2}\left\langle\frac{b(\alpha)\bar b(\alpha)}{\zeta-\alpha}\right\rangle,$$
$$B_+=-\frac{i}{2}\left\langle\frac{a(\alpha)\bar b(\alpha)}{\zeta-\alpha}\exp(-2i\alpha t)\right\rangle, \tag{14}$$
$$C_+=-\frac{i}{2}\left\langle\frac{\bar a(\alpha)b(\alpha)}{\zeta-\alpha}\exp(2i\alpha t)\right\rangle.$$

Since we have[4-6]

$$\lim_{t\to\infty}B_+\exp(2i\zeta t)={}^1/_2\pi a(\zeta)\bar b(\zeta)g(\zeta),$$
$$\lim_{t\to\infty}C_+\exp(-2i\zeta t)=-{}^1/_2\pi\bar a(\zeta)b(\zeta)g(\zeta), \tag{15}$$

we find from (13) the evolution equation which we have been seeking for the elements of the scattering matrix in the case of a continuous spectrum:

$$a_z-\frac{i}{2}\left[\left\langle\frac{b(\alpha)\bar b(\alpha)}{\zeta-\alpha}\right\rangle-i\pi|b|^2g(\zeta)\right]a=\varepsilon(\theta^{(1)}|\sigma_2\hat R|\theta^{(2)}),$$
$$b_z-\frac{i}{2}\left[8\zeta^2d_0+\left\langle\frac{a(\alpha)\bar a(\alpha)}{\zeta-\alpha}\right\rangle+i\pi|a|^2g(\zeta)\right]b \tag{16}$$
$$=-\varepsilon(\theta^{(1)}|\sigma_2\hat R e^{-2i\zeta t}|\bar\theta^{(1)}).$$

Here the notation $(\theta^{(i)}|f(t)|\theta^{(j)})$ means the integral $\int_{-\infty}^{\infty}{}^t\theta^{(i)}(t)f(t)\theta^{(j)}(t)dt$.

To find the formulas for the discrete spectrum, we need to make the replacement $\varepsilon\hat R\to\varepsilon\hat R-\zeta_z\sigma_3$, take the limit $\zeta\to\zeta_j$, and note that limits (15) disappear because of exponential factors. As a result we find

$$\zeta_{jz}=-\frac{\varepsilon}{\dot{a}_j}(\theta^{(1)}(\zeta_j)\,|\,\sigma_2\hat{R}\,|\,\theta^{(2)}(\zeta_j)),\tag{17}$$

$$\gamma_{jz}-\frac{i}{2}\left[8\zeta_j^2d_0+\left\langle\frac{a(\alpha)\bar{a}(\alpha)}{\zeta_j-\alpha}\right\rangle\right]\gamma_j$$
$$=\varepsilon\frac{\gamma_j}{\dot{a}_j}\Big[\,2i(\theta^{(1)}(\zeta_j)\,|\,\sigma_2\hat{R}t\,|\,\theta^{(2)}(\zeta_j))$$
$$-\frac{\partial}{\partial\zeta}(\theta^{(2)}(\zeta_j)\,|\,\sigma_2\hat{R}\,|\,\theta^{(1)}(\zeta)-\gamma_j(t)\theta^{(2)}(\zeta))_{\zeta=\zeta_j}\Big].\tag{18}$$

To reconstruct the potential $Q(t)$ from the known solution of the Riemann problem, we chose the matrix $\bar{\theta}$ for definiteness. This matrix can be represented by the asymptotic expansion[22]

$$\bar{\theta}(t,\zeta)=I+\frac{1}{2i\zeta}\bar{B}(t)+O(|\zeta|^{-2}).\tag{19}$$

We then find

$$Q(t)={}^1\!/_2[\sigma_3,\bar{B}(t)],\quad \mathscr{E}=2\bar{B}_{12}.$$

We wish to stress that Eqs. (13) and (16)–(18) describe the exact evolution of the scattering data. They are awkward for practical calculations, however, since the solutions θ and $\bar{\theta}$ of the Riemann problem in them depend on the unknown solution of the perturbed system (8). Further progress can be made by using an iterative scheme.

4. SOLITON SUBJECTED TO A PERTURBATION

We seek a solution for the envelope to first order in ε, in the form $\mathscr{E}=\mathscr{E}_s+\mathscr{E}_1$, where $\mathscr{E}_s$ is the soliton in the adiabatic approximation,[17] and $\mathscr{E}_1$ is a small quantity on the order of ε. The solution of the Riemann problem can then also be written in the form $\theta=\theta_s+\theta_1$, $\bar{\theta}=\bar{\theta}_s+\bar{\theta}_1$. Since the Jost coefficient is $b_s=0$ in the adiabatic approximation, the quantity b and therefore ρ and $\bar{\rho}$ are proportional to ε. When these comments are taken into account, expressions (12) with $\widetilde{N}=1$ reduce in the leading order to a system of two linear algebraic equations:

$$\theta_s(\zeta)=I-\frac{\zeta_1-\bar{\zeta}_1}{\zeta-\bar{\zeta}_1}\bar{\theta}_s(\zeta_1)\bar{F}_1,\quad \bar{\theta}_s(\zeta)=I-\frac{\bar{\zeta}_1-\zeta_1}{\zeta-\zeta_1}\theta_s(\zeta_1)F_1.$$

Solving these equations for $\bar{\theta}_s$, and carrying out an asymptotic expansion in accordance with (19), we find

$$\bar{B}_s=-2i(\zeta_1-\bar{\zeta}_1)(I-F_1)(I-\bar{F}_1F_1)^{-1}.$$

We introduce

$$\gamma_1=\exp(t_0+i\varphi_0+i\pi),\ \zeta_1=\xi_1+i\eta_1,\ y=2\eta_1t-t_0,\ \vartheta=2\xi_1t+\varphi_0.$$

We find the soliton solution

$$\mathscr{E}_s=4\eta_1\exp(-i\vartheta)\operatorname{sech}y\tag{20}$$

(the z dependence of y and ϑ is given below). We can write explicit expressions for the solutions θ_s and $\bar{\theta}_s$ of the Riemann problem.

a) For the continuous spectrum ($\xi=\operatorname{Re}\zeta$),

$$\theta_s(t,\xi)=(\xi-\bar{\zeta}_1)^{-1}H(y,\xi),\ \bar{\theta}_s(t,\xi)=(\xi-\zeta_1)^{-1}H(y,\xi),$$
$$H(y,\xi)=\begin{pmatrix}\xi-\xi_1-i\eta_1\operatorname{th}y & -i\eta_1e^{-i\vartheta}\operatorname{sech}y\\ -i\eta_1e^{i\vartheta}\operatorname{sech}y & \xi-\xi_1+i\eta_1\operatorname{th}y\end{pmatrix},\tag{21}$$

b) For a discrete spectrum,

$$\theta_s=\frac{1}{2}\begin{pmatrix}e^{-y} & -e^{-i\vartheta}\\ -e^{i\vartheta} & e^{y}\end{pmatrix}\operatorname{sech}y,\quad \bar{\theta}_s=\frac{1}{2}\begin{pmatrix}e^{y} & e^{-i\vartheta}\\ e^{i\vartheta} & e^{-y}\end{pmatrix}\operatorname{sech}y.\tag{22}$$

Using (20), we can write the following expression for the perturbation R in (9):

$$R=2d_0\mathscr{E}[2(p_1+p_2)\eta_1^2\operatorname{sech}^2y+p_1(\xi_1^2-\eta_1^2)-2ip_1\xi_1\eta_1\operatorname{th}y].\tag{23}$$

Now substituting (22) and (23) into (17), we easily find $\zeta_{1z}=0$, i.e., $\xi_1=\text{const}$ and $\eta_1=\text{const}$.

We now replace the complex quantity γ_1 by two real quantities, Δ_1 and τ_1, in accordance with $\gamma_1=\exp[i(\Delta_1-2\zeta_1\tau_1)]$. In these terms we have $\vartheta=\xi_1\eta_1^{-1}y+\Delta_1$ and $y=2\eta_1(t-\tau_1)$. Using (18) and (21), and using the notation $-\frac{1}{2}\langle(\zeta_1-\alpha)^{-1}\rangle=\omega_1+i\omega_2$, we find laws describing the evolution of τ_1 and Δ_1 with contributions on the order of ε:

$$\tau_{1z}+4d_0\xi_1-\omega_2/2\eta_1=4\varepsilon p_1d_0\xi_1,$$
$$\Delta_{1z}+4d_0(\xi_1^2+\eta_1^2)+\omega_1-\xi_1\eta_1^{-1}\omega_2=4\varepsilon d_0[p_1(\xi_1^2-\eta_1^2)-2p_2\eta_1^2].$$

Hence

$$\tau_1=\left[\frac{\omega_2}{2\eta_1}-4\xi_1d_0(1-\varepsilon p_1)\right]z+\frac{\tau_0}{2\eta_1},$$
$$\Delta_1=-4d_0[(1-\varepsilon p_1)\xi_1^2$$
$$+(1+\varepsilon(p_1+2p_2))\eta_1^2]z-(\omega_1z-\varphi_0)+\xi_1\eta_1^{-1}(\omega_2z+\tau_0)+\pi.$$

A small deviation from condition (4) thus leaves the soliton amplitude unchanged. Furthermore, the soliton velocity is independent of a variation of the refractive index n_2, while perturbations of both types contribute to a phase modulation.

The correction to the shape of the soliton is found by examining terms on the order of ε in (12). We have

$$\theta_1(\zeta)=-\frac{\zeta_1-\bar{\zeta}_1}{\zeta-\bar{\zeta}_1}\bar{\theta}_1(\zeta_1)\bar{F}_1+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\zeta}\bar{\theta}_s(\xi)\bar{\rho}(\xi),$$

$$\bar{\theta}_1(\zeta)=-\frac{\bar{\zeta}_1-\zeta_1}{\zeta-\zeta_1}\theta_1(\zeta_1)F_1-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\zeta}\theta_s(\xi)\rho^+(\xi).$$

The matrices θ_s and $\bar{\theta}_s$, which were found above, appear in the integrals. We thus have a system of two linear algebraic equations for θ_1 and $\bar{\theta}_1$. Solving these equations, and carrying out the asymptotic expansion

$$\bar{\theta}_1(t,\zeta)=(2i\zeta)^{-1}\bar{B}_1(t)+O(|\zeta|^{-2}),$$

we find an expression for $\mathscr{E}_1=2(B_1)_{12}$ with the well-known structure:[17]

$$\mathscr{E}_1=-\frac{2}{\pi}e^{-i\vartheta}\Big[\eta_1^2\operatorname{sech}^2y\int_{-\infty}^{\infty}\exp[i(\xi\eta_1^{-1}y$$
$$-\Delta_1+2\xi\tau_1)]\frac{b(\xi)\,d\xi}{(\xi-\zeta_1)(\xi-\bar{\zeta}_1)}$$
$$+\int_{-\infty}^{\infty}\exp[i(-\xi\eta_1^{-1}y+\Delta_1-2\xi\tau_1)]\frac{(\xi-\xi_1-i\eta_1\operatorname{th}y)^2}{(\xi-\zeta_1)(\xi-\bar{\zeta}_1)}\bar{b}(\xi)\,d\xi\Big].\tag{24}$$

The Jost coefficient $b(\xi)$ is found by solving Eq. (16). For the perturbation under consideration here, (23), in first order in ε, the solution can be written

$$b_z - \frac{i}{2}\left[8\xi^2 d_0 + \left\langle\frac{1}{\xi-\alpha}\right\rangle + i\pi g(\xi)\right] b$$
$$= 2i\varepsilon\pi d_0 (p_1+p_2)(\xi-\zeta_1)(\xi-\xi_1)$$
$$\times \exp[i(\Delta_1 - 2\xi\tau_1)] \operatorname{sech}\frac{\pi}{2\eta_1}(\xi-\xi_1). \tag{25}$$

For sufficiently large values of z we thus find

$$b(z,\xi) = -\frac{\varepsilon\pi}{2}(p_1+p_2)D^{-1}(g,\xi)(\xi-\zeta_1)(\xi-\xi_1)$$
$$\times \exp[i(\Delta_1 - 2\xi\tau_1)] \operatorname{sech}\frac{\pi}{2\eta_1}(\xi-\xi_1), \tag{26}$$

where

$$D(g,\xi) = (\xi-\xi_1)^2 + \eta_1^2 + \frac{1}{4d_0}\left(\omega_1 + \frac{\xi-\xi_1}{\eta_1}\omega_2\right)$$
$$+ \frac{1}{8d_0}\left\langle\frac{1}{\xi-\alpha}\right\rangle + \frac{i\pi}{8d_0} g(\xi).$$

We fix the lineshape of the resonant transition, assuming a Lorentz distribution:

$$g(\alpha) = \frac{1}{\pi}\frac{\Gamma}{(\alpha-\alpha_0)^2+\Gamma^2}$$

where Γ is the half-width, $2\alpha = (\omega_{12} - \omega)\Omega_0^{-1}$, and $2\alpha_0 = (\omega_0 - \omega)\Omega_0^{-1}$. We then find

$$D(g,\xi) = (\xi - \alpha_0 - i\Gamma)^{-1} D(\xi),$$

where

$$D(\xi) = \xi^3 + (\nu_1 - \alpha_0 - i\Gamma)\xi^2$$
$$+ [\nu_2^2 - (\alpha_0 + i\Gamma)\nu_1]\xi - 4\nu_2^2(\alpha_0 + i\Gamma) + (8d_0)^{-1},$$
$$\nu_1 = \frac{\omega_2}{4d_0\eta_1} - 2\xi_1, \quad \nu_2^2 = \xi_1^2 + \eta_1^2 + \frac{1}{4d_0}\left(\omega_1 - \frac{\xi_1}{\eta_1}\omega_2\right).$$

We write the polynomial $D(\xi)$ as

$$D(\xi) = \prod_{k=1}^{3}(\xi - x_k).$$

For typical parameter values, one root lies in the upper half-plane ($x_1 = x_1' + ix_1''$), while the two others lie in the lower half-plane ($x_2 = x_2' - ix_2''$, $x_3 = x_3' - ix_3''$). The correction $\mathscr{E}_1$ in (24) can then be found by the method of residues. We will not write out the corresponding expression here, which is quite lengthy. What is of basic interest here is the asymptotic behavior of the correction as $|y| \to \infty$. This asymptotic behavior determines the degree to which the soliton spreads out. Here are the results of the calculations.

(a) For $y \to \infty$,

$$\mathscr{E}_1 \to -\varepsilon(p_1+p_2)\left\{16\eta_1^3\left[(\zeta_1 - \alpha_0 + i\Gamma)\Big/\prod_{k=1}^{3}(\zeta_1 - \bar{x}_k)\right]e^{-y}\right.$$
$$+ 2\pi i \frac{(\bar{x}_1 - \alpha_0 + i\Gamma)(\bar{x}_1 - \zeta_1)^2}{(\bar{x}_1 - \bar{x}_2)(\bar{x}_1 - \bar{x}_3)}\exp\left(-\frac{x_1''}{\eta_1}y - i\frac{x_1' - \xi_1}{\eta_1}y\right)$$
$$\left.\times \operatorname{sech}\frac{\pi}{2\eta_1}(\bar{x}_1 - \xi_1)\right\}e^{-i\Phi},$$

(b) For $y \to -\infty$ (for definiteness, we are assuming $x_2'' < x_3''$),

$$\mathscr{E}_1 \to -\varepsilon(p_1+p_2)\left\{16\eta_1^3\left[(\zeta_1 - \alpha_0 + i\Gamma)\Big/\prod_{k=1}^{3}(\zeta_1 - \bar{x}_k)\right]e^{y}\right.$$
$$- 2\pi i \frac{(\bar{x}_2 - \alpha_0 + i\Gamma)(\bar{x}_2 - \zeta_1)^2}{(\bar{x}_2 - \bar{x}_1)(\bar{x}_2 - \bar{x}_3)}\exp\left(\frac{x_2''}{\eta_1}y - i\frac{x_2' - \xi_1}{\eta_1}y\right)$$
$$\left.\times \operatorname{sech}\frac{\pi}{2\eta_1}(\bar{x}_2 - \xi_1)\right\}e^{-i\Phi}.$$

When condition (4) is violated, the original symmetry with respect to y of the soliton shape is generally lost, and the degree of spreading is determined by the values of the imaginary parts of the roots x_1'' and x_2''. If $x_1'' > \eta_1$ and $x_2'' > \eta_1$, no spreading occurs, and the distortion of the soliton shape occurs within the initial width of the pulse. If, on the other hand, one or both of these conditions do not hold, the width of the correction, $\mathscr{E}_1$, is greater than the original width of the soliton, but no tail arises.

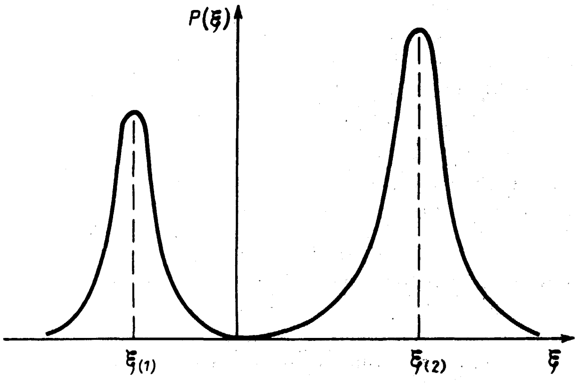

FIG. 1.

5. EMISSION BY A SOLITON SUBJECTED TO A PERTURBATION

At small values of $b(\xi)$, the spectral power density $p(\xi)$ of the emission by the soliton is given by[18]

$$p(\xi) = \frac{8}{\pi}\xi^2 \operatorname{Re}(\bar{b}b_z). \tag{27}$$

The parameter ξ is directly related to the wave number k of the linear waves which are emitted ($\xi = k/2$). Substituting expressions (25) and (26) into (27), we find

$$p(\xi) = \varepsilon^2\pi^2(p_1+p_2)^2\left[\xi^2(\xi-\zeta_1)(\xi-\xi_1)\Big/\prod_{k=1}^{3}(\xi - x_k)(\xi - \bar{x}_k)\right]$$
$$\times \operatorname{sech}\frac{\pi}{2\eta_1}(\xi - \xi_1).$$

Figure 1 shows the spectral density $p(\xi)$ for typical parameter values. The wave-number dependence is definitely not monotonic; there are two clearly defined peaks, $\sim\eta_1$ in width, which correspond to forward and backward emission. Away from the points $\xi_{(1)}$ and $\xi_{(2)}$, the function $p(\xi)$ falls off exponentially.

6. CONCLUSION

Condition (4), which is the condition for the existence of a soliton regime in the propagation of an ultrashort pulse through a resonant nonlinear waveguide, appears at first glance to be extremely restrictive, imposing stiff requirements on the carrier frequency of the pulse. It has been shown above that small deviations from this condition do not disrupt the important properties of the soliton. In particular, the amplitude and velocity of the soliton do not depend on a variation of the refractive index n_2. The spectral power density of the emission by the soliton has been found. Since the medium absorbs the radiation emitted by the soliton, the shape of the soliton remains localized.

The energy loss by the soliton as it moves along the waveguide can be offset through preliminary pumping of resonant atoms over a certain section of the waveguide.[16] This situation is also described by Eqs. (5), but with different boundary conditions. One might suggest that again in this case a slight deviation from the matching condition (4) would leave the important parameters of the soliton at an acceptable level.

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APPENDIX

Let us derive Eq. (13). We introduce $\chi = \Phi_z - V\Phi$. In the case of a continuous spectrum the function χ then satisfies the equation

$$\chi_t - U\chi = i\varepsilon \hat{R}\Phi. \tag{A1}$$

We assume $\Phi = \theta J h(z)$, where $h(z)$ is some function. From the condition $\chi \to 0$ as $t \to -\infty$ we find

$$h_z = (\theta_-^{-1} V_- \theta_- - \theta_-^{-1} \theta_{-z}) h,$$

where

$$V_- = A_- \sigma_3, \quad A_- = -\frac{i}{2}\left(4\zeta^2 d_0 + \frac{1}{2}\left\langle \frac{1}{\zeta - \alpha}\right\rangle\right).$$

As $t \to \infty$ we have $\Phi \to J\theta_+ h = JS\theta_- h$; hence

$$\chi \to J(S_z - SV_- - J^{-1}\tilde{V}_+ JS)\theta_- h, \tag{A2}$$

where $\tilde{V}_+ = V_- + V_+$ and V_+ is given in (14). Finally, for arbitrary t we assume $\chi = \theta J K(t)$, where $K(t)$ is some function to be determined. From (A1) we then find

$$K_t = i\varepsilon J^{-1}\theta^{-1}\hat{R}\theta J h,$$

In the limit $t \to \infty$ we then find

$$\chi \to i\varepsilon J\theta_+ \left(\int_{-\infty}^{\infty} J^{-1}\theta^{-1}\hat{R}\theta J\, dt \right) h. \tag{A3}$$

Comparing (A2) with (A3), we find Eq. (13), which we have been seeking.

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