# Structure and approximation of the Chapman–Enskog expansion for linearized Grad equations

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The structure of the Chapman–Enskog expansion for the stress tensor and for the heat-flux vector is found on the basis of linearized Grad equations. A method involving a partial summation of the Chapman–Enskog series, in order to eliminate the short-wavelength instability of the Barnett approximations, is proposed.

**1.** The problem of deriving hydrodynamic equations from the Boltzmann kinetic equation is a classic one. Still, several of the questions which arise here have not been finally resolved. In particular, just which equations should follow the Navier–Stokes approximation is not totally clear. In principle, the classic Chapman–Enskog method[1] makes it possible to refine the Navier–Stokes hydrodynamics. As was shown in Ref. 2, however, even the first corrections (the Barnett corrections and the super-Barnett corrections) result in a catastrophic degradation: a short-wavelength instability of sound waves occurs. The nonphysical properties of the Barnett approximations suggest that higher-order terms should be retained in the Chapman–Enskog expansion, in order to construct hydrodynamic equations which are not contradictory from the standpoint of the $H$ theorem. Similar situations are not uncommon in quantum field theory and statistical mechanics. Singularities (divergences) which arise in low orders of expansions in those other areas can frequently be eliminated by approximating the series as a whole in some manner: through a partial summation of infinite subsequences of diagrams, through the use of the Pade approximation, etc.

In the present paper we attempt to improve the Barnett approximations through a partial summation of the Chapman–Enskog series. The difficulties in determining the terms of this series from the Boltzmann equation are well known. For example, it was only comparatively recently that the Barnett and super-Barnett approximations were found for the very simple case of Maxwell molecules.[3] We will accordingly analyze the Chapman–Enskog expansion for the linearized Grad equations,[4,5] rather than for the Boltzmann equation. We know that in this case, at least, the differences between the results found from the Grad equations and those found from the Boltzmann equation are small. Use of the Grad equations has certain technical advantages, particularly when we work with the Chapman–Enskog series as a whole.

Let us briefly outline this paper. In Sec. 2 we determine the form of the coefficients of the Chapman–Enskog expansion for the linearized stress tensor and the linearized heat-flux vector. The expressions found as a result [expressions (2.7)] refine the corresponding result found by Grad[6] for the linearized Boltzmann equation. We discuss the linearized steady-state hydrodynamic equations. In Sec. 3 we propose a method for a systematic approximation of the Chapman–Enskog recurrence procedure. This method is essentially one of partially summing the series. We discuss some examples of the application of this method to the linearized Grad equations in order to eliminate the short-wavelength instability of the Barnett approximations.

**2.** We denote by $\rho_0$, $T_0$, and $\mathbf{u}_0 = 0$ the equilibrium values of the density, the temperature, and the flow velocity (in some appropriate Galilean coordinate system), while $\rho'$, $T'$, and $\mathbf{u}'$ are small deviations from the equilibrium values. In the moment equations which appear below, we find the viscosity coefficient $\mu$. It is convenient to write it in the form $\mu(T) = \eta(T)T$. The function $\eta(T)$ depends on the choice of a model for the interparticle interaction. In particular, we would have $\eta = \text{const}$ for Maxwell molecules, and $\eta \propto T^{-1/2}$ for hard spheres. Everywhere below, we use a system of units in which the Boltzmann constant is one. We introduce the dimensionless variables

$$\begin{gathered}\mathbf{u}=T_0^{-1/2}\mathbf{u}', \quad \rho=\rho_0^{-1}\rho', \quad T=T_0^{-1}T',\\ \mathbf{x}=\eta^{-1}(T_0)T_0^{-1/2}\rho_0\mathbf{x}', \quad t=\eta^{-1}(T_0)\rho_0 t'.\end{gathered} \tag{2.1}$$

Here $\mathbf{x}'$ represents the spatial coordinates, and $t'$ is the time. The 13-moment Grad equations, linearized near the equilibrium,[6] take the following form when written in terms of the variables (2.1):

$$\begin{gathered}\partial_t\rho=-\partial_i u_i, \quad \partial_t T=-{}^2/_3(\partial_i u_i+\partial_i q_i),\\ \partial_t u_k=-\partial_k\rho-\partial_k T-\partial_i\sigma_{ik} \quad (k=1,2,3),\\ \partial_t\sigma_{ik}=-(\partial_i u_k+\partial_k u_i-{}^2/_3\delta_{ik}\partial_m u_m)\\ -{}^5/_2(\partial_i q_k+\partial_k q_i-{}^2/_3\delta_{ik}\partial_m q_m)-\varepsilon^{-1}\sigma_{ik},\\ \partial_t q_k=-{}^5/_2\partial_k T-\partial_i\sigma_{ik}-{}^2/_3\varepsilon^{-1}q_k.\end{gathered} \tag{2.2}$$

Here and below, $\sigma_{ik}$ $(i, k = 1, 2, 3)$ is the traceless part of the stress tensor, $q_k$ is the heat flux vector, $\partial_i \equiv \partial/\partial x_i$, $\partial_t \equiv \partial/\partial_t$, a repeated index implies summation, $\varepsilon$ is a small parameter (the Knudsen number), and $\delta_{ik}$ is the Kronecker delta.

When applied to the system (2.2), the Chapman–Enskog method is the representation of $q_k$, $\sigma_{ik}$ by the series

$$\sigma_{ik}=\sum_{n=0}^{\infty}\varepsilon^{n+1}\sigma_{ik}^{(n)}, \quad q_k=\sum_{n=0}^{\infty}\varepsilon^{n+1}q_k^{(n)}. \tag{2.3}$$

The coefficients $\sigma_{ik}^{(n)}$, $q_k^{(n)}$ are determined by the recurrence procedure

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$$\sigma_{ik}^{(0)}=-(\partial_i u_k+\partial_k u_i-{}^2/_3\delta_{ik}\partial_m u_m),\quad q_k^{(0)}=-{}^{15}/_4\partial_k T,$$

$$\sigma_{ik}^{(n)}=-\left\{\sum_{m=0}^{n-1}\partial_t^{(m)}\sigma_{ik}^{(n-(m+1))}+\frac{2}{5}\left(\partial_i q_k^{(n-1)}+\partial_k q_i^{(n-1)}-\frac{2}{3}\delta_{ik}\partial_m q_m^{(n-1)}\right)\right\},\quad n\geqslant 1,$$

$$q_k^{(n)}=-\frac{3}{2}\left\{\sum_{m=0}^{n-1}\partial_t^{(m)}q_k^{(n-(m+1))}+\partial_i\sigma_{ik}^{(n-1)}\right\},\quad n\geqslant 1. \tag{2.4}$$

The operators $\partial_t^{(m)}$, for $m\geqslant 0$, do the following:

$$\partial_t^{(0)}\mathscr{D}\rho=-\mathscr{D}\partial_i u_i,\quad \partial_t^{(0)}\mathscr{D}T=-{}^2/_3\mathscr{D}\partial_i u_i,$$
$$\partial_t^{(0)}\mathscr{D}u_k=-\mathscr{D}\partial_k(T+\rho),$$
$$\partial_t^{(m)}\mathscr{D}\rho=0,\quad \partial_t^{(m)}\mathscr{D}T=-{}^2/_3\partial_i q_i^{(m-1)},$$
$$\partial_t^{(m)}\mathscr{D}u_k=-\mathscr{D}\partial_i\sigma_{ik}^{(m-1)},\quad m\geqslant 1. \tag{2.5}$$

Here and below, $\mathscr{D}$ is an arbitrary differential operator of the type $\partial_1^{l_1}\partial_2^{l_2}\partial_3^{l_3}$, $l_i\geqslant 0$, $\partial_i^0\equiv 1$.

According to (2.4) and (2.5), the coefficients of the series (2.3) can be expressed in terms of the spatial derivatives of $\rho$, $T$, and $\mathbf{u}$. We introduce

$$\overline{\partial_i u_k}=\partial_i u_k+\partial_k u_i-{}^2/_3\delta_{ik}\partial_m u_m,$$
$$\Gamma_{ik}=2(\partial_i\partial_k-{}^1/_3\delta_{ik}\Delta),\quad \Delta=\partial_m\partial_m. \tag{2.6}$$

The basic assertion of this section of the paper is as follows: The coefficients $\sigma_{ik}^{(n)}$, $q_k^{(n)}$ in (2.3), found from (2.4) and (2.5), are

$$\sigma_{ik}^{(2n)}=c_n\Delta^n\overline{\partial_i u_k}+d_n\Delta^{n-1}\Gamma_{ik}\partial_m u_m,$$
$$\sigma_{ik}^{(2n+1)}=a_n\Delta^n\Gamma_{ik}\rho+b_n\Delta^n\Gamma_{ik}T,$$
$$q_k^{(2n)}=\alpha_n\Delta^n\partial_k\rho+\beta_n\Delta^n\partial_k T,$$
$$q_k^{(2n+1)}=\varphi_n\Delta^n\partial_k\partial_m u_m+\psi_n\Delta^{n+1}u_k \tag{2.7}$$

for all $n\geqslant 0$. Here $c_n$, $d_n$, $a_n$, $b_n$, $\alpha_n$, $\beta_n$, $\varphi_n$, and $\psi_n$ are numerical coefficients.

Let us outline a proof by induction. A direct calculation yields

$$\sigma_{ik}^{(1)}=\Gamma_{ik}({}^1/_2T-\rho),\qquad q_k^{(1)}=-{}^{13}/_4\partial_k\partial_m u_m+{}^3/_2\Delta u_k. \tag{2.8}$$

Using $\sigma_{ik}^{(0)}$, $q_k^{(0)}$ from (2.4), we see that assertion (2.7) is proved for the case $n=0$. Let us assume that the structure (2.7) has been established for some $n$. Then for $n+1$ we have

$$\sigma_{ik}^{(2(n+1))}=-\left\{\sum_{m=0}^{2n+1}\partial_t^{(m)}\sigma_{ik}^{(2(n+1)-(m+1))}+\frac{2}{5}\left(\partial_i q_k^{(2n+1)}+\partial_k q_i^{(2n+1)}-\frac{2}{3}\delta_{ik}\partial_m q_m^{(2n+1)}\right)\right\}$$
$$=\left\{\sum_{p=0}^{n}c_p c_{n-p}-\frac{2}{5}\psi_n\right\}\Delta^{n+1}\overline{\partial_i u_k}+\Delta^n\Gamma_{ik}\partial_m u_m\cdot$$
$$\times\left\{\frac{1}{3}\sum_{p=0}^{n}[c_{n-p}(c_p+4d_p)+4d_{n-p}(c_p+d_p)]+a_n+\frac{2}{3}b_n+\frac{2}{3}\sum_{p=1}^{n}b_{n-p}(\varphi_{p-1}+\psi_{p-1})-\frac{2}{5}\varphi_n\right\}. \tag{2.9}$$

Clearly, the expression has the same structure as that of the corresponding coefficient $\sigma_{ik}^{(2m)}$ in (2.7). It is also a straightforward matter to verify that the expressions for $q_k^{(2(n+1))}$, $\sigma_{ik}^{(2(n+1)+1)}$, $q_k^{(2(n+1)+1)}$ have the form (2.7).

The structure of the Chapman–Enskog expansion (2.7) has some simple consequences. Truncating the series (2.3) at some finite order $n\geqslant 0$, and substituting the resulting expressions into the first five equations in (2.2), we find a closed system of equations for $\rho$, $T$, and $\mathbf{u}$. A steady-state solution of these equations for $n=2m$, with $m\geqslant 1$, can be found from the following equations (under appropriate boundary conditions):

$$\partial_i u_i=0,\quad \sum_{p=0}^{m}\varepsilon^{2p+1}c_p\Delta^{p+2}u_k=0,\quad k=1,2,3,$$
$$A_m(\Delta)\rho+B_m(\Delta)T=0,\quad C_m(\Delta)\rho+E_m(\Delta)T=0,$$
$$A_m(\Delta)=\sum_{p=0}^{m}\alpha_p\varepsilon^{2p+1}\Delta^{p+1},\quad B_m(\Delta)=\sum_{p=0}^{m}\beta_p\varepsilon^{2p+1}\Delta^{p+1},$$
$$C_m(\Delta)=\Delta+\frac{4}{3}\sum_{p=0}^{m-1}a_p\varepsilon^{2(p+1)}\Delta^{p+1},$$
$$E_m(\Delta)=\Delta+\frac{4}{3}\sum_{p=0}^{m-1}b_p\varepsilon^{2(p+1)}\Delta^{p+1}. \tag{2.10}$$

With $n=2m+1$, the steady-state equations can be found from (2.10) by replacing $m-1$ by $m$ in the sums in the operators $C_m(\Delta)$, $E_m(\Delta)$. In the case $n=0$ (this is the Navier–Stokes approximation), the steady-state equations lead to

$$\Delta T=0,\quad \Delta\rho=0,\quad \partial_i u_i=0,\quad \Delta^2 u_k=0. \tag{2.11}$$

Under this condition, only $q_k^{(0)}$, $q_k^{(1)}$, $\sigma_{ik}^{(0)}$, $\sigma_{ik}^{(1)}$, and $\sigma_{ik}^{(2)}$ in (2.7) are nonzero. Since we have $\partial_i\sigma_{ik}^{(1)}=\partial_i\sigma_{ik}^{(2)}=\partial_k q_k^{(1)}=0$, incorporating $\sigma_{ik}^{(1)}$, $\sigma_{ik}^{(2)}$, $q_k^{(1)}$ does not alter the conditions for a steady state of the Navier–Stokes approximation, (2.11). In a comparatively recent study, Galkin[7] established that the Chapman–Enskog series is degenerate for the linearized Boltzmann equation. For $n>0$, it generally does not follow from (2.10) that the series in (2.3) are degenerate.

In the time-varying case, the situation gets worse. Truncating series (2.3) at a finite order $n>0$ may result in a short-wavelength instability of the equilibrium point, as was shown in Ref. 2 for $n=1$ and 2 for one-dimensional equations (2.3). As we mentioned in Sec. 1, all orders must be taken into account in the Chapman–Enskog expansion. The relation (2.9) shows that the recurrence procedure for determining the numerical coefficients in (2.7) is rather cumbersome. In the following sections of this paper we propose some methods for solving Eqs. (2.4) approximately in order to approximate the series (2.3) as a whole.

**3.** Let us outline this new algorithm for an approximate solution of the recurrence system (2.4). We fix $k_0\geqslant 1$. Equations (2.4) are replaced by the approximate equations

$$\sigma_{ik}^{(n)}=-\left\{\sum_{m=0}^{n-1}\partial_t^{(m)}\sigma_{ik}^{(n-(m+1))}+\frac{2}{5}\left(\partial_i q_k^{(n-1)}+\partial_k q_i^{(n-1)}-\frac{2}{3}\delta_{ik}\partial_m q_m^{(n-1)}\right)\right\},\quad n=0,\ldots,k_0,$$

$$\sigma_{ik}^{(k_0+m)}\approx-\left\{\sum_{s=0}^{k_0-1}\partial_t^{(s)}\sigma_{ik}^{(k_0+m-(s+1))}+\frac{2}{5}\left(\partial_i q_k^{(k_0+m-1)}+\partial_k q_i^{(k_0+m-1)}-\frac{2}{3}\delta_{ik}\partial_s q_s^{(k_0+m-1)}\right)\right\},\quad m\geqslant 1 \tag{3.1}$$

with corresponding expressions for $q_k^{(n)}$. The operators $\partial_t^{(s)}$ $(s=0,\ldots,k_0-1)$ are found as in (2.5). We are thus restricting the discussion to a finite set of operators $\partial_t^{(s)}$ at the outset. The $k_0$th order is taken into account exactly in expansions (2.3), while all other orders, beginning with the $(k_0+1)$-st, are taken into account approximately. In the limit $k_0\to\infty$, the system (3.1) tends toward the system (2.4). It can be shown that (3.1) retains the structure of (2.7); the only changes are in the numerical values of the coefficients of the derivatives.

Equations (3.1) are generally simpler than the original equations, (2.4), and explicit solutions are possible in several cases, as we will see below. The next step of the algorithm is to sum the series (2.3) with the approximate values of the coefficients found from (3.1). The overall procedure is a recipe of a sort. For brevity, we will call the method (3.1) "regularization." We turn now to several examples of its use.

We begin with the very simple case of the linearized one-dimensional ten-moment Grad equations. The analogs of Eqs. (2.2) are

$$\partial_t\rho=-\partial_x u,\quad \partial_t T=-{}^2/_3\partial_x u,\quad \partial_t u=-\partial_x(T+\rho+\sigma),$$
$$\partial_t\sigma=-{}^4/_3\partial_x u-\varepsilon^{-1}\sigma. \tag{3.2}$$

Here $\sigma$ is the $xx$ component of the tensor $\sigma_{ik}$, and $x$ is a one-dimensional coordinate. Equations (3.2) can be found from the one-dimensional version of (2.2) by taking the limit $q\to 0$.

We now fix $k_0=1$. We go over to the variables $u$, $\theta=T+\rho$. The recurrence relations (3.1) can then be approximated by

$$\sigma^{(0)}=-{}^4/_3\partial_x u,\quad \sigma^{(1)}=-\partial_t^{(0)}\sigma^{(0)},\quad \sigma^{(n)}\approx-\partial_t^{(0)}\sigma^{(n-1)},\quad n\geqslant 2,$$
$$\partial_t^{(0)}\theta=-{}^5/_3\partial_x u,\quad \partial_t^{(0)}u=-\partial_x\theta. \tag{3.3}$$

From (3.3) we easily see that the $\sigma^{(n)}$ are

$$\sigma^{(2n)}=a_n\partial_x^{2n+1}u,\quad \sigma^{(2n+1)}=b_n\partial_x^{2n+2}\theta,\quad n\geqslant 0, \tag{3.4}$$

and the coefficients $a_n$ and $b_n$ are determined by the recurrence rule

$$a_n=b_n,\quad a_{n+1}={}^5/_3 a_n,\quad a_0=-{}^4/_3. \tag{3.5}$$

From (3.4) and (3.5) we find

$$\sigma^{(2n)}=({}^5/_3\partial_x^2)^n(-{}^4/_3\partial_x u),\quad \sigma^{(2n+1)}=({}^5/_3\partial_x^2)^n(-{}^4/_3\partial_x^2\theta). \tag{3.6}$$

Summing the series

$$\sum_{n=0}^{\infty}\varepsilon^{n+1}\sigma^{(n)}$$

with the coefficients (3.6), we find

$$\sigma_{1R}=R_1\sigma_1,\quad R_1=(1-{}^5/_3\varepsilon^2\partial_x^2)^{-1},$$
$$\sigma_1=-{}^4/_3(\varepsilon\partial_x u+\varepsilon^2\partial_x^2\theta). \tag{3.7}$$

The expression for $\sigma_1$ is the Barnett approximation of the non-diagonal part of the stress tensor $\sigma$.

We now fix $k_0=2$. In this case, Eqs. (3.1) become

$$\sigma^{(0)}=-{}^4/_3\partial_x u,\quad \sigma^{(1)}=-\partial_t^{(0)}\sigma^{(0)},\quad \sigma^{(2)}=-\partial_t^{(0)}\sigma^{(1)}$$
$$-\partial_t^{(1)}\sigma^{(0)},\quad \sigma^{(n)}\approx-\partial_t^{(0)}\sigma^{(n-1)}-\partial_t^{(1)}\sigma^{(n-2)},\quad n\geqslant 3,$$
$$\partial_t^{(1)}\theta=0,\quad \partial_t^{(1)}u={}^4/_3\partial_x^2 u. \tag{3.8}$$

The coefficients $\sigma^{(n)}$ are as in (3.4). The numbers $a_n$ and $b_n$ are determined by a recurrence procedure which is found from (3.5) by replacing 5/3 by 1/3. As a result we replace (3.7) by

$$\sigma_{2R}=R_2\sigma_1,\quad R_2=(1-{}^1/_3\varepsilon^2\partial_x^2)^{-1}. \tag{3.9}$$

The super-Barnett approximation $\sigma_2$ is

$$\sigma_2=\sigma_1-{}^4/_9\varepsilon^3\partial_x^3 u. \tag{3.10}$$

Substituting $\sigma^{(0)}$, $\sigma_1$, $\sigma_{1R}$, $\sigma_2$, and $\sigma_{2R}$ in place of $\sigma$ in the equations

$$\partial_t\theta=-{}^5/_3\partial_x u,\quad \partial_t u=-\partial_x(\theta+\sigma), \tag{3.11}$$

switching to the variables $t''=t/\varepsilon$, $x''=x/\varepsilon$, and using the representation

$$u=u_1\varphi,\quad \theta=\theta_1\varphi,\quad \varphi=\exp(\omega t''+ikx''), \tag{3.12}$$

we find the dispersion relations $\omega(k)$ for sound waves from the condition for the existence of a nontrivial solution of the system of linear equations for $\theta_1$, $u_1$. These dispersion relations are

$$\omega_{1,2}=-{}^4/_3k^2\pm k({}^4/_9k^2-{}^5/_3)^{1/2} \tag{3.13}$$

for the Navier–Stokes approximation $\sigma^{(0)}$,

$$\omega_{1,2}=-{}^2/_3k^2\pm k\left[\tfrac{4}{9}k^2-\tfrac{5}{3}(1+\tfrac{4}{9}k^2)\right]^{1/2} \tag{3.14}$$

for the Barnett approximation $\sigma_1$,

$$\omega_{1,2}=-\frac{2k^2}{3+5k^2}\pm i\frac{k}{2}\left(\frac{75k^4+41k^2+15}{25k^4+30k^2+9}\right)^{1/2} \tag{3.15}$$

for the regularized Barnett approximation $\sigma_{1R}$,

$$\omega_{1,2}=\frac{2}{9}k^2(k^2-3)\pm\left\{\frac{1}{91}[4k^4(3-k^2)^2-45k^2(4k^2+3)]\right\}^{1/2} \tag{3.16}$$

for the super-Barnett approximation $\sigma_2$, and

$$\omega_{1,2}=-\frac{2k^2}{3+k^2}\pm i\frac{k}{2}\left(\frac{100k^4+312k^2+180}{3k^4+18k^2+21}\right)^{1/2} \tag{3.17}$$

for the linearized super-Barnett approximation $\sigma_{2R}$.

Figure 1 shows dispersion curves for the Barnett approximation, (3.14) (the dashed line), and for the regular-

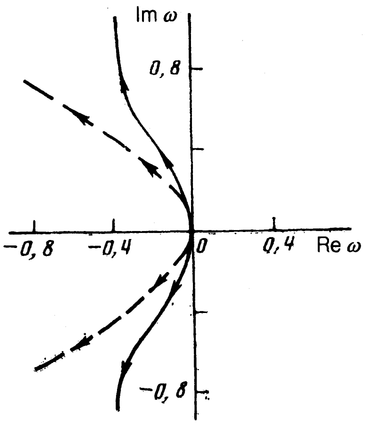


FIG. 1.

ized Barnett approximation, (3.15) (the solid line). The directions of the arrows correspond to an increase in the square of the wave vector, $k^2$. Figure 2 shows dispersion curves for the super-Barnett approximation, (3.16) (the dashed line), and for the regularized super-Barnett approximation (3.17). We see that the regularization eliminates the short-wavelength instability of the super-Barnett approximation.

As the next example we consider the one-dimensional version of the 13-moment system (2.2). The Barnett approximation $\sigma_1$, $q_1$ is

$$\sigma_1=-{}^4/_3\varepsilon\partial_x u-{}^4/_3\varepsilon^2\partial_x^2\rho+{}^2/_3\varepsilon^2\partial_x^2 T,$$
$$q_1=-\frac{15}{4}\varepsilon\partial_x T-{}^7/_4\varepsilon^2\partial_x^2 u. \qquad (3.18)$$

Substituting (3.18) into the one-dimensional equations in (2.2), and then proceeding as above, we find the dispersion relation

$$18\omega^3+69\omega^2k^2+\omega k^2(30+97k^2-14k^4)+15k^4(3+k^2)=0. \qquad (3.19)$$

It is for this case that the short-wavelength instability was established in Ref. 2. We fix $k_0=1$ in (3.1) and go over to one-dimensional equations. The approximate recurrence equations (3.1) then become

$$\sigma^{(0)}=-\frac{4}{3}\partial_x u,\quad \sigma^{(k)}\approx-\partial_t^{(0)}\sigma^{(k-1)}-\frac{8}{15}\partial_x q^{(k-1)},\quad k\geqslant 1,$$
$$q^{(0)}=-\frac{15}{4}\partial_x T,\quad q^{(k)}\approx-\frac{3}{2}\partial_t^{(0)}q^{(k-1)}-\frac{3}{2}\partial_x\sigma^{(k-1)},\quad k\geqslant 1,$$
$$\partial_t^{(0)}\rho=-\partial_x u,\quad \partial_t^{(0)}T=-{}^2/_3\partial_x u,\quad \partial_t^{(0)}u=-\partial_x(T+\rho), \qquad (3.20)$$

In the case $k=1$, the approximation becomes an exact equation. In the one-dimensional case, the structure of (2.7) is

$$\sigma^{(2n)}=c_n\partial_x^{2n+1}u,\quad \sigma^{(2n+1)}=a_n\partial_x^{2n+2}T+b_n\partial_x^{2n+2}\rho,$$
$$q^{(2n)}=\alpha_n\partial_x^{2n+1}T+\beta_n\partial_x^{2n+1}\rho,\quad q^{(2n+1)}=\varphi_n\partial_x^{2n+2}u. \qquad (3.21)$$

To determine the coefficients $c_n$, $a_n$, $b_n$, $\alpha_n$, $\beta_n$, and $\varphi_n$ from Eqs. (3.20) it is convenient to introduce the following entities: the space $X=R^3$, the vectors $x_n$ $(n\geqslant 0)$ in $X$ with the components $a_n$, $b_n$, $\varphi_n$; the space $Y=R^3$; and the vectors $y_n$ $(n\geqslant 0)$ in $Y$ with the components $c_n$, $\alpha_n$, $\beta_n$. We then find an analog of (3.5) from (3.20):

$$x_n=Sy_n,\quad y_{n+1}=Lx_n,\quad y_0=(-{}^4/_3,\ -{}^{15}/_4,\ 0), \qquad (3.22)$$

where the $3\times 3$ matrices $S$ and $L$ are

$$S=\begin{bmatrix}1 & -{}^8/_{15} & 0\\ 1 & 0 & -{}^8/_{15}\\ -{}^3/_2 & 1 & {}^3/_2\end{bmatrix},$$
$$L=\begin{bmatrix}{}^2/_3 & 1 & -{}^8/_{15}\\ -{}^3/_2 & 0 & {}^3/_2\\ 0 & -{}^3/_2 & {}^3/_2\end{bmatrix}. \qquad (3.23)$$

The solution of Eqs. (3.22) is

$$y_n=K^n y_0,\quad x_n=SK^n y_0,\quad K=LS. \qquad (3.24)$$

From this point on the procedure for constructing regular-

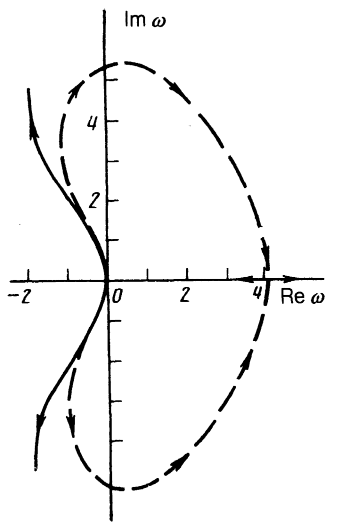


FIG. 2.

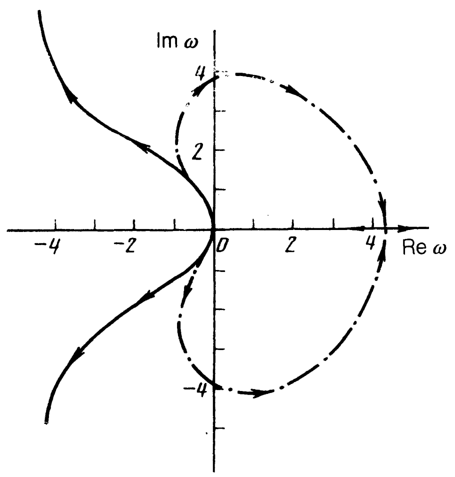


FIG. 3.

ized Barnett approximations $\sigma_{1R}$, $q_{1R}$ is analogous to the procedure used in the first example. The final result is

$$\sigma_{1R}=P_cR_1y_0\varepsilon\partial_xu+P_aSR_1y_0\varepsilon^2\partial_x^2T+P_bSR_1y_0\varepsilon^2\partial_x^2\rho,$$
$$q_{1R}=P_\alpha R_1y_0\varepsilon\partial_xT+P_\beta R_1y_0\varepsilon\partial_x\rho+P_\varphi SR_1y_0\varepsilon^2\partial_x^2u,$$
$$R_1=(1-\varepsilon^2K\partial_x^2)^{-1}. \qquad (3.25)$$

Here the $P_\mu$ project onto the corresponding axes in $X$ and $Y$ (e.g., $P_cy_k=c_k$). The dispersion relation for the approximation (3.25) is rather complicated, so we will not reproduce it here. Figure 3 is a sketch of the dispersion curves for the Barnett approximation (3.18) (the dashed line), and for the regularized Barnett approximation, (3.25) (the solid line). Only the acoustic branches are shown; the diffusion branches are essentially the same in each approximation.

As a final example we consider the result of regularization of the Barnett ($k_0=1$) approximations for Eqs. (2.2):

$$\sigma_{1Rik}=P_cR_1y_0\varepsilon\overline{\partial_iu_k}+P_dR_1y_0\Delta^{-1}\Gamma_{ik}\varepsilon\partial_su_s$$
$$+P_aSR_1y_0\varepsilon^2\Gamma_{ik}\rho+P_bSR_1y_0\varepsilon^2\Gamma_{ik}T,$$
$$q_{1Rk}=P_\alpha R_1y_0\varepsilon\partial_k\rho+P_\beta R_1y_0\varepsilon\partial_kT \qquad (3.26)$$
$$+P_\varphi SR_1y_0\varepsilon^2\partial_k\partial_su_s+P_\psi SR_1y_0\varepsilon^2\Delta u_k,$$
$$R_1=(1-\varepsilon^2K\Delta)^{-1}, \quad y_0=(0,\ -1,\ 0,\ -15/4).$$

The space of four-dimensional vectors $x_n=(a_n,b_n,\varphi_n,\psi_n)$, $y_n=(d_n,c_n,\alpha_n,\beta_n)$ was used in the derivation of (3.26) (a similar representation was used in the preceding example). We also used the $4\times4$ matrices $S$ and $L$ [analogs of (3.23)] given by

$$S=\begin{bmatrix} 1 & 1 & -2/5 & 0 \\ 1 & 1 & 0 & -2/5 \\ -2 & -1/2 & 3/2 & 1 \\ 0 & -3/2 & 0 & 0 \end{bmatrix},$$
$$L=\begin{bmatrix} 1 & 3/2 & 2/5 & 0 \\ 0 & 0 & 0 & 2/5 \\ -2 & 0 & 3/2 & 3/2 \\ 0 & -2 & 3/2 & 3/2 \end{bmatrix}. \qquad (3.27)$$

The matrix $K$ is equal to the product $LS$.

**4.** The method presented above for regularizing Barnett approximations is based on the ideas of the Padé approximate[8] and partial summation of series. It is not possible at the outset to say just whether this procedure will result in the required stability of the wave spectrum. This situation is typical for methods which use the Padé approximation: A real improvement in the original expansions may be achieved, but then again it may not. Several instructive examples in this connection are given in a monograph.[8]

In the case of the linearized Grad equations, the problem of the stability of the wave spectrum can be formulated without invoking any truncation of the sequence of operators $\partial_t^{(m)}$ as in (3.1). For the ten-moment equations, for example, the regularized stress tensor $\sigma_{nR}$ can be sought in the form

$$\sigma_{nR}=F_{1n}(\partial_x^2)\partial_xu+F_{2n}(\partial_x^2)\partial_x^2\theta.$$

The operator symbols $F_{1,2n}(-k^2)$ are written as power series in $-k^2$, in which the first $n$ coefficients are known. The functions $F_{1,2n}$ appear in the coefficients of the dispersion relation. We need to construct an approximation of the functions $F_{1,2n}$ which has the given part of a Taylor series and which leads to the correct positions of the roots of the dispersion relation. The dependence of the functions $F_{1,2n}$ on $\partial_x^2$ alone is of course a consequence of the simple structure of the form (2.7) and (3.4).

There is no particular difficulty in formally extending the procedure (3.1) to the nonlinear Grad equations. In general, however, it is not possible to determine structures of the type (2.7) and (3.4). It is, on the other hand, possible to distinguish terms of a common type [e.g., terms with a maximum nonlinearity $(\partial_xu)^{n+1}$ in the $n$th order] in each order of the expansion and to carry out a regularization procedure to eliminate the negative viscosity of the Barnett approximations.


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