Heat propagation in helium II with superfluid turbulence

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Those terms in the equations of dissipative two-fluid hydrodynamics formulated by Nemirovskiĭ and Lebedev which arise from superfluid turbulence are subjected to a comparative analysis. Their main features are identified. The system of equations for heat propagation in helium at rest is expanded in the Mach number of the relative velocity of the motion of the components. For second-sound waves, a supercritical heat flux represents an anisotropic medium with an anomalous dispersion, which does not disappear even for waves propagating across the flow. A region of strong absorption for low-frequency waves is identified. This region lies adjacent to the direction of the supercritical heat flux. Outside this region, the absorption and attenuation of sound are determined primarily by the steady-state heat flux. They depend only weakly on the parameters of the waves. In an analysis of the propagation of a square-wave heat pulse in a broad channel with a weak initial turbulence, a boundary-value problem is solved. The transient process which culminates in a steady-state heat flux is analyzed. The conditions for several experiments which would be capable of testing this theory are formulated.

There is flagging of research interest in the behavior of superfluid helium as a heat-transfer medium under conditions such that there is a supercritical thermal load. This interest arose, in particular, from problems involving the use of helium II in cryogenic devices. Comparisons of theoretical and experimental work have been the subject of many reports at recent conferences and meetings on cryogenics. The theoretical ideas based on the two-fluid model with a well-developed superfluid turbulence are reflected most comprehensively in the equations of motion of superfluid helium which have been formulated by Nemirovskiĭ and Lebedev.[1]

In Sec. 1 of this paper we analyze the equations formulated in Ref. 1, and we derive equations for heat transfer in helium II at rest. These equations are expanded in the Mach number of the velocity amplitude of the counterflow of the normal and superfluid components. Terms up to first order are retained in this expansion. In Sec. 2 we examine the propagation of second sound in the case of a supercritical steady-state heat flux. In Sec. 3 we take up the two-dimensional problem of the propagation of stepped heat pulses in the case of homogeneous and weak initial turbulence.

1. HEAT PROPAGATION EQUATIONS

In order to analyze the effect of superfluid turbulence, it is useful to single out in the equations of motion formulated for superfluid helium by Nevirovskiĭ and Lebedev[1] the leading parts of those terms which are related to this turbulence. Since the vortices of the superfluid turbulence draw energy from the kinetic energy of the superfluid component, it is natural to assume that the vortex energy density is much lower than the kinetic-energy density of the superfluid component. We also assume that the work density per unit time of the force of the friction with the frozen system of vortices is much higher than the corresponding change in the vortex energy density. Under these assumptions approximate equations of motion of superfluid helium can be written as follows (see the Appendix for the notation):

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0, \tag{1}$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial P}{\partial x_i} + \frac{\partial (\rho_n V_{ni} V_{nk} + \rho_s V_{si} V_{sk})}{\partial x_k} = 0, \tag{2}$$

$$\frac{\partial \rho\sigma}{\partial t} + \operatorname{div}(\rho\sigma \mathbf{V}_n) = \frac{\mathbf{f}\mathbf{V}_{ns}}{T}, \tag{3}$$

$$\frac{\partial \mathbf{V}_s}{\partial t} + (\mathbf{V}_s \nabla)\mathbf{V}_s + \nabla\mu = \frac{\mathbf{f}}{\rho_s}, \tag{4}$$

$$\frac{\partial Y}{\partial t} + \operatorname{div}(Y\mathbf{V}_L) = |\mathbf{V}_{ns}| Y^{3/2} - \nu Y^2. \tag{5}$$

Equation (5) is the renormalized Vinen equation. We need to supplement these equations with the thermodynamic relation[2]

$$dP = \rho\, d\mu + \rho\sigma\, dT + \frac{\rho_n}{2}\, dV_{ns}^2. \tag{6}$$

Here $Y = \alpha^2 L$, L is the total length of the vortex filaments in a unit volume, α is a dimensionless empirical parameter of the Vinen equation, $\mathbf{V}_L = b\,\mathbf{V}_{\rm ns}$ is the average velocity of the vortex formation, $\mathbf{V}_{\rm ns} = \mathbf{V}_n - \mathbf{V}_s$, $b = b(T)$ is the known function of the temperature,[3] $\nu = \nu(T)$ is a renormalized empirical parameter of the Vinen equation, $\mathbf{f} = A\rho_s\rho_n\nu^2 Y \mathbf{V}_{\rm ns}$ is the force density of the mutual friction between the normal and superfluid components, $A = A(T)$ is the Gorter–Melink constant, and ρ, σ, and $\bar{\rho}_n = \rho_n/\rho$ are known (tabulated) functions of the temperature T and the pressure P. The dependence of ρ and σ on $V_{\rm ns}^2$ is found from the Maxwell relation, which follow from (6). The dependence of $\bar{\rho}_n$ on $V_{\rm ns}^2$ is not known, but in practice it can be ignored.[2]

The time scales in Eqs. (1)–(4) are the periods of the sound waves or the time taken by the sound to traverse a length scale of the system. A time scale in Eq. (5) is $\tau = \nu/V_{\rm ns}^2$. We wish to stress that the turbulent-friction force is proportional to the parameters A and ν^2 and also to the product $YV_{\rm ns}$. The time scale of the variation in Y may be much longer than that for $V_{\rm ns}$. In the steady state, in which Y reaches $Y_c = V_{\rm ns}^2/\nu^2$, we have $\mathbf{f} = A\rho_s\rho_n V_{\rm ns}^2 \mathbf{V}_{\rm ns}$. For $V_{\rm ns} \gg 1$ cm/s this quantity is given by a known expression.[4,5]

We will be interested below in problems involving heat

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propagation under conditions such that there is no mass flow, i.e., with

$$j=\rho(\bar{\rho}_n V_n+\bar{\rho}_s V_s)=0.$$

At $T>0.8$ K, this case corresponds quite accurately to the propagation of sec-ond-sound waves.[2] In this case, if we ignore the dependence of ρ on T, P, and $V_{\rm ns}^2$, and if we consider only the functional dependences $\sigma=\sigma(T,V_{\rm ns}^2)$ and $\bar{\rho}_n=\bar{\rho}_n(T)$, we can reduce Eqs. (1)–(4) to two equations for T and $\mathbf{V}_{\rm ns}$ by using (6), making the substitutions $\mathbf{V}_s=-\bar{\rho}_n\mathbf{V}_{\rm ns}$ and $\mathbf{V}_n=\bar{\rho}_s\mathbf{V}_{\rm ns}$, and eliminating P and μ. Solving these equations for $\partial T/\partial t$ and ∇T, and using (5), we find a system of equations describing heat propagation in superfluid helium at rest. We write this system of equations in the approximation linear in $V_{\rm ns}/C_2$ [$C_2=(\bar{\rho}_s\sigma^2/\bar{\rho}_n\sigma_T)^{1/2}$ is the second-sound velocity] and in dimensionless form:

$$\frac{\partial G}{\partial y}+\operatorname{div}\mathbf{U}+\varepsilon\left[\bar{\rho}_n q-\frac{\bar{\rho}_s}{2}(1-q)\right]\frac{\partial U^2}{\partial y}$$

$$=\varepsilon 2\gamma\frac{\bar{L}}{\bar{L}_I}U^2\left[\bar{\rho}_s+\frac{\sigma}{\sigma_T T}(\bar{\rho}_s-\bar{\rho}_{nT}T)\right], \tag{7}$$

$$\nabla G+\frac{\partial\mathbf{U}}{\partial y}+\varepsilon\bar{\rho}_s[\nabla U^2+(1-q)\mathbf{U}\operatorname{div}\mathbf{U}]=-2\gamma\frac{\bar{L}}{\bar{L}_I}\mathbf{U}, \tag{8}$$

$$\frac{\partial\bar{L}}{\partial y}+\varepsilon\operatorname{div}[\bar{L}b\mathbf{U}]=\varepsilon_L(|\mathbf{U}|\bar{L}^{3/2}-\bar{L}^2). \tag{9}$$

Here

$y=C_2t/h$, $U=V_{\rm ns}/V$, $G=(T-T_0)\sigma_T C_2/\sigma\bar{\rho}_s V$,
$\bar{L}=Y/Y_a$, $Y_a=V^2/\nu^2$, $\varepsilon_L=h/\tau_L C_2$, $\tau_L=\nu/V^2$,
$2\gamma=\rho A\nu^2 Y_I h/C_2$, $\varepsilon=V/C_2\ll 1$,
$q=\sigma\bar{\rho}_{nT}/\sigma_T\bar{\rho}_n\simeq 1$, $\sigma_T=\partial\sigma/\partial T$, $\bar{\rho}_{nT}=d\bar{\rho}_n/dT$,

T_0 is the initial temperature, V is the amplitude of the characteristic velocity of the countercurrent of the components, h is a length scale by which the spatial coordinates are divided (normalized), and Y_I and $\bar{L}_I$ are the unnormalized and normalized initial values of the turbulence. All the normalizing quantities which depend on the temperature are taken at T_0. In practice we would have $\varepsilon\sim 10^{-3}$–10^{-1}.

2. PROPAGATION OF SECOND SOUND IN A SUPERCRITICAL HEAT FLUX

Let us use Eqs. (7)–(9) in the zeroth approximation in ε to analyze the propagation of a small-amplitude, monochromatic, plane second-sound wave in the case of a steady-state supercritical two-dimensional heat flux, in which the velocity of the counterflow of the components is V. For uniform waves (Re $\mathbf{K}$/Re K = Im $\mathbf{K}$/Im K), the dispersion relation relating the frequency ω and the wave vector $\mathbf{K}$ takes the simple form

$$C_2^2K^2=\frac{\omega[\omega-i\rho A\nu/\tau_L][(\omega-i/2\tau_L)(\omega-i\rho A\nu/\tau_L)-\rho A\nu/\tau_L^2]}{(\omega-i/2\tau_L)(\omega-i\rho A\nu/\tau_L)-(\rho A\nu/\tau_L^2)\sin^2\varphi}, \tag{10}$$

where φ is the angle between the direction of the radiation and the heat flux.

Analysis of this dispersion relation shows that for low frequencies there is a strong-absorption region ($|\mathrm{Im}\,K|>\mathrm{Re}\,K$) adjacent to the direction of the flux. For $\rho A\nu\gg 1$, the frequency region with waves which are not strongly absorbed ($\mathrm{Re}\,K>|\mathrm{Im}\,K|$) and the phase velocity of these waves are given by

$$\omega>\omega_1=\frac{(\rho A\nu)^{1/2}}{\tau_L}[\cos^2\varphi-{}^9/_4(\rho A\nu+2)]^{1/2},$$

$$C'_{2\rm ph}>C'_{2\rm ph}(\omega_1)=C_2(4/\rho A\nu)^{1/4}[\cos^2\varphi-{}^9/_4(\rho A\nu+2)]^{1/4}.$$

Under the conditions $|\omega|>\rho A\nu/\tau_L$ and $\rho A\nu\gg 1$, the solution of (10) in both the case $\mathrm{Im}\,\omega\equiv 0$ (traveling waves) and the case $\mathrm{Im}\,K\equiv 0$ (standing waves) gives us some simple equations which determine the basic behavior as a function of V, φ, and ω or K. The sound velocity C_2' is given by

$$C_2'=C_2\left[1\mp\frac{p^2}{8}\left(1-\frac{4\cos^2\varphi}{\rho A\nu}\right)\right], \tag{11}$$

where $p=\rho A\nu/\tau_L\omega$ in the case $\mathrm{Im}\,\omega\equiv 0$ and $p=\rho A\nu/\tau_L KC_2$ in the case $\mathrm{Im}\,K\equiv 0$ (the minus sign in front of the p^2 corresponds to the phase velocity $C'_{2\rm ph}$, while the plus sign corresponds to the group velocity C'_{2g}). The expressions for the damping rates K_2 and ω_2 are

$$K_2=\frac{\rho A\nu}{2\tau_L C_2}\frac{C'_{2\rm ph}}{C_2},\quad \omega_2=\frac{\rho A\nu}{2\tau_L}\left(1+\frac{\cos^2\varphi}{2\tau_L^2K^2C_2^2}\right). \tag{12}$$

In the case $\varphi=\pi/2$, Eq. (10) becomes quadratic. From its solutions we find $C'_{2\rm ph}=C_2 2^{1/2}/[1+(1+p^2)^{1/2}]^{1/2}$ and $\mathrm{Re}\,K>|\mathrm{Im}\,K|$ for all ω in the case $\mathrm{Im}\,\omega\equiv 0$, while for $\mathrm{Im}\,K\equiv 0$ we find $C'_{2\rm ph}=C_2(1-p^2/4)^{1/2}$. For $K>\rho A\nu/2^{1/2}\tau_L C_2$ we find $\mathrm{Re}\,\omega>\mathrm{Im}\,\omega$, and the damping rates are again given by (12).

It can be seen from (11)–(12) that for second sound a supercritical heat flux represents an anisotropic medium with an anomalous dispersion. The dispersion is anomalous because of the absorption of sound, which is due primarily to the intensity of the steady-state heat flux. The anisotropy arises because the longitudinal component of the second sound initiates turbulence oscillations according to (9) (Ref. 1). At $\omega\tau_L\gg 1$ the turbulence oscillations are weak, and they lag behind the oscillations in the velocity of the normal component by a phase of nearly $\pi/2$. According to (11), this circumstance reduces the anomalous dispersion and thus increases the phase velocity. Under the condition $\omega\tau_L\leqslant 1$, the turbulence and velocity oscillations are comparable in amplitude and nearly in phase, so there is a region of strong absorption.

The same problem was taken up in Ref. 1. There, a more complicated dispersion relation was solved in the case $\mathrm{Im}\,K\equiv 0$ (no expansion in ε was carried out). An erroneous expression was derived for the functional dependence $\omega(K,\varphi)$. According to (11), in contrast with the results of Ref. 1, there is also dispersion for sound propagating perpendicular to the steady-state heat flux.

Solving the problem under the condition $\mathrm{Im}\,\omega\equiv 0$ in the linear approximation in ε adds, in (11), only a term which represents a drift of the sound caused by the heat flux $V[\bar{\rho}_s-\bar{\rho}_n+(1-q)]\cos\varphi$ (this term was found in Ref. 6) and, in (12), a term $-(\rho A\nu/2\tau_L C_2)2\varepsilon[\bar{\rho}_s+(\bar{\rho}_s-\bar{\rho}_{nT}T)\sigma/\sigma_T T-b/2]\cos\varphi$. The coefficients of all the terms determined by the superfluid turbulence are of order unity.

It follows from (11) that the three second-sound phase velocities measured in a heat flux [these velocities are $C_{l\parallel}$ and $C_{l\perp}$, which are the velocities of low-frequency "longitudinal" ($\varphi = 0$) and "transverse" ($\varphi = \pi/2$) sound, and $C_{h\parallel}$, the velocity of high-frequency "longitudinal" sound, where $\omega_h \gg (C_2/8\bar{\rho}_s V)^{1/2}\rho A\nu/\tau_L$ and $\rho A\nu/\tau_L < \omega_l \ll (C_2/8\bar{\rho}_s V)^{1/2}\rho A\nu/\tau_L$] must satisfy the relation

$$\frac{4}{\rho A\nu} = 1 - \frac{C_{l\parallel} - C_{h\parallel}}{C_{l\perp} - C_2}.$$

This relation can serve as a test of theoretical ideas or as a method for determining $A\nu$.

3. PROPAGATION OF A HEAT PULSE IN A BROAD CHANNEL

Let us consider the longitudinal propagation of a two-dimensional square heat pulse in a broad channel (i.e., in one in which we can ignore the lateral walls) of unit length (in terms of the unnormalized coordinate of length, h) under the condition $\varepsilon_L \ll 1$ ($h \ll \tau_L C_2$). Under the condition $y \leqslant 1/\varepsilon_L$, in the zeroth approximation in ε and ε_L, we thus have $\bar{L} \simeq \bar{L}_I$ from (9), and from Eqs. (7) and (8) we find

$$\frac{\partial G}{\partial y} + \frac{\partial U}{\partial x} = 0, \tag{13}$$

$$\frac{\partial G}{\partial x} + \frac{\partial U}{\partial y} = -2\gamma U. \tag{14}$$

These equations describe the initial propagation of the pulse in the superfluid helium in the case of well-developed superfluid turbulence. The attenuation is proportional to the initial turbulence level Y_I (or to the corresponding normalized quantity $\bar{L}_I$). Eliminating G from (13) and (14), we find a telegraphist's equation for U:

$$\frac{\partial^2 U}{\partial y^2} + 2\gamma\frac{\partial U}{\partial y} - \frac{\partial^2 U}{\partial x^2} = 0. \tag{15}$$

We assume that there is a heat source $W = \rho\sigma T\bar{\rho}_s V\theta(y)$ at the closed end of the channel at $x = 0$; here $\theta(y)$ is the unit step function. The $x = 1$ end of the channel is open, connected to a vessel holding helium. We assume that before the heat pulse is applied, for $y \leqslant 0$, well-developed superfluid turbulence, of intensity $Y_I < Y_a$ ($\gamma \neq 0$), is present in the channel, distributed uniformly in x; we assume that the temperature is T_0 everywhere; and we assume[1] $V_{ns} = 0$. Using (13) and (14) under these assumptions, and noting that the front of the pulse propagates along the characteristic $y - x = 0$, decaying in accordance with the value of γ (Ref. 7), we can write the following boundary conditions for our problem:

$$U(0,y) = \theta(y), \quad U(x,y)|_{x=y} = e^{-\gamma y}\theta(y), \quad \frac{\partial U}{\partial x}(1,y) = 0, \tag{16}$$

$$\frac{\partial G}{\partial x}(0,y) = -2\gamma\theta(y), \quad G(x,y)|_{x=y} = e^{-\gamma y}\theta(y), \quad G(1,y) = 0. \tag{17}$$

The conditions on the characteristic $y - x = 0$ have been replaced by the initial conditions $U(x, 0) = 0$ and $G(x, 0) = 0$, while the conditions at $x = 1$ are a qualitatively justified extrapolation of the corresponding conditions for the steady-state problem. The problem under consideration here thus reduces to one of finding a solution of Eq. (15) in the region $0 \leqslant x \leqslant 1$, $y \geqslant x$ under the boundary conditions (16) and then using the solution found for U to find a solution for G on the basis of Eq. (13) and the boundary conditions (17).

We partition the range of U (and thus that of G) into triangular subregions by means of the characteristics $y + x = 2m$ and $y - x = 2m$ (m is an integer). We denote these subregions by a sequence of integers in such a way that in the odd-numbered $(2m - 1)$-subregions adjacent to the part of the contour $x = 0$ we have $2(m - 1) + x < y < 2m - x$, while in the even-numbered $(2m)$-subregions adjacent to the part of the contour $x = 1$ we have $2m - x < y < 2m + x$.

The solutions are of the form

$$\begin{aligned} U_{2m-1} &= R_1 + \sum_{k=1}^{m-1} (R_{2k+1} + R_{2k}), \\ G_{2m-1} &= Q_1 + \sum_{k=1}^{m-1} (Q_{2k+1} + Q_{2k}) \end{aligned} \tag{18}$$

in the odd-numbered subregions and

$$\begin{aligned} U_{2m} &= \sum_{k=1}^{m} (R_{2k-1} + R_{2k}), \\ G_{2m} &= \sum_{k=1}^{m} (Q_{2k-1} + Q_{2k}) \end{aligned} \tag{19}$$

in the even-numbered subregions. Here $R_{2k+1} = F_k$, $Q_{2k+1} = H_k$ for $y_k = 2k + x$, $k \geqslant 0$; and $R_{2k} = -F_k$, $Q_{2k} = H_k$ for $y_k = 2k - x$, $k \geqslant 1$ (below, the series in F_k is term-by-term differentiable), where

$$\begin{aligned} F_k &= (-1)^k e^{-\gamma y}\Big\{ I_0[\gamma(y^2 - y_k^2)^{1/2}] + 2\sum_{n=1}^{\infty}\Big(\frac{y - y_k}{y + y_k}\Big)^{n/2} \\ &\quad \times I_n[\gamma(y^2 - y_k^2)^{1/2}]\Big\}\theta(y - y_k) \\ &= (-1)^k\Big\{ e^{-\gamma y_k} + \gamma y_k \int_{y_k}^{y} e^{-\gamma z}\frac{I_1[\gamma(z^2 - y_k^2)^{1/2}]}{(z^2 - y_k^2)^{1/2}}dz\Big\}\theta(y - y_k), \\ H_k &= (-1)^k\Big\{ e^{-\gamma y} I_0[\gamma(y^2 - y_k^2)^{1/2}] \\ &\quad + 2\gamma\int_{y_k}^{y} e^{-\gamma z} I_0[\gamma(z^2 - y_k^2)^{1/2}]dz\Big\}\theta(y - y_k), \end{aligned}$$

and I_n are modified Bessel functions. On the characteristics $y + x = 2m$ and $y - x = 2m$ the solutions have discontinuities:

$$\begin{aligned} U_{2m} - U_{2m-1} &= -(-1)^m e^{-\gamma y}, \\ G_{2m} - G_{2m-1} &= (-1)^m e^{-\gamma y}, \quad y = 2m - x, \end{aligned} \tag{20}$$

$$\begin{aligned} U_{2m+1} - U_{2m} &= (-1)^m e^{-\gamma y}, \\ G_{2m+1} - G_{2m} &= (-1)^m e^{-\gamma y}, \quad y = 2m + x. \end{aligned} \tag{21}$$

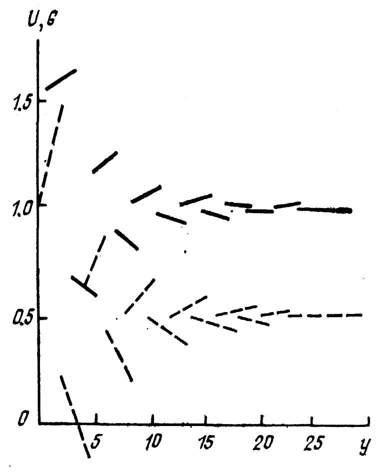

FIG. 1. Abrupt oscillations in the normalized heat flux $U(1,y)$ (solid lines) and in the normalized temperature excursion $G(0,y)$ (dashed lines) for the value $\gamma = 0.25$.

Solutions (18)–(19) represent the propagation and evolution of the primary wave, described by R_1 and Q_1, and of the reflected waves, described by R_{2k}, Q_{2k} and R_{2k+1}, Q_{2k+1} at $k \geqslant 1$. The quantities R_{2k}, Q_{2k} describe waves which are reflected at $x = 1$ when waves R_{2k-1}, Q_{2k-1} are incident, while R_{2k+1}, Q_{2k+1} describe waves which are reflected at $x = 0$ when waves R_{2k}, Q_{2k} are incident. Equations (20) and (21) describe the propagation of the reflected wave- front along the characteristics. The reflection conditions are $R_{2k} = R_{2k-1}$, $Q_{2k} = -Q_{2k-1}$ at $x = 1$ and $R_{2k+1} = -R_{2k}$, $Q_{2k+1} = Q_{2k}$ at $x = 0$. The amplitude of the wavefront decays at $e^{-\gamma y}$; the sign of the wave is determined by the expressions $\operatorname{sign} R_{k+1} = \operatorname{sign} \sin[(2k+1)\pi/4]$, $\operatorname{sign} Q_{k+1} = \operatorname{sign} \cos[(2k+1)\pi/4]$, where k is the index of the reflection. We see that the cycle of reflections has a period of 4.

Analysis of the resulting solutions shows that the temperature of the superfluid helium behind the front of the primary wave increases by an amount $\Delta T \simeq W/\rho\sigma_T T C_2$ ($G \simeq 1$). When the wavefront reaches the open end of the channel, a reflected wave appears. This wave tends to oppose a temperature rise. During subsequent reflections, there are damped abrupt oscillations in U and G with a period of 4 (Fig. 1). The oscillation amplitude decays as $e^{-\gamma y}$, and we find $U(x,y) \to 1$ and $G(x,y) \to 2\gamma(1-x)$. In other words, the quantities tend toward the known steady-state solutions. This process is illustrated well by the quantities $\delta U = U - 1$ and $\delta G = G - 2\gamma(1-x)$, which can be written as follows for $\gamma \ll 1$ and $\gamma 2m < 0.5$:

$$\delta U_{2m-1} \approx (-1)^m \gamma x \left(1 - \gamma \frac{y}{2}\right),$$

$$\delta G_{2m-1} \approx -(-1)^m [1 - \gamma(2(2m-1) - y)],$$

$$\delta U_{2m} \approx -(-1)^m \left[1 - \gamma 2m \left(1 - \gamma \frac{y}{2}\right)\right],$$

$$\delta G_{2m} \approx -(-1)^m 2\gamma(1-x) \left(1 - \frac{3}{4} \gamma 2m\right).$$

These expressions show that there are basically damped oscillations with an initial amplitude of 1: oscillations in the normalized temperature in the odd-numbered subregions adjacent to the source ($x = 0$) and oscillations in the normalized heat flux in the even-numbered subregions adjacent to the open end of the channel ($x = 1$). These temperature oscillations can be interpreted as mutual damped oscillations in the concentration of the components with an initial amplitude $\Delta\bar\rho_n \simeq \bar\rho_n \bar\rho_s V/C_2$, which are not initially accompanied by oppositely directed flows of the components. According to (20)–(21) these oscillations stem from discontinuities in the concentration and the flow at the wavefront and reflections of this wavefront.

Solutions (18)–(19) are based on the boundary conditions (16)–(17) at $x = 1$ and also on the reflection conditions at the open end of the channel, which follow from those boundary conditions. These boundary conditions are valid experimentally for a steady-state heat flux in this approximation. Although there is qualitative justification for extrapolating these conditions into the region $y \geqslant 1$ for a pulsed flux, an experimental test is required.

Solutions (18)–(19) are valid under the condition that a significant increase in $\bar L$, which is proportional to the damping according to (8), occur over a time $y > 1/\gamma$. The change in $\bar L$ is described by the Vinen equation (9). Setting $U = 1$ in it, we find a single-valued relationship between $\bar L$ and y in the zeroth approximation in ε (we assume $\varepsilon \ll \varepsilon_L$):

$$\ln\left(\frac{\bar L_I^{-1/2} - 1}{\bar L^{-1/2} - 1} \exp(\bar L_I^{-1/2} - \bar L^{-1/2})\right) = \frac{1}{2} \varepsilon_L (y - y_I) = \frac{1}{2} \frac{(t - t_I)}{\tau_L}, \tag{22}$$

where $\bar L_I < 1$ corresponds to the time y_I. At small values of $\bar L$ ($\bar L_I \ll 1$), relationship (22) can be approximated by $\bar L_I^{1/2} \varepsilon_L (y - y_I) \simeq 2(1 - \bar L_I^{1/2}/\bar L^{1/2})$. The condition for the applicability of solutions (18)–(19) then becomes (we are assuming $y_I = 0$)

$$\bar L_I^{1/2} > 2(\rho A \nu)^{-1}. \tag{23}$$

This condition depends only on the parameters A and ν of the theory. Let us assume that this condition holds. Then the solutions (18)–(19) reach $U = 1$, $G = 2\gamma(1-x)$ over a time $y \sim 1/\varepsilon_L$.

At times $y > 1/\varepsilon_L$ the problem is described by Eqs. (7)–(9). Analysis of solutions of these equations shows that, in the zeroth approximation in ε and ε_L, we have

$$U = 1, \quad G = 2\gamma(1-x)\bar L/\bar L_I, \tag{24}$$

where $\bar L$ is given by (22), and under the condition $y \gg 1/\varepsilon_L$ we have $\bar L \simeq 1$.

The nature of the solution of this problem does not change substantially in nature if ε_L is not small and if condition (23) does not hold. In other words, this process is first described by (18)–(19) and ultimately by (24) in all cases.

The observed magnitude of the process (i.e., the temperature excursion near the source) is characterized by the following quantities: the initial temperature drop $T_I = (\sigma/\sigma_T)(\bar\rho_s V/C_2)$ (this expression has solid experimental support[9]); the oscillation period $\tau_k = 4h/C_2$; the time scale of the decay of the oscillations, $\tau_d = \tau_L 2/\rho A \nu \bar L_I$; the time scale of the variation in the normalized turbulence,

$\tau_L = \nu/V^2$ (the increase in $\bar{L}$ from 0.1 to 0.9 occurs over a time $\sim 10\tau_L$, approximately linearly); and the intermediate temperature after the decay of the oscillations (for $2/\rho A\nu\bar{L}_I < 1$), $T_{\rm int} = T_I\tau_k/2\tau_d = T_k\bar{L}_I$, where $T_k = h\rho\bar{\rho}_n AV^3/\sigma$ is the known temperature in the case of a steady-state process.[5] If there is no initial turbulence ($\bar{L}_I = 0$), and if the heat fluxes are subcritical or only slightly supercritical, the process by which a Poiseuille flow is established as a result of the viscosity of the normal component is correspondingly characterized by T_I, by the oscillation decay time $\tau_p = \rho\bar{\rho}_n d^2/16\bar{\rho}_s\eta$ (d is the channel diameter, and η is the viscosity), and by the temperature once the Poiseuille flow has been established, $T_p = T_I\tau_k/2\tau_p$.

Experiments by Peshkov and Tkachenko[5] on the propagation of a heat pulse through a channel ($h = 8$ m) with unexcited helium II have revealed rapid formation of a weak temperature gradient as a result of the Poiseuille flow (see Fig. 3 in Ref. 5). The temporal characteristics of the transient process were $\tau_k = 1.6$ s and $\tau_p = 0.8$ s, and this process was not observed. If the length of the channel had instead been 1 m, then at a heat flux density $W = 0.045$ W/cm^2 one would have been able to observe, over a time ~ 2 s, damped oscillations around the temperature $T_p = 0.03$ mK with a period $\tau_k = 0.2$ s and an initial amplitude $T_I = 0.25$ mK. In addition, if there had been an initial turbulence in the channel it would have been possible to observe the transient process resulting in the formation of a temperature gradient by the turbulence. With $\bar{L}_I = 0.2$ and $W = 0.045$ W/cm^2, for example, we would have $\tau_k = 0.2$ s, $\tau_d = 0.4$ s, $10\tau_L \simeq 3$ s (see Fig. 12 on p. 48 in Ref. 4), $T_I = 0.25$ mK, $T_{\rm int} = 0.06$ mK, and $T_k = 0.3$ mK.

A comparison of the results of an experiment of that sort with the results found in this section of the paper might reveal whether boundary conditions (16)–(17) are appropriate for the experiments, i.e., whether the conditions of reflection at an open end of the channel, adopted here, are valid. Alternatively, such a comparison might be of assistance in refining the theory and in identifying different reflection conditions.

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APPENDIX

The following terms, associated with the superfluid-turbulence characteristic L, were introduced in Ref. 1, where hydrodynamic equations for a superfluid turbulence were formulated: in Eq. (2),

$$\frac{\partial(P_L\delta_{ik}+\pi_{ik})}{\partial x_k} = \frac{\partial\tau_{ik}}{\partial x_k},$$

in Eq. (3),

$$R/T - \operatorname{div}\mathbf{\Sigma} = A\rho_s\rho_n V_{ns}^2(\beta^2/\alpha^2)L[1 - (1-L/L_c)\varkappa\alpha^4/\beta^3 A\rho_n]/T - \operatorname{div}[S_L(\mathbf{V}_L - \mathbf{V}_n)],$$

in Eq. (4),

$$\frac{\mathbf{f}}{\rho_s} = \{A\rho_s\rho_n V_{ns}^2(\beta^2/\alpha^2)L[1 - (1-L^{1/2}/L_c^{1/2})\varkappa\alpha^4/\beta^3 A\rho_n] - \tau_{ik}\,\partial V_{ni}/\partial x_k + L\varkappa(\partial\rho_s/\partial T) \times [\ln(l/a_0)/\ln(\delta/a_0)]((\mathbf{V}_L - \mathbf{V}_n), \nabla T)\}\mathbf{V}_{ns}/\rho_s V_{ns}^2,$$

Here $dP_L = -\varepsilon_b dL$ is an additional term in (6) (Ref. 1), $\varepsilon_b = (\rho_s(h/m)^2/4\pi)\ln(\delta/a_0) = \rho_s\varkappa$ is the energy per unit length of a vortex thread, τ_{ik} is the viscous stress tensor, $S_L \cong -L\varkappa\partial\rho_s/\partial T\ln(\delta/l)/\ln(\delta/a_0)$, $L_c = V_{ns}^2\alpha^2/\beta^2$ is the limiting steady-state value of L in the coordinate system moving with the turbulence drift, δ is the characteristic distance between vortices, a_0 is the nominal radius of a core, and l is the mean free path of the excitations.

As in Ref. 1, we set $\tau_{ik} = 0$. From the assumptions formulated at the beginning of this paper we find the following:

$$\varkappa(\alpha^2/\beta^2)(V_{ns}^2/V_s^2) \cong \varkappa\alpha^2/\beta^2 \ll 1, \tag{A1}$$

i.e., $\alpha/\beta \ll 10^3$ s/cm^2, from the first and

$$\varkappa(\alpha^2/\beta^2)(\alpha^2/\beta A\rho_n)\bar{L}^{1/2}(1-\bar{L}^{1/2}) \ll 1 \tag{A2}$$

from the second. Using (A1) with $\beta\rho_n A/\alpha^2 \geqslant 1/4$, we see that (A2) definitely holds. We also assume

$$(\partial\rho_s/\partial T)T/\rho_n\rho_s A|V_{ns}|d_T \lesssim 1, \quad (\partial\rho_s/\partial T)T/\rho_n\rho_s A|V_{ns}|d_\Sigma \lesssim 1,$$

where d_T and d_Σ are the sizes of the temperature drop and the source (Σ) drop, which are found from the conditions $d_T|\nabla T|/T \simeq 1$ and $d_\Sigma \operatorname{div}\Sigma/|\Sigma| \simeq 1$. These conditions usually hold quite well. A special analysis of these conditions is required near T_λ and in an analysis of thermal shock waves.

Under these assumptions it is legitimate to retain the terms associated with L in Eqs. (3) and (4) alone. Here we have, approximately,

$$f = A\rho_s\rho_n(\beta^2/\alpha^2)LV_{ns} = A\rho_s\rho_n\nu^2 YV_{ns}, \quad (\nu = \beta/\alpha^2),$$

$$R/T - \operatorname{div}\Sigma = fV_{ns}/T.$$

The energy flux associated with the vortex formation has also been ignored in the equation expressing energy conservation.

[1)] A similar problem was analyzed in Ref. 8. Equations like (13) and (14) were used there. As can be seen from (7) and (8), the right sides are proportional to $\bar{L}U^2$ and $\bar{L}U$, respectively.[1] According to (9), although $\bar{L}$ does tend toward $\bar{L}_c = U^2$ in the coordinate system moving with the turbulence drift, it does so slowly. In Ref. 8, the right-hand sides were erroneously written as proportional to U^4 and U^3. According to (9), that behavior is valid only at times $y \gg 1/\varepsilon_L$. In accordance with the discussion above, the solution found in Ref. 8 is based on incorrect equations and is physically meaningless.

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