Effect of Hall currents on magnetic-field dissipation

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The dissipation of a strong magnetic field in a conducting sphere is studied ("strong" means $\omega_B\tau \gg 1$, where ω_B is the electron gyrofrequency, and τ is the electron relaxation time). Nondissipative Hall currents can substantially accelerate the decay of the magnetic field. The reason is that the Hall drift leads to the formation of regions with high current density and pronounced irregularities of the magnetic field, in which accelerated dissipation occurs. The symmetry of the field may change temporarily in the course of the evolution. For example, a field which initially has mirror symmetry with respect to the equatorial plane may become asymmetric because of the Hall effect. After a certain time, when dissipative effects become dominant, the field reverts to its original configuration (although with a far lower strength).

1. In a strong magnetic field, the conductivity and resistance of a material are anisotropic and are described by tensors σ and $\mathfrak{R}$. In a coordinate system with z axis parallel to the magnetic field **B**, these tensors are (Ref. 1, for example)

$$\sigma=\begin{pmatrix}\sigma_\perp & \sigma_\wedge & 0\\ -\sigma_\wedge & \sigma_\perp & 0\\ 0 & 0 & \sigma_\parallel\end{pmatrix},\quad \mathfrak{R}=\sigma^{-1}=\begin{pmatrix}R_\perp & R_\wedge & 0\\ -R_\wedge & R_\perp & 0\\ 0 & 0 & R_\parallel\end{pmatrix},$$

$$R_\parallel=\frac{1}{\sigma_\parallel}=\frac{1}{\sigma_0},\quad R_\perp=\frac{\sigma_\perp}{\sigma_\perp{}^2+\sigma_\wedge{}^2},$$
$$R_\wedge=-\frac{\sigma_\wedge}{\sigma_\perp{}^2+\sigma_\wedge{}^2},\tag{1}$$

where the subscripts $\parallel$ and $\perp$ mean the components parallel and perpendicular to the magnetic field, and $\wedge$ means the so-called Hall component. The quantity σ_0 is the conductivity in the absence of a magnetic field. The Hall current arises in a material from the drift of charge in the direction perpendicular to the electric and magnetic fields. This is a dissipation-free current, since it does not contribute to an increase in the entropy density of the medium, $\dot{Q}$:

$$\dot{Q}=\mathbf{jE}=R_\parallel j_\parallel{}^2+R_\perp j_\perp{}^2.\tag{2}$$

Here **j** is the current density, and **E** is the electric field.

In the relaxation-time approximation (Ref. 2, for example), the expressions for the components of the tensors σ and $\mathfrak{R}$ take the simple form

$$\sigma_\parallel=\sigma_0=\frac{e^2n\tau}{m},\quad \sigma_\perp=\frac{\sigma_0}{1+\omega_B{}^2\tau^2},\quad \sigma_\wedge=-\frac{\sigma_0\omega_B\tau}{1+\omega_B{}^2\tau^2},$$
$$R_\parallel=R_\perp=1/\sigma_0,\quad R_\wedge=B/nce,\tag{3}$$

where $e=|e|$, m, $\omega_B=eB/mc$, and n are the charge, effective mass, gyrofrequency, and density of the electrons. The quantity τ is a relaxation time. Under the condition $\omega_B\tau \ll 1$ we have $\sigma_\parallel \approx \sigma_\perp \gg \sigma_\wedge$, and the magnetic field has only a slight effect on the charge transport. If $\omega_B\tau \gg 1$ holds, then we have $\sigma_\parallel \gg \sigma_\wedge \gg \sigma_\perp$, and the magnetic field is important. The Hall resistance $R_\wedge$ does not depend on the relaxation time. The reason is that the Hall currents are nondissipative.

It follows from Eq. (2) that the Hall resistance does not directly cause attenuation of the electric current. In a magnetized material ($\omega_B\tau \gg 1$), however, the Hall drift can substantially alter the geometry of both the current and the magnetic field. The Hall currents thus indirectly affect the rate of dissipation, which depends strongly on the current configuration. This effect of the Hall current on dissipative processes is the subject of the present paper. We examine the decay of strong magnetic fields—strong enough to magnetize the material—inside a uniform sphere of radius a. The longest-lived mode of a weak ($\omega_B\tau \ll 1$) magnetic field in a sphere is known (Ref. 1, for example) to decay exponentially with a time scale $\sim a^2\sigma_0/c^2$.

We will show that in a magnetized material nondissipative Hall currents substantially alter the picture of the field dissipation, causing significantly more rapid decay. The field decay law may be nonexponential. In addition, Hall currents can temporarily cause a significant change in the magnetic field configuration in the course of the evolution.

2. The evolution of a quasisteady magnetic field in a conducting medium is described by the induction equation, which can be written as follows in the absence of motion:

$$\frac{\partial\mathbf{B}}{\partial t}=-\frac{c^2}{4\pi}[\nabla(\mathfrak{R}[\nabla\mathbf{B}])].\tag{4}$$

In the relaxation-time approximation, this equation can be rewritten as

$$\frac{\partial\mathbf{B}}{\partial t}=-\frac{c^2}{4\pi}\left[\nabla\left(\frac{1}{\sigma_0}[\nabla\mathbf{B}]-\frac{1}{cen}[\mathbf{B}[\nabla\mathbf{B}]]\right)\right].\tag{5}$$

We assume for simplicity that the sphere is homogeneous and that $\sigma_0=\text{const}$. In an actual situation, of course, the current decay could lead to nonuniform heating of the sphere. Since the relaxation time and density of the electrons may depend on the temperature, the conductivity can, in general, become inhomogeneous as the magnetic field evolves. If the specific heat of the sphere material is sufficiently large, however, the inhomogeneity of σ_0 will be unimportant and can be ignored.

We consider the decay of an axisymmetric toroidal magnetic field which is directed along the azimuthal unit vector $\mathbf{e}_\varphi$: $\mathbf{B}=B(r,\theta)\mathbf{e}_\varphi$, where r, θ, φ are spherical coordinates. For a field with this configuration, the induction equation (5) can be put in the form

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$$\frac{\partial b}{\partial \xi}=\frac{1}{x^2}\frac{\partial^2}{\partial x^2}(x^2 b)+\frac{1}{x^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(b\sin\theta)\right]+\frac{\alpha}{x}\left(\operatorname{ctg}\theta\frac{\partial b^2}{\partial x}-\frac{1}{x}\frac{\partial b^2}{\partial\theta}\right), \tag{6}$$

where we have introduced the dimensionless time $\xi = c^2 t/4\pi\sigma_0 a^2$ and the radial coordinate $x = r/a$. In addition, we are using $b = B/B_0$, where B_0 is a normalizing value of the magnetic field (this value is set by the initial conditions, as discussed below), and $\alpha = e\tau B_0/mc$. For a toroidal magnetic field the following boundary conditions must hold:

$$b=0 \text{ for } x=1,\ b=0 \text{ for } x=0. \tag{7}$$

The first of these boundary conditions is a consequence of the continuity of the field at the surface. For a toroidal field we would have $\nabla\times\mathbf{B}\neq 0$, and this field could not exist in vacuum. We must therefore have $B = 0$ at the surface of the sphere. The second boundary condition follows from the fact that the current density remains bounded as $x\to 0$.

For a weak magnetic field ($\alpha < 1$), Eq. (6) reduces to a linear equation and can be solved without difficulty. The general solution can be written as an expansion in the normal modes $B^{(i)}$. The fundamental mode $B^{(0)}$ is the one which is damped most slowly. For this mode we can write

$$B^{(0)}\propto\frac{1}{\lambda^{1/2}x}\left(\frac{\sin(\lambda^{1/2}x)}{\lambda^{1/2}x}-\cos(\lambda^{1/2}x)\right)\exp(-\lambda\xi)\sin\theta, \tag{8}$$

where $\lambda\approx 20.19$. The time scale t_m on which the fundamental toroidal mode decays is

$$t_m\approx 0{,}62\sigma_0 a^2/c^2. \tag{9}$$

The magnetic configuration which corresponds to a dipole field outside the sphere decays more slowly by a factor of about 2 (Ref. 2, for example):

$$t_{md}\approx 1{,}27\sigma_0 a^2/c^2. \tag{10}$$

3. We have solved Eq. (6) numerically. This equation depends on one parameter α, which is the value of the magnetization parameter $\omega_B\tau$ at $B = B_0$. Calculations were carried out for four values of α: 5, 25, 50, and 200. An upper limit was imposed on the value of the parameter α by the computer facilities available (as α increases, a progressively finer mesh is required to prevent numerical instability during the calculations). At $\alpha = 200$, an instability nevertheless did occur in one stage of the calculation. It was suppressed by artificial smoothing at a certain time.

In the calculations, Eq. (6) was approximated by a standard explicit finite-difference scheme (Ref. 3, for example). The dimensionless timestep ξ was chosen to satisfy the Courant condition.[3] The number of nodes along x and $y = \cos\theta$ for each value of α is shown in Table I, along with the step along ξ.

The initial field configuration was the same in each case considered:

$$B|_{t=0}=\frac{B_0}{\lambda^{1/2}x}\left[\frac{\sin(\lambda^{1/2}x)}{\lambda^{1/2}x}-\cos(\lambda^{1/2}x)\right]\sin\theta. \tag{11}$$

The dependence on x and θ in this configuration corresponds to the fundamental toroidal mode in an unmagnetized sphere [see (8)]. At $t = 0$ the field is symmetric with respect to the equatorial plane. The maximum field, $B_m = 0.436B_0$, is reached at the equator, at a distance $x = 0.463$ from the center of the sphere. We will use the constant B_0 in (11) as the normalizing value in Eq. (6). The initial condition was chosen as in (11), for convenience in comparison with the case $\alpha < 1$. In the case of a weak field, the magnetic configuration (11) decays with the time scale (9), while remaining qualitatively the same [see (8)].

TABLE I.

α	Number of nodes along x	Number of nodes along y	Step along t/t_m
25	11	81	$2{,}5\cdot 10^{-5}$
50	11	81	$2{,}5\cdot 10^{-5}$
200	21	101	$4{,}0\cdot 10^{-6}$

Figures 1–3 show the evolution of the magnetic configuration for the cases $\alpha = 25$, 50, and 200. Shown here are contour lines of the magnetic field. The times given in the figure captions are in units of the dissipative time t_m. We first note the nature of the drift of the magnetic field due to the Hall effect. The field diffuses out of the lower hemisphere into the upper one. This change in the field configuration becomes more pronounced, and occurs more rapidly, as the parameter α is increased. One result of this drift is that the point in the meridional plane corresponding to the maximum value of the field, B_m, moves away from the equatorial plane into the upper hemisphere, simultaneously approaching the polar axis.

Table II shows the coordinates (x_m, y_m) of the point where the field is a maximum and the time of the drift to this point when the field distortion is greatest (the initial values here are $x_m = 0.46$, $y_m = 0$).

Because of the field drift, a region with a field variation more pronounced than elsewhere in this sphere forms near the surface of the upper hemisphere. In other words, the current density is relatively high there. The current dissipation is relatively high in this region, so the damping of the magnetic field is more rapid than in the case $\alpha < 1$. The Hall drift is important only at comparatively large values of the parameter α. Corresponding calculations were carried out for $\alpha = 5$, but in this case the asymmetry of the field with respect to the equatorial plane resulting from the Hall currents was slight.

It is fairly easy to see the physical reasons for the drift of the field from one hemisphere into the other. The Hall current leads to the excitation of an electric field component $\mathbf{E}_\wedge$ in the medium. This component is perpendicular to both the current $\mathbf{j}$ and the magnetic field $\mathbf{B}$:

$$\mathbf{E}_\wedge=\Lambda[\mathbf{jB}],\quad \Lambda=R_\wedge/B=1/nce. \tag{12}$$

In the case of a toroidal magnetic field, the contour lines of the current lie in meridional planes, passing through the symmetry axis of the field: $j = (j_r, j_\theta, 0)$. We assume that the field is initially symmetric with respect to the equatorial plane. Let us examine the change in the magnetic flux over a short time interval Δt as a result of the Hall effect. From

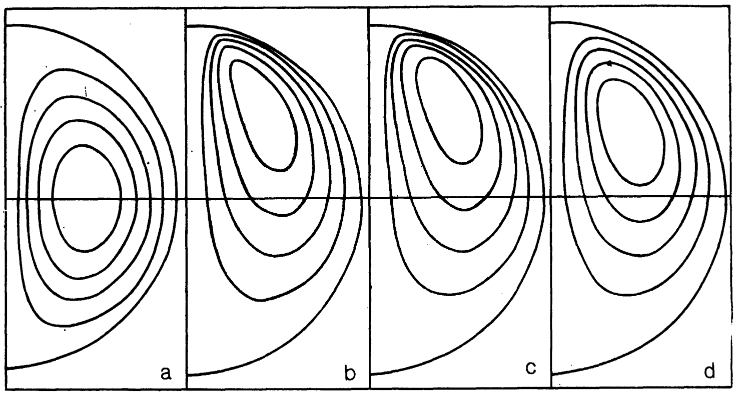

FIG. 1. Contours of constant magnetic field in the meridional plane for $\alpha = 25$ and at various times t/t_m: a—0.0; b—0.4; c—0.65; d—1.7.

Maxwell's equation $\partial \mathbf{B}/\partial t = -c\nabla \times \mathbf{E}$ we see that the Hall component (12) of the electric field leads to a change

$$\Delta \mathbf{B}_\wedge = -c\Delta t[\nabla \mathbf{E}_\wedge] \tag{13}$$

in the magnetic field over the time interval Δt. The change caused by $\Delta \mathbf{B}_\wedge$ in the magnetic flux through a contour in the meridional plane bounded by the symmetry axis of the field and a surface semicircle is zero:

$$\int \Delta \mathbf{B}_\wedge \, d\mathbf{s} = -c\Delta t \oint \mathbf{E}_\wedge \, d\mathbf{l} = 0. \tag{14}$$

Here $d\mathbf{s}$ is a surface element of the semicircle bounded by the contour, and $d\mathbf{l}$ is a length element of the contour (at the surface of this sphere we have $E_\wedge = 0$; on the symmetry axis we have $d\mathbf{l} \| \mathbf{j}$). Relation (14) is a consequence of the nondissipative nature of Hall currents. We find the following result for the change in the flux enclosed by a contour which lies in the upper hemisphere and which is bounded by the symmetry axis, by part of the surface circle, and by the intersection with the equatorial plane:

$$\int_{(+)} \Delta \mathbf{B}_\wedge \, d\mathbf{s} = -c\Delta t \oint \mathbf{E}_\wedge \, d\mathbf{l} = \Delta t \int_0^a u_\theta B \, dr. \tag{15}$$

Here $\mathbf{u}$ is the electron current velocity, given by $\mathbf{u} = \mathbf{j}/ne$. The change in the flux enclosed by the corresponding contour in the lower hemisphere is

$$\int_{(-)} \Delta \mathbf{B}_\wedge \, d\mathbf{s} = -\int_{(+)} \Delta \mathbf{B}_\wedge \, d\mathbf{s} = -\Delta t \int_0^a u_\theta B \, dr. \tag{16}$$

Since the current is continuous we have $\int_0^a u_\theta dr = 0$, but integrals of the type (15) and (16) may be nonzero since B depends on r. The drift of the field from one hemisphere into the other is therefore due simply to a drift of the frozen-in magnetic field as a result of the current motion of the electrons at the velocity $\mathbf{u}$. The drift direction is determined by the sign of the Hall resistance. It is this drift which causes the field to lose its initial symmetry with respect to the equator.

As time elapses, there is a progressive damping of the field due to ohmic dissipation. This decrease in the field leads to a decrease in the magnetization parameter $\omega_B \tau \sim \alpha b$ and to weakening of the Hall drift. After a certain time, the field weakens to the extent that the product αb becomes ~ 1. The drift of the field from one hemisphere into the other plays essentially no role here, and the field begins a slow approach to symmetry (Figs. 1–3). The reason is that the higher harmonics of the field which appeared in an earlier stage (as a result of the drift) decay more rapidly than the fundamental mode. The evolution of the various harmonics in this stage is now determined by Joule dissipation alone. As a result, the fundamental mode of the toroidal field dominates again after a certain time, and the field becomes symmetric with respect to the equatorial plane.

Figure 4 shows the time evolution of the maximum field

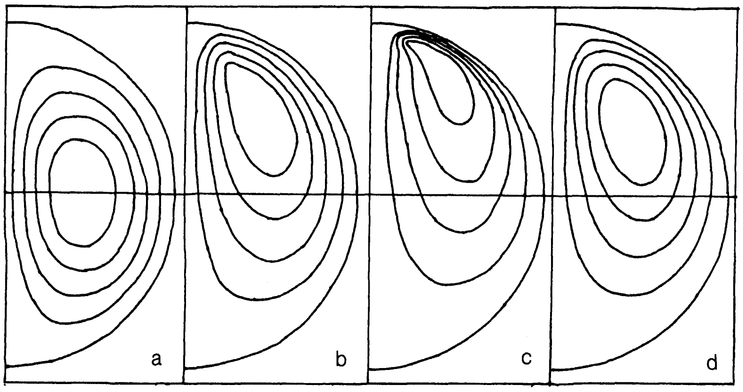

FIG. 2. The same as in Fig. 1, but for $\alpha = 50$ and the following times t/t_m: a—0.0; b—0.14; c—0.25; d—1.9.

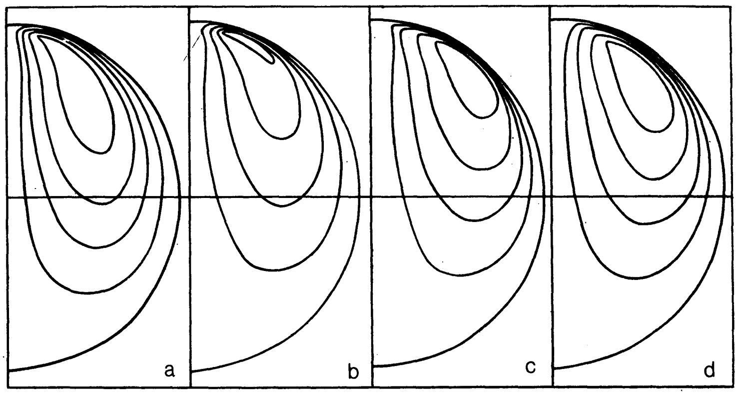

FIG. 3. The same as in Fig. 1, but for $\alpha = 200$ and the following times t/t_m: a—0.038; b—0.05; c—0.25; d—0.5.

in the sphere, B_m. The field reaches its maximum value in the upper hemisphere, as we have already mentioned. At comparatively large values of α, the change in the maximum field is nonmonotonic. In the initial stage, B_m may in fact increase. This increase results from the drift of the field out of the lower hemisphere into the upper one under the influence of the Hall current. Later on (Fig. 4), the field dissipation is more rapid, the larger the parameter α. The reason lies in the formation of pronounced field irregularities in the upper hemisphere and enhancement of the current there. Under the condition $\alpha \gg 1$, the field decrease in this stage is no longer described by a simple exponential law with a time scale t_m [see (9)].

The total magnetic-field energy inside the sphere, $E = 1/8\pi \int B^2 dV$, decays monotonically, of course (Fig. 5). With increasing α, this decay of the field accelerates significantly. As in the B_m case, this acceleration stems from the appearance of a high electric current near the surface of the upper hemisphere as a result of the Hall effect. The time scale $t_m(\alpha)$ over which the field energy decreases by a factor of e from its initial value decreases by a factor of about 5 as the parameter α is raised from 0 to 200. Over this range of α the $t_m(\alpha)$ dependence is described approximately by

$$t_m(\alpha) \approx t_m/(1+0{,}0044\alpha^{5/4}). \qquad (17)$$

The error in the expression (17) is less than 10%. The curves in Fig. 5 cannot be described by a simple exponential law with a time scale $t_m(\alpha)$. The field energy falls off comparatively rapidly only in the initial stage, in which the field is capable of strongly magnetizing the electron gas. After a long time, when the Hall drift has become unimportant, and the higher harmonics of the field decay, the fundamental toroidal mode becomes predominant (this mode is symmetric with respect to the equator). This mode decays exponentially with a time $t_m > t_m(\alpha)$.

TABLE II.

α	x_m	y_m	t/t_m
25	0,7	0,775	0,31
50	0,9	0,925	0,23
200	0,95	0,94	0,045

4. Let us summarize the basic results of this study. This analysis has shown that under certain conditions the Hall drift can cause a striking acceleration of the decay of the magnetic field in a conducting medium. The reason is that the drift causes a substantial change in the configuration of the electric current. Regions with pronounced irregularities of the magnetic field and correspondingly high current densities can form. In our example, these regions form near the surface in the upper hemisphere. Because of the high current density here, there are an accelerated dissipation and thus an effective heat evolution. The accelerated dissipation in certain regions of the conducting object leads to a faster overall dissipation of the field. The time scale of the decrease in the field energy itself decreases with increasing value of the magnetization parameter α [see (17)]. In the course of the evolution, as the total energy of the magnetic field falls off monotonically, the maximum field B_m may increase noticeably during certain stages.

What we regard as a striking feature of the field evolution in this example is the temporary deviation from a mirror symmetry with respect to the equatorial plane. As we mentioned earlier, this asymmetry results from a drift of the

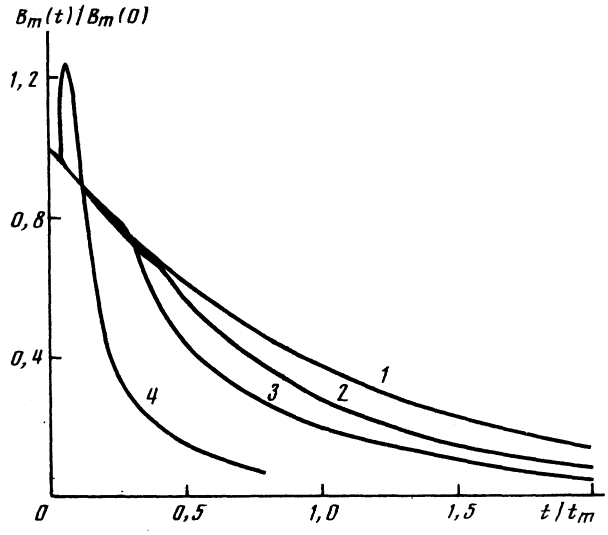

FIG. 4. Time evolution of the maximum field B_m, divided by the initial field $B_m(0)$. 1—$\alpha = 0$; 2—$\alpha = 25$; 3—$\alpha = 50$; 4—$\alpha = 200$.

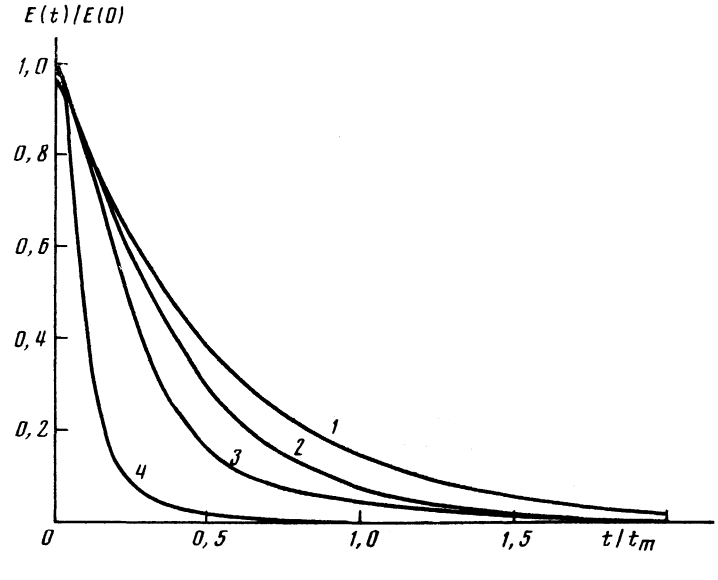

FIG. 5. Time evolution of the total energy E of the magnetic field inside the sphere, divided by its initial value $E(0)$. 1—$\alpha = 0$; 2—$\alpha = 25$; 3—$\alpha = 50$; 4—$\alpha = 200$.

magnetic field out of one hemisphere into the other caused by the Hall current. As the field dissipation proceeds, and the velocity of the Hall drift decreases, this drift becomes progressively weaker. After a long time, when the conducting sphere has become unmagnetized, the magnetic field can be written as the sum of normal modes, each of which is evolving independently. The higher harmonics are damped over a time scale shorter than that for the fundamental mode. As a result, the fundamental mode becomes the dominant one after a certain time. This fundamental mode is symmetric with respect to the equator.

In deriving Eq. (6) we used values found for $R_\perp$ and $R_\wedge$ in the relaxation-time approximation. It is simple to verify that the basic equation would have the same form even if we were to abandon that approximation, provided that $R_\perp$ is independent of B and that $R_\wedge$ is proportional to B. In this case we would have $\alpha = (R_\wedge / BR_\perp) B_0$ in Eq. (6).

Under laboratory conditions the Hall current can influence magnetic-field dissipation even at small values of B_0. In metals at sufficiently low temperatures, for example, the relaxation time is $\tau \sim 10^{-9}$ s. The condition $\alpha > 1$ thus holds even for $B_0 > 10^2$ G. As can be seen from our calculations, the effect of the Hall current on the dissipation becomes particularly strong for $\alpha \geqslant 50$, i.e., for $B_0 \geqslant 5 \cdot 10^3$ G. At such fields, both of the effects which we have been discussing here—the accelerated field decay and the breaking of the symmetry with respect to the central plane—may be manifested in laboratory experiments. This symmetry breaking may result in a difference in the heating of the upper and lower hemispheres. It is not difficult to show that the field dissipation has the consequence that the difference between the temperatures of the upper and lower hemispheres will be of order $\Delta T \sim B^2 / 8\pi\rho C_p$, where ρ is the density and C_p is the specific heat. Assuming $p \sim 10$ g/cm^3, $C_p \sim 10^6$ erg/(g·deg), and $B \sim 10^4$ G, we find $\Delta T \sim 1$ K. Note, however, that the local temperature difference between certain points which are symmetric with respect to the central plane may be considerably larger.

Hall currents can of course lead to accelerated dissipation of more than the toroidal field. The case of a toroidal field was analyzed here because of its simplicity. Magnetic configurations in which $\mathbf{j}$ and $\mathbf{B}$ are not parallel are fairly common. For example, we would have $\mathbf{j} \perp \mathbf{B}$ for a dipole configuration. Admittedly, in this case we would have $\mathbf{j} \| \mathbf{e}_\varphi$ and the breaking of the symmetry of the field with respect to the equatorial plane would not occur. As in the case of a toroidal field, however, the Hall drift would accelerate the field decay.

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