Magnetic-field penetration into a nonuniform Josephson junction

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The critical state of a linear, nonuniform Josephson junction with a periodic array of pinning centers is studied. Profiles of the magnetic field penetrating into the junction are constructed. The correlation phase at all points in the junction is dealt by simultaneous numerical solution of a chain of sine-Gordon equations with matching conditions at the pinning centers. When the distances between the defects are large, and when the pinning force is strong, the behavior found can be described approximately by the phenomenological Bean formula. In the opposite case, there are two possibilities: Either the front of the profile stretches out to an extreme extent, or a uniform distribution of vortices is established instantaneously in the junction if the external field exceeds a certain threshold.

If an external field above the lower critical field is applied to a hard type-II superconductor, a rapid relaxation is followed by the establishment of a uniform distribution of magnetic flux. This flux then undergoes a very slow relaxation to a uniform distribution which minimizes the free energy. The state which is initially established is called the "critical" state and is usually described by phenomenological theories.[1–4] Below we examine a corresponding situation for a nonuniform Josephson junction, without resorting to phenomenological approaches.

We have previously determined the critical profiles of the magnetic induction [and thus the dependence of the critical current on the magnetic field, $j_c(H)$] from the requirement that the drop in the density of vortices at each pinning center be the maximum which can be maintained by the given center.[5,6] We ignored the correlation among the phases at different pinning centers, under the assumption that, say, the distance between irregularities was large. The actual critical profiles for the final state between pinning centers should therefore be less steep than those found in Refs. 5 and 6. Below we report numerical calculations of corresponding extreme profiles of the vortex number density $n(x)$ [or $B(x) \equiv \overline{H}(x)$] in a linear, nonuniform junction, with allowance for the correlation among the phases at different pinning centers. We essentially carried out a numerical solution of a chain of sine-Gordon equations coupled by matching conditions at the nonuniformities, and we then selected the extreme solution.

We consider a linear Josephson junction which is intersected at uniform intervals L by Josephson junctions of finite length $2l$ (Fig. 1)[7,8] (in principle, we could choose other nonuniformities[9]). Between intersections the system is described by the sine-Gordon equation

$$\delta^2 \partial^2\theta/\partial x^2 = \sin\theta,$$

where δ is the Josephson length. On each side of each notch, the derivatives of the phases and their jumps are related by

$$\partial\theta_m^{(3)}/\partial x = \partial\theta_m^{(1)}/\partial x \equiv \theta_m', \qquad \theta_m^{(1)} - \theta_m^{(3)} = 2l\theta_m'. \tag{1}$$

We assume that an external field is applied from the right. We seek the extreme profile which corresponds to a metastable state with the highest value of the thermodynamic potential, working in the following way. We fix the position of the profile's front: $n(x<0) = 0$, where $x = 0$ corresponds to node $m = 0$. If $L \gg \delta$, we can assume that a solution which decays with distance into the sample is

$$\theta(x) = 4\,\mathrm{arctg}\exp[(x - x_0)/\delta]$$

for $-L < x < 0$. The value x_0 determines the phase $\theta_0^{(3)} \equiv \theta(x \to -0)$ and its derivative; it therefore determines the complete form of the solution on the right: $m = x = 0$. This may be an extremely complex deterministic-chaotic solution. We vary x_0 to seek a solution for which the magnetic field H_N and the vortex density n_N reach maxima on the surface, i.e., at a certain given notch index N. The steepest front is realized in this case. In this case, the minimum number of vortices penetrates from the surface into the junction at a fixed value of the external field.

For each specific x_0 we find $\{\theta_0^{(3)}, \theta_0'\}$. Using (1) we find $\{\theta_0^{(1)}, \theta_0'\}$. Using these results and the known expressions for the solution of the sine-Gordon equation,[10] we find $\{\theta_1^{(3)}, \theta_1'\}$, etc. As a result, $n_N[x_0]$, the penetrating flux, and the corresponding free energy are all extremely nonsmooth functions of the parameter x_0. Small changes in x_0 may result in extremely large changes in the values of these functions (compare curves 1 and 2 in Fig. 2, which are two solutions corresponding to approximately equal values of x_0). For this reason, it is essentially impossible to accurately determine the upper local minimum of the free energy by simply trying several values of x_0. On the other hand, by going through this trial-and-error procedure a sufficient

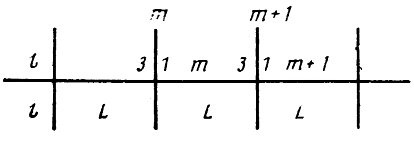

FIG. 1. Linear Josephson junction (the *xz* plane) with a periodic array of irregularities. L is the period of the structure, and $2l$ is the length of the transverse Josephson junctions, which are parallel to the *yz* plane.

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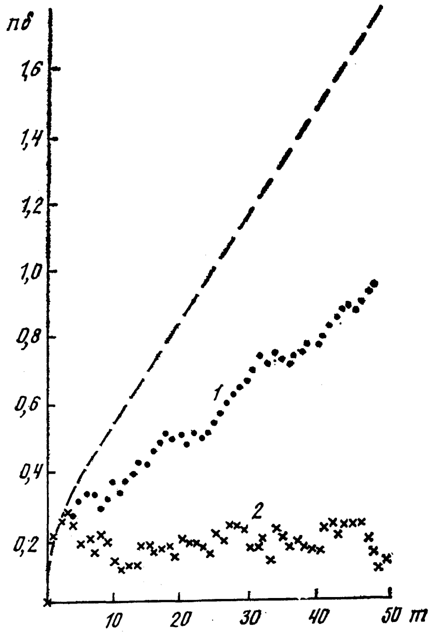

FIG. 2. Profile of the vortex number density $n\delta = 2\overline{H}/\pi^2 H_{c1}$ in a junction with $L/\delta = 100$ and $l/\delta = 0.1$. The external field is applied from the right. 1—Extreme profile found in a calculation with an initial condition $x_0/\delta = -0.78250438$; 2—representative profile found for approximately the same value, $x_0/\delta = -0.7835$; dashed line—profile found without consideration of phase correlations.[5,6] Here $\overline{H}_m$ is the average field over various regions of the junction. The profiles of the magnetic field at the nodes, H_m, are similar.

number of times, we can approach the extreme profile which we need.

We adopt the following specific values of the parameters: $L/\delta = 100$, $l/\delta = 0.1$, and $N = 50$. Figure 2 shows a profile found by the procedure outlined above after ~ 15000 trials. Corresponding to this profile is the value $x_0/\delta = -0.78250438$. In other words, the last vortex is forced out to the left of pinning center $m = 0$ almost completely. Also shown in this figure is a profile calculated by ignoring the correlations among the phases at the various pinning centers:[5,6]

$$\overline{H} = \frac{\pi}{2} H_{c1} \frac{l}{\delta} \frac{x - y_0}{L}, \tag{2}$$

where H_{c1} is the critical field of a linear Josephson junction. Expression (2) was derived for the region

$$2\Phi_0/\pi^2 \delta d = H_{c1} \ll \overline{H} \ll H_s = \Phi_0 2/ld,$$

i.e., for $2/\pi^2 \ll n\delta \ll \delta/2l$, where $d = 2\lambda_L + d'$ is the effective thickness of the junction, ϕ_0 is the flux quantum, and λ_L is the London penetration depth.

It follows from Fig. 2 that in this region the profile constructed with allowance for phase correlation is approximately linear. In other words, the field penetration is of the Bean type.[1,3] The correlation among the phases at the various pinning centers should be ignored during the solution of this problem only if there are very large jumps in the number of vortices at each center:[5,6] $\Delta n_m L \sim Ll/\delta^2 \gg 1$, where $\Delta n_m = n_m - n_{m-1}$, and n_m is the number density of vortices in gap m (Fig. 1). Consequently, as L increases, the extreme profile which we find should become a progressively better approximation of the profile shown by the dashed line in Fig. 2, remaining under the latter at any finite L. It should also be noted that for a finite number of trials the profile of n_m which is found will be slightly smoother than that for an infinite number, and the actual critical profile will lie between curve 1 and the dashed line in Fig. 2.

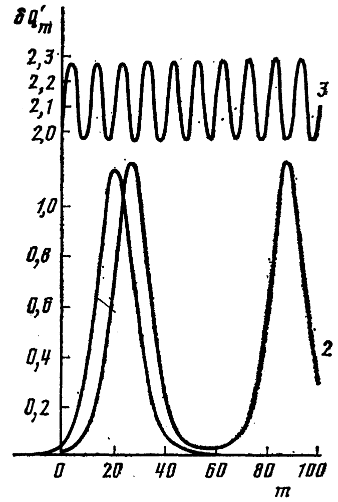

FIG. 3. Values of the field H_m at the nodes of a dense Josephson structure $(L/\delta = l/\delta = 0.1)$ for various initial conditions at the point $m = 0$. 1—Solitary solution; 2,3—examples of essentially periodic solutions.

At small values of L and l, on the other hand, we were not able to find any significant deviation of the solutions from periodic solutions or solitary solutions, regardless of the initial conditions which we specified. This assertion is correct at least over distances of $100L$ (some representative solutions of this sort are shown in Fig. 3 for $L/\delta = l/\delta = 0.1$). For a junction with a periodic irregularity in this regime, in contrast with the first which we considered, we thus observed no traces of "deterministic chaotic behavior" or solutions of the type shown in Fig. 2. What can we conclude from such observations at small values of L/δ and l/δ? The numerical procedure cannot rule out the possibility that stochastic behavior will nevertheless set in far from $x = 0$. We can point out two possibilities, but we cannot choose between them by means of our approach.

1. This observation remains valid for any scale greater than that studied ($100L$). Actually, the only solutions which are possible are either periodic or solitary. The meaning here is that at small values of L/δ and l/δ magnetic vortices will immediately penetrate the entire sample once the external magnetic field exceeds the effective lower critical field. There will be no critical state at all. The region in which the transition is made to this behavior from the behavior characteristic of the first regime is $\Delta n_m L \sim Ll/\delta^2 \sim 1$. This parameter also appears in the Frenkel'–Kontorova model:

$$E = \left(\frac{1}{16} \frac{\delta}{L} E_J \right) \sum_m \left[\frac{(\theta_{m+1} - \theta_m)^2}{2} + \left(\frac{2Ll}{\delta^2} \right) (1 - \cos\theta_m) \right]. \tag{3}$$

In the case $L \ll l \ll \delta$, that model is equivalent to the system under consideration here[1)] (Refs. 7 and 8; E_J is the Josephson energy). However, we know[11,12] that specifically this parameter specifies the thresholds for the depinning of incommensurable structures in the Frenkel'–Kontorova model.

2. In this regime, the variations in the soliton number density occur over a large scale, so they cannot be observed (because the front of the magnetic-induction profile is stretched out to an extreme degree). To estimate the length scales over which we would expect variations in the density of vortices in the critical state, let us estimate how closely two extreme vortices which have penetrated into the interior of the junction (i.e., at the boundary of the front) can approach each other. This distance of closest approach can be found from the balance struck by the repulsive force between two vortices with the minimum pinning force at the irregularities of the junction, since the last vortex has a single neighbor. (We are assuming that the distance between vortices is large.) We can therefore estimate the possible jump in the number density of vortices from $n = 0$. We assume $L \ll l \ll \delta$. We describe the system by the Frenkel'–Kontorova model, (3). The energy of an individual vortex in a periodic structure with $2Ll/\delta^2 \ll 1$ depends on the coordinate of the center of this vortex in the following way:[13]

$$E_v(x) \Big/ \left(\frac{1}{16}\frac{\delta}{L}E_J\right) = 8\left(\frac{2Ll}{\delta^2}\right)^{1/2} + 16\pi^2 \exp\{-\pi^2/[2Ll/\delta^2]^{1/2}\}\cos\left(2\pi\frac{x}{L}\right). \tag{4}$$

The energy of the interaction between two solitons separated by a distance $\mathcal{L} \gg \delta(L/2l)^{1/2}$, i.e., separated by a distance greater than their width, is given by[14]

$$E_i(\mathcal{L}) \Big/ \left(\frac{1}{16}\frac{\delta}{L}E_J\right) = 32\left(\frac{2Ll}{\delta^2}\right)^{1/2} \exp\left[-\left(\frac{2Ll}{\delta^2}\right)^{1/2}\frac{\mathcal{L}}{L}\right]. \tag{5}$$

The distance which we are seeking is found by equating the intervortex repulsive force $\partial E_i/\partial\mathcal{L}$ to the maximum force pinning a vortex in the lattice, found from (4):

$$\frac{32}{L}\frac{2Ll}{\delta^2}\left(\frac{1}{16}\frac{\delta}{L}E_J\right)\exp\left[-\left(\frac{2Ll}{\delta^2}\right)^{1/2}\frac{\mathcal{L}}{L}\right] = \frac{32\pi^3}{L}\left(\frac{1}{16}\frac{\delta}{L}E_J\right)\exp\left[-\frac{\pi^2}{(2Ll/\delta^2)^{1/2}}\right]. \tag{6}$$

As a result we find

$$\frac{\mathcal{L}}{L} \sim \pi^2\frac{\delta^2}{2Ll} \gg 1.$$

In particular, for the parameter values used above, $L/\delta = l/\delta = 0.1$, we find $\mathcal{L}/[\delta(L/2l)^{1/2}] \sim 10^2$ and $\mathcal{L}/L \sim 10^3$. These values go beyond the capabilities of our numerical calculations (Fig. 3).

The second version looks preferable, since in models of the Frenkel'-Kontorova type only certain of the quasiperiodic uniform states are depinned.

[1)] Strictly speaking, only the case $L \ll \delta \ll l$ was studied in Refs. 7 and 8. However, the equivalence of the two models for the situation under consideration here can be proved by a method completely analogous to that used in Ref. 8.

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