Semiclassical theory of electromagnetic processes in a plane wave and a constant field

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General expressions for the probability of radiation by a relativistic particle are derived in a semiclassical theory for the case in which the particle is moving in an external field which is a superposition of a locally constant field and a monochromatic plane wave. The wave propagation direction is arbitrary. The Compton effect and the two-photon production of a pair of particles in an external field are discussed as applications.

1. INTRODUCTION

Processes which occur in electromagnetic fields of complex configuration have attracted considerable interest. One such field is a superposition of a plane wave and a constant field. Radiative processes are analyzed through the use of the solutions (exact or semiclassical) of the wave equations in such a field. One should bear in mind that exact solutions are available only in the case in which a plane wave is propagating along a magnetic field. Solutions of the Klein-Gordon and Dirac equations for this case were derived some time ago by Redmond,[1] who also analyzed the classical motion of the particle. The Green's functions of scalar and spinor particles in a field of this sort were found by Batalin and Fradkin[2] by a functional-integration method. The mass operator in a field with a Redmond configuration was derived by Milstein and one of the present authors for scalar[3] and spinor[4] particles by an operator approach. Below we use the corresponding wave functions to calculate the matrix elements of a mass operator which contain much physical information (radiation probabilities and level shifts). In particular, we discuss radiation effects which occur near cyclotron resonance and the scattering of a photon by an electron (the Compton effect) in a magnetic field. Several other studies in this area are reviewed in Ref. 5.

The present paper is based on the semiclassical theory of radiation and of pair production which has been derived by one of the present authors.[6] In the region of semiclassical motion there is no need to work from a solution of the Dirac (or Klein-Gordon) equations; it is sufficient to know simply the classical trajectories of the particles. We will be discussing the strongly relativistic case, $\gamma = \varepsilon/m \gg 1$, in which the velocity of a particle varies only slightly over the formation time of the process. In covariant notation this condition is

$$\left| \frac{(F_{\mu\nu}p^\nu)^2}{m^2} \right| \gg |F_{\mu\nu}F^{\mu\nu}|,\ |F_{\mu\nu}{}^*F^{\mu\nu}|,$$

where $F^{\mu\nu}$ is the electromagnetic field tensor of the field in which the particle is moving ($F^{\mu\nu} = F^{\mu\nu}_{\rm ex} + f^{\mu\nu}$, where $F^{\mu\nu}_{\rm ex}$ is the external field, and $f^{\mu\nu}$ is the field of the wave), $F^*_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, and p^ν is the momentum of the particle. If we also require that the variation in the external field $F_{\rm ex}$ over the formation length be small, the problem reduces to one of processes in a uniform external field $F_{\rm ex}$ and the field of a plane wave f which is propagating in an arbitrary direction. For definiteness, we assume that this wave is monochromatic. The expressions derived in this paper are exact in the case

$$F^2 = F^*F = 0.$$

In this case, the electromagnetic field is a nonmonochromatic plane wave, in which the problem can be solved exactly. In Sec. 2 we derive some general equations for the radiation probability per unit time. In Sec. 3 we analyze the Compton effect in an external field. In Sec. 4 we discuss the production of an electron-positron pair. The expressions derived here are "snapshots" of the processes. When the integral properties of these processes are to be determined, these snapshots should be averaged over the trajectory of the particle or with the corresponding distribution function.

2. GENERAL EXPRESSIONS

The calculations can be carried out best on the basis of the semiclassical theory of radiation and pair production.[6] The radiation probability in this theory is ($\hbar = c = 1,\ e^2 = \alpha = 1/137$)

$$dw_\gamma = \frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}|M|^2, \qquad M = \int dt R(t)\exp[ik'x(t)], \qquad (1)$$

where $k' = \varepsilon k/\varepsilon'$, $k = k(\omega,\mathbf{k})$ is the 4-momentum of the photon, ε and m are the energy and mass of the particle, $\varepsilon' = \varepsilon - \omega$, $x(t) = (t,\mathbf{r}(t))$, t is the time, and $\mathbf{r}(t)$ is the coordinate along the classical trajectory of the particle. For spinor particles we can write, to within relativistic accuracy ($1/\gamma = m/\varepsilon \ll 1$),

$$R = \varphi_f^+(A + i\boldsymbol{\sigma}\mathbf{B})\varphi_i, \qquad A = \frac{1}{2}\left(1 + \frac{\varepsilon}{\varepsilon'}\right)\mathbf{e}^*\mathbf{a},$$

$$\mathbf{B} = \frac{\omega}{2\varepsilon'}[\mathbf{e}^*\mathbf{b}], \qquad \mathbf{a} = \mathbf{v} - \frac{\mathbf{k}}{\omega}, \qquad \mathbf{b} = \frac{\mathbf{k}}{\omega\gamma} - \mathbf{a}, \qquad (2)$$

where $\mathbf{v} = \mathbf{v}(t)$ is the classical velocity of the particle, $\varphi_{f(i)}$ is a two-component spinor which describes the polarization of the particle, and $\mathbf{e}$ is the polarization vector of the photon.

After integration over photon emission angle and summation over the polarizations of the final particles, expressions (1) and (2) can be written in a form convenient for calculations, in which all cancellations of the leading terms have already been carried out (see Ref. 7, for example, for the case $\zeta = 0$):

$$\frac{d\omega_\gamma}{d\omega}=\frac{i\alpha}{8\pi\gamma^2}\int\int\frac{dt\,d\tau}{\tau-i0}\left[4+\left(\frac{\varepsilon}{\varepsilon'}+\frac{\varepsilon'}{\varepsilon}\right)(\mathbf{\Delta}_1-\mathbf{\Delta}_2)^2+\frac{i\omega}{\varepsilon}\boldsymbol{\zeta}\mathbf{N}\right]$$

$$\times\exp\left\{-\frac{i\tau}{l_\omega}\left(1+\frac{1}{\tau}\int_{t_1}^{t_2}\mathbf{\Delta}^2(t')dt'\right)\right\},\qquad(3)$$

$$\mathbf{\Delta}(t')=\frac{1}{m}(\mathbf{p}(t')-\boldsymbol{\pi}),\qquad \boldsymbol{\pi}=\frac{1}{\tau}\int_{t_1}^{t_2}\mathbf{p}(t)dt,\qquad l_\omega=\frac{2\varepsilon\varepsilon'}{m^2\omega},$$

$$\mathbf{N}=2\left(2+\frac{\omega}{\varepsilon'}\right)[\mathbf{\Delta}_2\mathbf{\Delta}_1]+2[(\mathbf{\Delta}_1-\mathbf{\Delta}_2)\mathbf{v}(t)],$$

where $t_{2,1}=t\pm\tau/2$, and $\mathbf{p}(t)=\varepsilon\mathbf{v}(t)$ is the momentum of the particle. Obviously, the quantity $\Delta(t)$ is unaffected by the substitution $\mathbf{p}(t)\to\mathbf{p}(t)+\mathbf{p}_0$, where $\mathbf{p}_0$ is the time-independent momentum. To pursue the analysis we need explicit expressions for the momentum $\mathbf{p}(t)$ and for the vector $\Delta(t)$ as functions of the time in a field with the configuration under consideration here. We will carry out specific calculations in the frame of reference in which the monochromatic plane wave, with a wave vector $q=q(q_0,\mathbf{q})$, is propagating in the direction $\mathbf{n}=\mathbf{q}/q_0$, opposite the electron. One can always find a relativistic frame of reference ($\gamma\gg1$) in which the condition $q_0\ll\varepsilon$ holds. This is a necessary condition if we wish to treat the plane wave as classical. Solving the equations of motion of the particle in the electromagnetic field, we find, to within the prescribed accuracy,

$$\mathbf{p}_\perp(t)/m=\boldsymbol{\Omega}t+\boldsymbol{\xi}(t),\qquad \boldsymbol{\xi}(t)=\boldsymbol{\xi}_2\sin(\nu t+\varphi_0)+\boldsymbol{\xi}_1\cos(\nu t+\varphi_0),$$

$$\boldsymbol{\Omega}=\frac{e}{m}\mathbf{F}_\perp,\qquad \mathbf{F}_\perp=\mathbf{E}-\mathbf{n}(\mathbf{nE})+[\mathbf{Hn}],\qquad \nu=2q_0,$$

$$\boldsymbol{\xi}\mathbf{n}=0.\qquad(4)$$

Here **E** and **H** are the electric and magnetic fields, both independent of the time, and the orthogonal vectors $\xi_{1,2}$ characterize the intensity $\xi_0^2=(\xi_1^2+\xi_2^2)/2$ and polarization of the wave. The corresponding Stokes parameters are

$$\lambda_3=\frac{\xi_1^2-\xi_2^2}{\xi_1^2+\xi_2^2},\qquad \lambda_2=\frac{[\boldsymbol{\xi}_1\boldsymbol{\xi}_2]\mathbf{n}}{\xi_0^2}.\qquad(5)$$

Evaluating the corresponding combinations which appear in the expression for the radiation probability (3), we find

$$\mathbf{\Delta}_2-\mathbf{\Delta}_1=\boldsymbol{\Omega}\tau+2\sin\frac{\nu\tau}{2}\boldsymbol{\eta}(t),$$

$$\boldsymbol{\eta}(t)=\boldsymbol{\xi}_2\cos(\nu t+\varphi_0)-\boldsymbol{\xi}_1\sin(\nu t+\varphi_0),$$

$$\int_{t_1}^{t_2}\Delta^2(t')dt'=\Omega^2\frac{\tau^3}{12}+\frac{4}{\nu^2}\boldsymbol{\Omega\eta}(t)\left(\sin\frac{\nu\tau}{2}-\frac{\nu\tau}{2}\cos\frac{\nu\tau}{2}\right)$$

$$+\xi_0^2\left[\tau+\frac{2}{\nu^2\tau}(\cos\nu\tau-1)+\frac{\lambda_3}{\nu}\cos2(\nu t+\varphi_0)\right.$$

$$\left.\times\left(\sin\nu\tau+\frac{2}{\nu\tau}(\cos\nu\tau-1)\right)\right],$$

$$(\mathbf{\Delta}_2-\mathbf{\Delta}_1)^2=\Omega^2\tau^2+4\boldsymbol{\Omega\eta}(t)\tau\sin\frac{\nu\tau}{2}$$

$$+4\xi_0^2(1-\lambda_3\cos2(\nu t+\varphi_0))\sin^2\frac{\nu\tau}{2},$$

$$[\mathbf{\Delta}_2\mathbf{\Delta}_1]=\left(\cos\frac{\nu\tau}{2}-\frac{2}{\nu\tau}\sin\frac{\nu\tau}{2}\right)\left\{\tau[\boldsymbol{\Omega\xi}(t)]-2\lambda_2\mathbf{n}\xi_0^2\sin\frac{\nu\tau}{2}\right\}.\qquad(6)$$

We make the replacements

$$\nu t+\varphi_0=\varphi,\qquad \tau\to l_0\tau,\qquad l_0=\frac{2\varepsilon}{m^2}=2\gamma\lambda\!\!\!^{-}_c$$

and we assume

$$\Omega l_0\approx\frac{2e\mathbf{F}_\perp\varepsilon}{m^3}=2\boldsymbol{\chi},\qquad \nu l_0=\frac{4q_0\varepsilon}{m^2}\approx\frac{2qp}{m^2}\equiv s.\qquad(7)$$

As a result, we find the following expression for the photon emission probability per unit time:

$$\frac{d^2w_\gamma(t)}{dt\,dx}=\frac{dW_\gamma(t)}{dx}=\frac{i\alpha m^2}{2\pi\varepsilon}\int_{-\infty}^{\infty}\frac{d\tau}{\tau-i0}\left[1+\left(\frac{\varepsilon'}{\varepsilon}+\frac{\varepsilon}{\varepsilon'}\right)m_0\right.$$

$$\left.+ix\boldsymbol{\zeta}\mathbf{m}\right]\exp[-iu\tau\Phi(\varphi,\tau)],$$

$$u=\frac{\omega}{\varepsilon'},\qquad x=\frac{\omega}{\varepsilon},$$

$$m_0=\chi^2\tau^2+2\boldsymbol{\chi\eta}(\varphi)\tau\sin\frac{s\tau}{2}+\xi_0^2(1-\lambda_3\cos2\varphi)\sin^2\frac{s\tau}{2},$$

$$\qquad(8)$$

$$\mathbf{m}=2\left(1+\frac{u}{2}\right)\left\{[\boldsymbol{\chi\xi}(\varphi)]\left(\tau\cos\frac{s\tau}{2}-\frac{2}{s}\sin\frac{s\tau}{2}\right)+\frac{1}{2}\lambda_2\xi_0^2\mathbf{v}\right.$$

$$\left.\times\left(\sin s\tau+\frac{2}{s\tau}(\cos s\tau-1)\right)\right\}-[\boldsymbol{\chi}\mathbf{v}]\tau-[\boldsymbol{\eta}(\varphi)\mathbf{v}]\sin\frac{s\tau}{2},$$

$$\Phi=\frac{1}{3}\chi^2\tau^2+\frac{8}{s^2\tau}\boldsymbol{\chi\eta}(\varphi)\left(\sin\frac{s\tau}{2}-\frac{s\tau}{2}\cos\frac{s\tau}{2}\right)$$

$$+\xi_0^2\left[1+\frac{2}{s^2\tau^2}(\cos s\tau-1)+\frac{\lambda_3}{s\tau}\cos2\varphi\right.$$

$$\left.\times\left(\sin s\tau+\frac{2}{s\tau}(\cos s\tau-1)\right)\right]+1.$$

The quantities $\mathbf{v}(t)$, $\boldsymbol{\chi}(t)$, and $\zeta(t)$ in this expression are determined by the classical trajectory of the particle. If the wave period is smaller than or on the order of the formation time of the process [which is determined by the characteristic values of the variable τ in the integral in (8)], this expression must be averaged over the phase of the wave, φ.

To write expression (8) in invariant form, we introduce 4-vectors which characterize the wave:

$$a^\mu(\varphi)=a_2^\mu\sin\varphi+a_1^\mu\cos\varphi,$$

$$b^\mu(\varphi)=\frac{da^\mu(\varphi)}{d\varphi},\qquad \varphi=qx,$$

$$q^2=qa_1=qa_2=a_1a_2=0,\qquad f^{\mu\nu}=q^\mu b^\nu-q^\nu b^\mu,\qquad(9)$$

$$\xi_{1,2}^2=-e^2a_{1,2}^2/m^2.$$

Here $a^\mu(\varphi)$ is the vector potential, and $f^{\mu\nu}$ is the electromagnetic field tensor of the wave. To relativistic accuracy, the combinations which appear in (8) are

$$\chi^2\approx-\frac{e^2}{m^6}(F^{\mu\nu}p_\nu)^2,\qquad \boldsymbol{\chi\eta}\approx-\frac{e^2}{m^4(qp)}F^{\mu\nu}p_\nu f_{\mu\rho}p^\rho,\qquad(10)$$

$$(\boldsymbol{\zeta\chi}\mathbf{v})\approx\frac{e}{m^3}s^\mu F_{\mu\nu}^*p^\nu,\qquad (\boldsymbol{\zeta\eta}\mathbf{v})\approx\frac{e}{m(pq)}s^\mu f_{\mu\nu}^*p^\nu,$$

$$(\boldsymbol{\zeta\chi\xi})\approx\frac{e^2}{m^3}s^\mu F_{\mu\nu}^*a^\nu,$$

$$-\lambda_2\xi_0^2(\boldsymbol{\zeta}\mathbf{n})=\frac{e^2}{m(pq)}\varepsilon_{\mu\nu\rho\sigma}s^\mu a_1^\nu a_2^\rho q^\sigma,$$

where s^μ is the spin 4-vector, $F^{\mu\nu}$ is the tensor of the static electromagnetic field, and the asterisk (*) denotes a dual tensor.

Integrating (8) over $x=u/(1+u)$, we find the total probability for radiation per unit time, W_γ. This probability is related in a known way, by a dispersion relation, to the correction to the mass of the particle.[8] Using that relation, we find

$$\frac{\Delta m}{m}=\frac{\alpha}{2\pi}\int_0^1 dx\int_0^\infty\frac{d\tau}{\tau}\left\{\left[1+\left(\frac{\varepsilon}{\varepsilon'}+\frac{\varepsilon'}{\varepsilon}\right)m_0+ix(\boldsymbol{\zeta}\mathbf{m})\right]\right.$$
$$\times\exp[-iu\tau\Phi(\tau)]-\exp(-iu\tau)\ . \qquad (11)$$

In our approximation, the part of this expression which depends on the electron spin, $\Delta m_\zeta(t)=\boldsymbol{\zeta}(t)\mathbf{M}(t)$, determines the radiation corrections to the equation of motion of the spin:

$$\frac{d\boldsymbol{\zeta}_{rad}}{dt}=2\frac{[\mathbf{M}\boldsymbol{\zeta}]}{\gamma}.$$

If the plane wave is circularly polarized ($\lambda_3=0$, $\lambda_2=\pm1$), the expressions for $m_\mu(\varphi,\tau)$ and $\Phi(\varphi,\tau)$ become

$$m_0^{cr}=\chi^2\tau^2+2\chi\eta\tau\sin\frac{s\tau}{2}+\xi_0^2\sin^2\frac{s\tau}{2},$$
$$\boldsymbol{\zeta}\mathbf{m}^{cr}=2\left(1+\frac{u}{2}\right)\lambda_2(\boldsymbol{\zeta}\mathbf{v})\left[(\boldsymbol{\chi}\boldsymbol{\eta})\left(\tau\cos\frac{s\tau}{2}-\frac{2}{s}\sin\frac{s\tau}{2}\right)\right.$$
$$\left.+\frac{\xi_0^2}{2}\left(\sin s\tau+\frac{2}{s\tau}(\cos s\tau-1)\right)\right]-(\boldsymbol{\zeta}\boldsymbol{\chi}\mathbf{v})\tau-(\boldsymbol{\zeta}\boldsymbol{\eta}\mathbf{v})\sin\frac{s\tau}{2},$$
$$(12)$$
$$\Phi^{cr}=\frac{1}{3}\chi^2\tau^2+\frac{8}{s^2\tau}(\boldsymbol{\chi}\boldsymbol{\eta})\left(\sin\frac{s\tau}{2}-\frac{s\tau}{2}\cos\frac{s\tau}{2}\right)$$
$$+\xi_0^2\left(1+\frac{2}{s^2\tau^2}(\cos s\tau-1)\right)+1.$$

If we set the wave intensity equal to zero ($\xi_0^2=0$), Eqs. (8) and (11) become the expression for the radiation probability and the correction to the mass in a constant external field in the semiclassical approximation.[9] In the case $\chi=0$, these expressions are the same as the corresponding expressions in the field of a monochromatic plane wave.[10,11]

The parameter ξ_0 characterizes the change induced in the momentum of the particle by the wave field in comparison with the mass of the particle over a time on the order of the wave oscillation period. At large values $\xi_0\gg1$, the radiation is formed in a time much shorter than this period ($\tau\ll1/s$). In this case, we can carry out an expansion in the quantity $s\tau$ in Eqs. (8), (11), and (12). We find as a result that the process is determined by the local value of the field intensity with the corresponding value of the parameter χ_1:

$$\chi_1(t)=\chi(t)+\eta(\varphi)\frac{s}{2},\qquad \chi_1^2=-\frac{e^2}{m^6}[(F^{\mu\nu}+f^{\mu\nu})p_\nu]^2. \qquad (13)$$

The radiation is of the same nature in the case $\xi_0\lesssim1$, but at large values of the parameter $\mu=\chi/s$ ($\mu\gg1$). Under the condition $\mu\ll1$ ($\xi_0^2\lesssim1$), the momentum transferred by the static field over the wave oscillation period is much smaller than the mass of the particle ($\tau\sim1/s$, $\chi\tau\ll1$), and corresponding expansions must be carried out in $\chi\tau$. Retaining the leading terms of this expansion, we find the corrections to the probability for the process in a plane wave which result from the constant field.

3. COMPTON EFFECT IN A STATIC FIELD

In the case $\xi_0\ll1$, we can expand the exponential factor in expression (8) in the parameter ξ (η) and ultimately retain terms $\propto\xi_0^2$. The zeroth terms in the expansion in ξ_0 in Eq. (8) then give us the probability for radiation in a constant field $F_{\mu\nu}$, and the corrections $\propto\xi_0^2$ describe Compton scattering in this field. Let us examine this Compton scattering in more detail.

Making use of the relationship between the electric and magnetic fields in the wave, on the one hand, and the wave energy, on the other, we express the parameter ξ_0^2 in terms of the photon density and frequency:

$$\frac{E^2+H^2}{8\pi}=\frac{1}{4\pi}\left(\frac{m\xi_0}{e}\right)^2=q_0n_{ph},$$
$$\xi_0^2=\frac{4\pi\alpha n_{ph}}{m^2q_0}\approx\frac{16\pi\alpha n_{ph}\varepsilon}{m^4s}. \qquad (14)$$

We also note that the term proportional to ξ_0^2 in the radiation probability in (8),

$$dW_\gamma=dW_\gamma^{ex}+\xi_0^2dW_\gamma^{\xi}$$

is related to the cross section for the Compton effect by

$$\xi_0^2\,dW_\gamma^{\xi}=2n_{ph}\,d\sigma_c,\qquad d\sigma_c=\frac{8\pi\alpha\varepsilon}{m^4s}dW_\gamma^{\xi}. \qquad (15)$$

As a result we find the following expression for the cross section for Compton scattering in a constant external field:

$$\frac{d\sigma_c}{dx}=\frac{4\alpha^2}{m^2si}\int_{-\infty}^{\infty}\frac{d\tau}{\tau-i0}e^{-i\rho(\tau)}\left\{(1+\beta\chi^2\tau^2)F+\frac{4i}{s^3}u\beta\chi^2\Lambda_1zg_2(z)\right.$$
$$+\frac{\beta}{2}g_3(z)+2\left(1+\frac{u}{2}\right)\lambda_2(\boldsymbol{\zeta}\mathbf{v})x$$
$$\times\left[\frac{4u}{s^3}\chi^2g_1(z)+\frac{i}{z}g_2(z)\right]-x(\boldsymbol{\zeta}\boldsymbol{\chi}\mathbf{v})\left[i\frac{z}{s}F\right.$$
$$\left.\left.-\frac{2u}{s^2}\Lambda_1g_2(z)\right]+\frac{2u}{s^2}x(\boldsymbol{\zeta}\boldsymbol{\chi})\lambda_3g_2(z)\sin2\varphi_1\right\}, \qquad (16)$$

where

$$\rho(\tau)=u\tau\left(1+\chi^2\frac{\tau^2}{3}\right),\qquad \beta=\frac{\varepsilon}{\varepsilon'}+\frac{\varepsilon'}{\varepsilon},\qquad z=s\tau,$$
$$F=\frac{8u^2}{s^4}\chi^2\Lambda_1g_1(z)+\frac{2iu}{sz}g_3(z)+iu\frac{z}{s},$$
$$g_1(z)=1+\frac{z^2}{4}+\left(\frac{z^2}{4}-1\right)\cos z-z\sin z, \qquad (17)$$
$$g_2(z)=1-\cos z-\frac{z\sin z}{2},\qquad g_3(z)=\cos z-1,$$
$$\Lambda_1=1+\lambda_3\cos2\varphi_1,$$

and φ_1 is the angle between $\boldsymbol{\chi}$ and $\boldsymbol{\xi}_1$. Setting $\lambda_{2,3}=0$ in (16) and (17), we find the cross section for the Compton effect for the case of unpolarized initial photons. The integrand in (16) has no singularities as $\tau\to0$, so the integral has its usual meaning. In deriving the asymptotic behavior in the case of a weak field, however, it is convenient to use a contour representation of (16).

The overall effect of the external field on the Compton scattering is determined by the value of the parameter $\mu=\chi/s$. For $\mu\ll1$ the external field is unable to cause any substantial change in the motion of the charged particles over the time scales of the process, $T\sim 1/q_0$:

$$\frac{\Delta p_\perp}{m}\sim\frac{eF_\perp T}{m}\sim\frac{eF_\perp}{mq_0}\sim\frac{\chi}{s}\equiv\mu\ll 1. \tag{18}$$

Even if the condition $\mu\ll 1$ holds, near the maximum of the spectral distribution, at $u\approx s$, the effect of the external field becomes important when $\delta = 1-u/s$ is sufficiently small. The reason is that the value $u=s$ corresponds to Compton backscattering, and a photon with a frequency ω near the extreme value $\varepsilon s/(s+1)$ is formed over a time

$$\frac{T}{\delta}\sim\frac{1}{q_0\delta},\quad \left(\tau\sim\frac{1}{s\delta}\right).$$

The field-dependent term in the phase $\rho(\tau)$ in (17) can then be ignored only if

$$u\chi^2\tau^3\sim\frac{\chi^2}{s^2\delta^3}=\frac{\mu^2}{\delta^3}\ll 1. \tag{19}$$

The parameter $\mu\delta^{-3/2}$ thus serves as a measure of the effect of the external field on the spectrum of the Compton scattering near the limiting frequency. If the parameter μ is small ($\mu\ll 1$), but δ is also, so we have $\delta\lesssim\mu^{2/3}$, the leading term in (16) simplifies substantially:

$$\frac{d\sigma_c}{dx}=\frac{\alpha^2}{im^2s}\int_{-\infty}^{\infty}\frac{d\tau}{\tau-i0}\exp i[z-\rho(\tau)]\,[\beta-x(2+u)\lambda_2(\zeta v)]. \tag{20}$$

Using tabulated integrals,[12] we find

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\tau}{\tau-i0}\exp i[z-\rho(\tau)]$$

$$=\begin{cases}\dfrac{1}{3^{1/2}\pi}\displaystyle\int_a^\infty K_{1/3}(y)\,dy, & u\geqslant s,\\[2ex] 1-\dfrac{1}{3}\displaystyle\int_a^\infty (J_{1/3}(y)+J_{-1/3}(y))\,dy, & u\leqslant s,\end{cases}$$

$$a=\frac{2u}{3\chi}\left|\frac{s}{u}-1\right|^{3/2}\approx\frac{2s}{3\chi}\left|1-\frac{u}{s}\right|^{3/2}. \tag{21}$$

We see from the last expression that the field "smears out" the limiting frequency in the region $u>s$, replacing it by an exponential decay at $a\gg 1$. At $u=s$, the height of the spectral curve is a third of the maximum value in the absence of a field, and it is independent of the strength of the external field. In the region $u<s$, with $a\gg 1$, the field has only a slight effect on the Compton scattering. This conclusion agrees with the qualitative analysis above.

Since

$$\int_0^\infty dy\int_{2y^{3/2}/3\mu}^\infty dz\left[\frac{3^{1/2}}{\pi}K_{1/3}(z)-J_{1/3}(z)-J_{-1/3}(z)\right]=0,$$

the external field basically causes only a redistribution of the spectral probability for the process, leaving its overall value unchanged, in this frequency region. Correspondingly, in evaluating the corrections to the total cross section we can carry out an expansion in $\chi\tau$ in (16) at all values of x. We retain the first terms of the expansion in μ and $\chi\tau$ and we evaluate the integral over τ by means of the theory of residues. It is then an elementary matter to evaluate the integral over x. As a result we find the following expression for the cross section for the Compton effect with field corrections:

$$\begin{aligned}\sigma_c=\frac{\pi r_e^2}{s}\Big\{&2\left(1-\frac{4}{s}-\frac{8}{s^2}\right)L+1+\frac{16}{s}-\frac{1}{z^2}\\&+\lambda_2(\zeta v)\left[2\left(1+\frac{2}{s}\right)L\right.\\&\left.-5+\frac{2}{z}-\frac{1}{z^2}\right]+4\mu^2\left[\frac{16}{s^2}(L-s)-2+\frac{59}{3z}-\frac{59}{3z^2}+\frac{13}{z^3}\right.\\&-\frac{1}{z^4}-\frac{2}{z^5}+2\lambda_3\cos(2\varphi_1)\left(\frac{8}{s^2}(L-s)-2\frac{11}{z}-\frac{11}{z^2}+\frac{9}{z^3}-\frac{3}{z^4}\right)\\&+\lambda_2(\zeta v)\left(\frac{4}{s}(L-s)+4-\frac{7}{3z}-\frac{19}{3z^2}+\frac{11}{3z^3}\right.\\&\left.\left.+\frac{3}{z^4}-\frac{2}{z^5}\right)\right]+4(\zeta\chi v)\left[\frac{1}{z^2}\right.\\&\left.-\frac{1}{z^3}\right]-8\lambda_3[(\zeta\chi v)\cos(2\varphi_1)+(\zeta\chi)\sin(2\varphi_1)]\\&\times\left(\frac{1}{s^2}(L-s)+\frac{1}{z}-\frac{1}{z^2}+\frac{1}{2z^3}\right)\Big\}.\end{aligned} \tag{22}$$

Here $z=1+s$, $L=\ln(1+s)$, λ_3 and λ_2 are the degree of linear polarization and the degree of circular polarization, respectively, of the initial photon, and ζ is the polarization of the initial electron. With $\chi=0$, expression (22) becomes the cross section for the polarized Compton effect (see, for example, Refs. 10 and 11). The corrections to the Klein-Nishina formula for unpolarized photons were found in Ref. 13. They agree with (22) if we set $\lambda_2=\lambda_3=0$ in the latter.

In the other limiting case, in which the photon formation length in the constant field,

$$l_c=\frac{m}{eF_\perp}\left(1+\frac{\chi}{u}\right)^{1/3},$$

is much shorter than the wavelength $\lambda=1/q_0$, the field of the wave can be assumed to remain constant over this distance. This situation corresponds to an expansion in powers of $s\tau$ in (8) and (12):

$$s\tau\sim\frac{l_c}{\lambda}=\frac{mq_0}{eF_\perp}\left(1+\frac{\chi}{u}\right)^{1/3}=\frac{s}{\chi}\left(1+\frac{\chi}{u}\right)^{1/3}\ll 1. \tag{23}$$

The probability for the process is determined by the expressions for magnetobremsstrahlung in the resultant field, (13). For unpolarized electrons this probability is

$$\frac{dW_\gamma}{dx}=\frac{\alpha m^2}{3^{1/2}\pi\varepsilon}\left(\beta K_{2/3}(z_1)-\int_{z_1}^\infty K_{1/3}(y)\,dy\right), \tag{24}$$

$$\beta=\frac{\varepsilon'}{\varepsilon}+\frac{\varepsilon}{\varepsilon'},\quad z_1=\frac{2u}{3\chi_1},\quad \chi_1=\chi+\eta(\varphi)\frac{s}{2}.$$

The condition for the applicability of expression (24), in accordance with (23), is that the value of the parameter μ_1 be large:

$$\mu_1=\frac{\chi_1}{s}(\mu+\xi_0)\gg 1.$$

If the wave intensity is sufficiently small ($\xi_0\ll\mu$) in this case, we can carry out an expansion in the wave field in (24), and we can switch to the cross section for the Compton process in accordance with (14) and (15). As a result we find, for example, the following expression for the unpolarized Compton effect:

$$\frac{d\sigma_c}{dx}=\frac{r_e^2}{2\cdot 3^{1/2}\mu^2 s}\left\{\left[(\beta-1)z^2+\frac{4}{9}\beta\right]K_{2/3}(z)+\frac{2}{3}zK_{1/3}(z)\right\},$$

$$z=\frac{2u}{3\chi}. \qquad (25)$$

In deriving (25) we used recurrence relations for the Bessel functions. With the help of these functions, we can write the total cross section for Compton scattering as

$$\sigma_c=\frac{r_e^2}{6\cdot 3^{1/2}\mu\chi}\int_0^\infty\left(\frac{4}{9}+z^2\right)\frac{5u^2+7u+5}{(1+u)^3}K_{2/3}(z)du. \qquad (26)$$

In the case $\chi\ll 1$, values $u\sim\chi\ll 1$ contribute to the integral in (26), and the cross section itself is

$$\sigma_c=\frac{5\pi}{4\cdot 3^{1/2}}\frac{r_e^2}{\mu},\qquad \chi\ll 1,\qquad \mu\gg 1. \qquad (27)$$

Under the condition $\chi\gg 1$, values $u\sim 1$ contribute, and we can use the asymptotic expressions for the Bessel functions at small values of their argument ($z\ll 1$). In this case we find

$$\sigma_c=\frac{28}{81}\frac{\Gamma(2/3)}{3^{1/3}}\left(\frac{s}{\chi^{2/3}}\right)^2\frac{\pi r_e^2}{s},\qquad \chi\gg 1. \qquad (28)$$

It can be seen from (28) that in the case $\chi\gg 1$, $s\gg 1$ the expansion parameter is the quantity $s/\chi^{2/3}$, in agreement with (23).

4. PHOTOPRODUCTION OF AN ELECTRON-POSITRON PAIR

The spectral probability for pair production is found from the radiation probability through the replacements $\varepsilon\to-\varepsilon$, $\omega\to-\omega$, $z\to-z$, $\omega^2d\omega\to-\varepsilon^2d\varepsilon$ (Refs. 9 and 11, for example). In this case we find

$$u\to-(1-x)^{-1},\qquad \mu=\frac{\chi}{s}\to\frac{\varkappa}{4\Lambda},$$

$$\varkappa^2=-\frac{e^2}{m^6}(F^{\mu\nu}k_\nu)^2,$$

$$\frac{u}{s}\to(4x(1-x)\Lambda)^{-1}=\frac{\mathrm{ch}^2 y}{\Lambda},$$

$$x=\frac{\varepsilon}{\omega}=\frac{1+\mathrm{th}\, y}{2},\qquad \Lambda=\frac{qk}{2m^2}.$$

According to the analysis above, in the case $\varkappa\ll\Lambda$ the field has a strong effect on pair production by two photons only near the threshold, with $\cosh^2 y\approx\Lambda$. In this case the cross section for the two-photon process is

$$\frac{d\sigma_{\gamma\gamma}}{dy}=\frac{\pi r_e^2}{\Lambda}\left[1-\frac{1}{2\,\mathrm{ch}^2 y}+\frac{1}{\Lambda}\left(1-\frac{\mathrm{ch}^2 y}{\Lambda}\right)\right]\left[\vartheta(\Lambda-\mathrm{ch}^2 y)\left(1-\frac{1}{3}\int_b^\infty dx\,(J_{1/3}(x)+J_{-1/3}(x))\right)+\frac{\vartheta(\mathrm{ch}^2 y-\Lambda)}{3^{1/2}\pi}\int_b^\infty K_{1/3}(x)dx\right)\Bigg],$$

$$b=\frac{8\,\mathrm{ch}^2 y}{3\varkappa}\left|\frac{\Lambda}{\mathrm{ch}^2 y}-1\right|^{3/2}, \qquad (29)$$

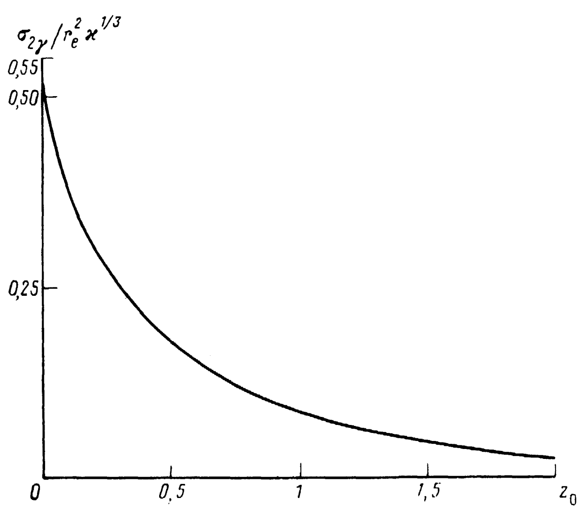

FIG. 1.

where $\vartheta(x)=1$ if $x>0$ or $\vartheta(x)=0$ if $x<0$.

Let us consider the case $\Lambda<1$. In other words, we consider the situation outside the kinematic region of pair production in the absence of a field, with an exponential suppression of the one-photon process in the constant field ($\varkappa\ll 1$). Using (20), (21), and (29), we find the following expression for the cross section for the two-photon process:

$$\frac{d\sigma_{\gamma\gamma}}{dy}\approx\frac{r_e^2}{2\cdot 3^{1/2}}\int_{b_1}^\infty K_{1/3}(z)dz,$$

$$b_1=\frac{8}{3\varkappa}(\Delta+y^2)^{3/2},\qquad \Delta=1-\Lambda\ll 1,\qquad y\ll 1. \qquad (30)$$

Integrating by parts in (30), we find the total cross section under the condition $\Lambda<1$:

$$\sigma_{\gamma\gamma}=r_e^2\left(\frac{\Delta}{3}\right)^{1/2}\int_{z_0}^\infty\left[\left(\frac{z}{z_0}\right)^{2/3}-1\right]^{1/2}K_{1/3}(z)dz,$$

$$z_0=\frac{8\Delta^{3/2}}{3\varkappa}. \qquad (31)$$

In the limiting cases $z_0\gg 1$ and $z_0\ll 1$ we have, respectively,

$$\sigma_{\gamma\gamma}\approx\frac{\pi r_e^2}{6}\frac{\Delta^{1/2}}{z_0^{5/6}}e^{-z_0},\qquad z_0\gg 1,$$

$$\sigma_{\gamma\gamma}\approx\frac{r_e^2}{4}(6\varkappa)^{1/3}\left(\frac{\pi}{3}\right)^{1/2}\Gamma\left(\frac{5}{6}\right). \qquad (32)$$

It follows from (31) and (32) that near the threshold ($\Delta\ll 1$) the quantity $\rho_{\gamma\gamma}/\varkappa^{1/3}$ has a scaling form which depends on only z_0, i.e., on only a certain combination of Δ and $\varkappa$. This function is plotted in Fig. 1. If the channel for the two-photon process is open ($\Lambda-1\gtrsim 1$), the corrections to the total cross section are small in proportion to $(\varkappa/\Lambda)^2$.

In the other limit $\varkappa\gg\Lambda$, the differential cross section for two-photon pair production can be found from (25) with the help of the replacements specified above. In this case the total cross section can be written

$$\sigma_{\gamma\gamma}=\frac{2r_e^2\Lambda}{3^{1/2}\varkappa^2}\int_0^1\frac{9-v^2}{1-v^2}\left(\frac{4}{9}+\lambda^2\right)K_{2/3}(\lambda)dv,$$

$$\lambda=(3\varkappa(1-v^2)/8)^{-1}. \qquad (33)$$

For $\varkappa \gg 1$, we can use the asymptotic expression for $K_{2/3}(\lambda)$ at small values of its argument in the integral over v. As a result we find

$$\sigma_{\tau\tau} \approx \frac{20}{63}\left(\frac{2}{3}\right)^{1/3} \frac{\Gamma(5/6)}{\Gamma(7/6)} \frac{\Lambda}{\varkappa^{4/3}} \pi r_e^2. \tag{34}$$

CONCLUSION

The results derived above are based on two important assumptions: that the transverse motion of the particle is relativistic and that the field F_{ex} is uniform over the formation length of the processes which we have been discussing. Under these assumptions, the expressions derived depend on the local values of the coordinate, the velocity, and the spin of the particle. Radiation effects can thus be incorporated in the equations of motion. It therefore becomes possible, in particular, to determine the characteristics of the radiation emitted by a particle throughout its motion in the field.

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