1/*n* Expansion for the Klein-Gordon equation

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An effective method is developed for the calculation of the coefficients in the $1/N$ expansion of the discrete spectrum of the Klein-Gordon equation. The method, based on using the $\hbar$ expansion and quantization conditions, gives rise to simple recursion relations from which coefficients of arbitrarily high order can be obtained both for the ground and the radially excited states. As an example we discuss the calculation of energy eigenvalues for the Coulomb interaction and the funnel-type potential in the $1/n$-expansion scheme.

1. INTRODUCTION

One of the most effective methods for studying the bound-state spectra in nonrelativistic quantum mechanics is the $1/N$-expansion method.[1,2] There exist several variants,[3–5] differing only in the choice of the parameters A and B in the relation $\hbar^2 l(l+1) = \Lambda^2 + \hbar A\Lambda + \hbar^2 B$ and in the treatment of nodes of the wave function. In this semi-classical approach the energy level is searched for in the form of the expansion $\varepsilon = \varepsilon^{(0)} + \varepsilon^{(1)}/\Lambda + \varepsilon^{(2)}/\Lambda^2 + \ldots$ The zeroth approximation corresponds here to the energy of the classical particle at the minimum of the effective potential.

The successful application of the $1/N$ expansion to various problems of nonrelativistic physics has stimulated interest in extending the method to the relativistic case. Such a generalization for the case of the Klein-Gordon equation was started with the study of the Coulomb interaction in the limit of large space dimensions ($N \to \infty$).[6,7] The case of the motion of a scalar particle in the field of a spherically symmetric potential, transforming like the time component of a four-vector, was considered soon afterwards.[8–10]

It should be noted that what was studied in Refs. 8–10 was in effect the equation obtained from the Klein-Gordon equation after discarding some of its terms, and similar in form to the Schrödinger equation, to which one of the standard $1/N$-expansion schemes was then directly applied.[3,4] In this process the calculations of excited states become quite unwieldy, which is characteristic of the manner in which the nodes of the wave function are treated in the standard scheme.

Recently a new effective technique for finding the coefficients in the $1/N$ expansion for bound states of the Schrödinger equation was proposed.[11] Based on making explicit use of the semiclassical nature of the $1/N$ method in the form of an expansion in the Planck constant, it permits the application of the quantization condition to take into account the nodes of the wave function. The resultant recursion relations then take on a simple form for both the ground and excited states and permit, in principle, the determination of the $1/N$-expansion coefficients to arbitrarily high order.

The purpose of the present article is the extension of the method developed in Ref. 11 to the relativistic case in the framework of the Klein-Gordon equation. In contrast to Refs. 8–10, the discussion is carried out for the most general spherically symmetric potential having both scalar and vector parts, without reducing the initial equation to a Schrödinger form. The method can be applied directly to any $1/N$-expansion scheme. As an example we consider its application to the determination of the coefficients in the $1/n$ expansion.[2–5]

2. $\hbar$-EXPANSION FOR THE KLEIN-GORDON EQUATION

For a scalar particle in the field of a vector potential $V(r)$ constituting the time component of a four-vector, and a scalar potential included, following the "dynamical model,"[12–14] in the mass term $m(r)$, the radial part of the Klein-Gordon equation has the form

$$\hbar^2 U''(r) = \left\{ m^2(r)c^2 - \frac{1}{c^2}[E - V(r)]^2 + \frac{\hbar^2 l(l+1)}{r^2} \right\} U(r). \quad (1)$$

Ensuring the possibility of applying the resultant formulas to an arbitrary $1/N$-expansion scheme, we write in view of ambiguities in the passage to classical mechanics

$$\hbar^2 l(l+1) = \Lambda^2 + \hbar A\Lambda + \hbar^2 B. \quad (2)$$

After substitution of the logarithmic derivative $\mathbb{C}(r) = \hbar U'(r)/U(r)$ Eq. (1) goes over then into the nonlinear Riccati equation

$$\hbar\mathbb{C}'(r) + \mathbb{C}^2(r) = \frac{1}{r^2}(\Lambda^2 + \hbar A\Lambda + \hbar^2 B) + m^2(r)c^2 - \frac{1}{c^2}[E - V(r)]^2, \quad (3)$$

We shall look for the eigenfunctions and eigenvalues of this equation in the form of asymptotic series in the Planck constant

$$\mathbb{C}(r) = \sum_{k=0}^{\infty} \mathbb{C}_k(r)\hbar^k, \quad (4)$$

$$E = \sum_{k=0}^{\infty} E_k \hbar^k. \quad (5)$$

Substituting the expansions (4) and (5) in (3) and equating coefficients of equal powers of $\hbar$ we obtain

$$\mathbb{C}_0^2(r) = \frac{\Lambda^2}{r^2} + m^2(r)c^2 - \frac{1}{c^2}[E_0 - V(r)]^2,$$

$$\mathbb{C}_0'(r) + 2\mathbb{C}_0(r)\mathbb{C}_1(r) = \gamma_1 \left(\frac{r_0}{r}\right)^2 + \frac{2E_1 V(r)}{c^2} - \frac{2E_0 E_1}{c^2},$$

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$$\mathbb{C}_{k-1}'(r)+\sum_{i=0}^{k}\mathbb{C}_i(r)\,\mathbb{C}_{k-i}(r)$$

$$=\gamma_k\left(\frac{r_0}{r}\right)^2+\frac{2E_kV(r)}{c^2}-\frac{1}{c^2}\sum_{i=0}^{k}E_iE_{k-i}, \tag{6}$$

where $\gamma_1=A\Lambda/r_0^2$, $\gamma_2=B/r_0^2$, $\gamma_3=\gamma_4=\ldots=0$.

The resultant system of equations can be solved successively in the case of ground states, provided E_0 is known.

In the $\hbar\to 0$ limit, which specifies the zeroth approximation, E_0 is determined as the smallest possible value of the energy of a relativistic particle performing classical motion on a stable circular orbit, namely

$$E_0=V_{\text{eff}}(r_0)=V(r_0)+m(r_0)c^2\left[1+\left(\frac{\Lambda}{r_0m(r_0)c}\right)^2\right]^{1/2}. \tag{7}$$

The radius r_0 of the orbit represents the position of the minimum of the effective potential and is calculated as the real positive root of the equation

$$r_0^3m'(r_0)c^2+r_0^3V'(r_0)\left[1+\left(\frac{\Lambda}{r_0m(r_0)c}\right)^2\right]^{1/2}=\frac{\Lambda^2}{m(r_0)}. \tag{8}$$

Now, knowing E_0, we obtain $\mathbb{C}_0(r)$ from the first equation in the system (6). Substitution of E_0, $\mathbb{C}_0'(r_0)$, and $\mathbb{C}_0(r_0)=0$ into the second equation permits the evaluation of E_1, which, in turn, gives $\mathbb{C}_1(r)$, etc.

For excited states, however, it is necessary to take into account the nodes of the wave function. The $\hbar$-expansion under discussion simplifies this problem, allowing the application of the quantization condition and the formalism of the theory of functions of a complex variable.

In the case of excited states with radial quantum number n_r the wave function $U(r)$ of Eq. (1) has precisely n_r real zeros. Then, according to the theorem on the logarithmic derivative, the Zwaan-Dunham[15,16] quantization conditions apply to the function $\mathbb{C}(r)$

$$\frac{1}{2\pi i}\oint\mathbb{C}(r)\,dr=n_r\hbar,\qquad n_r=0,1,2,\ldots \tag{9}$$

In view of the ambiguities in the transition to classical mechanics the quantization conditions (9) must be supplemented by a rule for the transition to the classical limit. Keeping in mind that the method under discussion is viewed as complementing the WKB approximation we demand that

$$\hbar\to 0,\qquad n_r=\text{const},\qquad \hbar n_r\to 0. \tag{10}$$

Now, after the substitution of the expansion (4), the quantization conditions are rewritten in the form

$$\frac{1}{2\pi i}\oint\mathbb{C}_1(r)\,dr=n_r,$$

$$\frac{1}{2\pi i}\oint\mathbb{C}_k(r)\,dr=0,\quad k\neq 1. \tag{11}$$

Further application of the residue theorem to the explicit form of the functions $\mathbb{C}_k(r)$ solves the problem of the description of radially excited states.

3. RECURRENCE RELATIONS

Following Ref. 11 we change to the new variable $x=(r-r_0)/r_0$ and represent $\mathbb{C}_0(r)$ in the form of the series:

$$\mathbb{C}_0(x)=-\omega x(1+a_1x+a_2x^2+\ldots)^{1/2}=x\sum_{\alpha=0}^{\infty}\mathbb{C}_\alpha^0x^\alpha. \tag{12}$$

The minus sign ensures here that the boundary conditions are satisfied, and we have introduced the notation

$$\omega^2=\frac{3\Lambda^2}{r_0^2}+\frac{2E_0V_2}{c^2}+c^2(2m_0m_2+m_1^2)-\frac{1}{c^2}(2V_0V_2+V_1^2), \tag{13}$$

$$a_\alpha=\frac{1}{\omega^2}\left[(-1)^\alpha\frac{3+\alpha}{r_0^2}\Lambda^2+\frac{2E_0V_{\alpha+2}}{c^2}+c^2\sum_{i=0}^{\alpha+2}m_im_{\alpha+2-i}-\frac{1}{c^2}\sum_{i=0}^{\alpha+2}V_iV_{\alpha+2-i}\right], \tag{14}$$

where $V_\alpha=r_0^\alpha V^{(\alpha)}(r_0)/\alpha!$ and $m_\alpha=r_0^\alpha m^{(\alpha)}(r_0)/\alpha!$ are coefficients in the expansions

$$V(x)=\sum_{\alpha=0}^{\infty}V_\alpha x^\alpha,\qquad m(x)=\sum_{\alpha=0}^{\infty}m_\alpha x^\alpha, \tag{15}$$

and the coefficients $\mathbb{C}_\alpha^0$ are connected with ω and a_α by the relations

$$\mathbb{C}_0^0=-\omega,\quad \mathbb{C}_1^0=-{}^1/_2\omega a_1,$$
$$\mathbb{C}_\alpha^0=\frac{1}{2\omega}\left[\sum_{j=1}^{\alpha-1}\mathbb{C}_j^0\mathbb{C}_{\alpha-j}^0-\omega^2a_\alpha\right]. \tag{16}$$

It is evident from the expression (12) that the point $x=0$ is a simple zero of the function $\mathbb{C}_0(x)$. Consequently the function

$$\mathbb{C}_k(x)=\frac{1}{2\mathbb{C}_0(x)}\left[\frac{\gamma_k}{(1+x)^2}+\frac{2E_kV(x)}{c^2}-\frac{1}{c^2}\sum_{i=0}^{k}E_iE_{k-i}-\frac{1}{r_0}\mathbb{C}_{k-1}'(x)-\sum_{i=0}^{k-1}\mathbb{C}_i(x)\,\mathbb{C}_{k-i}(x)\right] \tag{17}$$

has at that point poles of order $(2k-1)$ and can be expanded in its neighborhood in a Laurent series

$$\mathbb{C}_k(x)=x^{1-2k}\sum_{\alpha=0}^{\infty}\mathbb{C}_\alpha^kx^\alpha. \tag{18}$$

Such an expansion, upon substitution into the quantization conditions (11), makes it possible to express, using the residue theorem, the coefficients $\mathbb{C}_\alpha^k$ directly in terms of the radial quantum number n_r:

$$\mathbb{C}_0^1=n_r/r_0,\quad \mathbb{C}_{2k-2}^k=0,\quad k\neq 1. \tag{19}$$

From the expression (17), using the expansion (18), we obtain

$$\frac{3-2k+\alpha}{r_0}\mathbb{C}_\alpha^{k-1}+\sum_{j=0}^{k}\sum_{\beta=0}^{\alpha}\mathbb{C}_\beta^j\mathbb{C}_{\alpha-\beta}^{k-j}$$

$$= \left[(-1)^{\alpha}(\alpha-2k+3)\gamma_k + \frac{2E_k}{c^2} V_{\alpha-2k+2} \right]$$

$$\times \theta(\alpha-2k+2) - \delta_{\alpha,2k-2} \frac{1}{c^2} \sum_{j=0}^{k} E_j E_{k-j}, \qquad (20)$$

where we have introduced the Heaviside function $\theta(\alpha)$ and the Kronecker symbol $\delta_{\alpha,\beta}$.

Substitution of the quantization conditions (19) splits the recursion relation (20) in accordance with the value of the index α. For $\alpha \neq 2k-2$ there remains only the part determining the coefficients $\mathbb{C}_\alpha^k$ in the expansion of the wave function:

$$\mathbb{C}_\alpha^{\,k} = \frac{1}{2\mathbb{C}_0^{\,0}} \left\{ \left[(-1)^{\alpha}(\alpha-2k+3)\gamma_k + \frac{2E_k}{c^2} V_{\alpha-2k+2} \right] \theta(\alpha-2k+2) \right.$$

$$\left. - \frac{3-2k+\alpha}{r_0} \mathbb{C}_\alpha^{\,k-1} - \sum_{j=1}^{k-1} \sum_{\beta=0}^{\alpha} \mathbb{C}_\beta^{\,j} \mathbb{C}_{\alpha-\beta}^{\,k-j} - 2 \sum_{\beta=1}^{\alpha} \mathbb{C}_\beta^{\,0} \mathbb{C}_{\alpha-\beta}^{\,k} \right\}. \qquad (21)$$

In the case $\alpha = 2k-2$ we arrive at the recursion relation for the coefficients of the $\hbar$-expansion of the energy:

$$E_k = \frac{c^2}{2(E_0-V_0)} \left(\gamma_k - \frac{1}{r_0} \mathbb{C}_{2k-2}^{\,k-1} - \sum_{j=0}^{k} \sum_{\beta=0}^{2k-2} \mathbb{C}_\beta^{\,j} \mathbb{C}_{2k-2-\beta}^{\,k-j} \right.$$

$$\left. - \frac{1}{c^2} \sum_{j=1}^{k-1} E_j E_{k-j} \right). \qquad (22)$$

Thus, formulas (21) and (22) determine completely the coefficients in the $\hbar$-expansion of the eigenfunctions and eigenvalues of the Klein-Gordon equation. Here the values of the initial coefficients, needed for the recursion relations to work, are determined by the potential parameters $(E_0, \mathbb{C}_0^0, \mathbb{C}_\alpha^0)$ and the quantization conditions $(\mathbb{C}_0^1, \mathbb{C}_{2k-2}^k)$.

4. $1/n$ EXPANSION FOR THE KLEIN-GORDON EQUATION

Knowing the $\hbar$-expansion coefficients for the energy $E = E_0 + \hbar E_1 + \hbar^2 E_2 + \ldots$, one easily reproduces the terms in the standard $1/N$-expansion schemes: $E = \varepsilon^{(0)} + \varepsilon^{(1)}/\Lambda + \varepsilon^{(2)}/\Lambda^2 + \ldots$ Indeed, as is easily seen, the parameter Λ enters in the initial equation in the form of the combination $\hbar/\Lambda$. Consequently

$$\hbar^k E_k = \varepsilon^{(k)}/\Lambda^k \qquad (23)$$

under the condition that $\varepsilon^{(0)} = E_0$.

The concrete choice of the version of the $1/N$ expansion is specified, as was mentioned earlier, by the choice of the coefficients A and B [see Eq. (2)] that determine Λ and fix the zeroth approximation E_0 in the $\hbar$ expansion.

As an example we consider the application of formulas (21) and (22) to finding the energy in the $1/n$-expansion scheme.[2,5]

4.1. The Coulomb interaction

The Coulomb potential $V(r) = -\beta/r$, where $\beta = Ze^2$, is interesting because its exact solution is known. Setting $m(r) = m = 1$ we obtain from Eqs. (7) and (8)

$$r_0 = \frac{\Lambda^2}{\beta} \left[1 - \left(\frac{\beta}{c\Lambda} \right)^2 \right]^{1/2}, \quad E_0 = c^2 \left[1 - \left(\frac{\beta}{c\Lambda} \right)^2 \right]^{1/2}. \qquad (24)$$

Then, substituting $A = -1-2n_r$ and $B = n_r(n_r+1)$, which correspond to the value $\Lambda = \hbar(n_r + l + 1)$ accepted in the $1/n$ expansion, we obtain from the recursion relation (22), after expansion in a series in $1/c^2$,

$$E = c^2 - 2\left(\frac{\beta}{2\Lambda}\right)^2 - \frac{1}{c^2}\left(\frac{\beta}{2\Lambda}\right)^4 \times \left[2 + 8\left(\frac{\hbar\varkappa}{2\Lambda}\right) + 8\left(\frac{\hbar\varkappa}{2\Lambda}\right)^2 + \ldots \right]$$

$$- \frac{1}{c^4}\left(\frac{\beta}{2\Lambda}\right)^6 \left[4 + 24\left(\frac{\hbar\varkappa}{2\Lambda}\right) + 72\left(\frac{\hbar\varkappa}{2\Lambda}\right)^2 + \ldots \right]$$

$$- \frac{1}{c^6}\left(\frac{\beta}{2\Lambda}\right)^8 \left[10 + 80\left(\frac{\hbar\varkappa}{2\Lambda}\right) + 336\left(\frac{\hbar\varkappa}{2\Lambda}\right)^2 + \ldots \right] - \ldots, \qquad (25)$$

where $\varkappa = 2n_r + 1$, which coincides with the expansion of the exact solution[6]

$$E = c^2 \left\{ 1 + \frac{4\beta^2}{\hbar^2 c^2} \left[2n_r + 1 + \left((2l-1)^2 - \frac{4\beta^2}{\hbar^2 c^2} \right)^{1/2} \right]^{-2} \right\}^{-1/2} \qquad (26)$$

in a series in powers of $\hbar$ and $1/c^2$.

It follows from (25) that the characteristic peculiarity of the $1/n$ expansion for the Klein-Gordon equation with the Coulomb potential is the absence of a nonrelativistic part in the coefficients E_k, $k \geqslant 1$.

We remark that reduction of the Klein-Gordon equation to a Schrödinger-like form[8-10] changes the coefficient of $(\hbar^2/c^6)$ in (25) from 336 to 464. The coefficients in the terms of higher order in $\hbar$ and $1/c^2$ are also changed.

4.2. Funnel-type potential

The Klein-Gordon equation (1) for a funnel-type potential, widely used in the description of hadron spectra, takes the form

$$U''(r) = \left[\left(m + \frac{kr}{2} \right)^2 - \left(E + \frac{2}{3}\frac{q}{r} \right)^2 \right] U(r), \qquad (27)$$

where $\hbar = c = 1$.

We shall demonstrate with this example the speed with which the $1/n$ expansion converges in the relativistic case.

The calculations were performed using formulas (21) and (22). The minimum of the effective potential was found as the real positive root of the equation

$$\frac{r^3 k}{2} + \frac{2}{3} qr \left[1 + \left(\frac{\Lambda}{r(m+rk/2)} \right)^2 \right]^{1/2} - \frac{\Lambda^2}{m+rk/2} = 0. \qquad (28)$$

The parameters of the potential were taken from Ref. 17 and had the following values: $m = 1.37$ GeV, $k = 0.2086$ GeV^2, $q = 0.39$.

In Table I are shown the results of the calculations of successive sums of terms of the $1/n$ expansion for doubled energy eigenvalues, as used in hadronic spectroscopy.[17] It is seen from Table I that the accuracy of the description of the

TABLE I. Precision of the $1/n$ expansion for the funnel potential.

k	$\xi_{nl}^{(k)}$			
	$1S$	$1P$	$1D$	$2P$
1	3,085967	3,523560	3,809142	3,984558
2	3,073860	3,520865	3,807685	3,970603
3	3,070793	3,520711	3,807646	3,969732
4	3,069347	3,520720	3,807650	3,969100
5	3,068207	3,520719	3,807650	3,968882
6	3,067568	3,520714	3,807650	3,968790
7	3,067349	3,520714	3,807650	3,968752
8	3,067128	3,520714	3,807650	3,968734
9	3,066898	3,520714	3,807650	3,968726
10	3,066973	3,520714	3,807650	3,968722
ξ_{nl}^{exact}	3,067	3,520714	3,807650	3,968718

Note. The $\xi_{nl}^{(k)}$ are sums of successive terms in the $1/n$ expansion: $\xi_{nl}^{(k)} = 2(\varepsilon^{(0)} + \varepsilon^{(1)}/n + \ldots + \varepsilon^{(k)}/n^k)$; ξ_{nl}^{exact} is the numerical solution of the Klein-Gordon equation. The value of ξ_{nl}^{exact} for the $1S$ state was taken from Ref. 17.

energy spectrum of the Klein-Gordon equation with the help of the sum of the first terms in the $1/n$ expansion increases, just as in the case of the Schrödinger equation, with increasing orbital quantum number and decreases with increasing radial quantum number. The least accurate description is for the state with $l = 0$.

CONCLUSION

We developed here a method for the determination of the coefficients in the $1/N$ expansion for the Klein-Gordon equation with a potential having scalar and four-vector parts. Based on utilization of the $\hbar$ expansion and quantization conditions, it gives rise to simple recursion relations for both ground and radially excited states. Passage from one version of the $1/N$ expansion to another leaves the recursion relations unchanged and reduces only to a replacement of the initial coefficients. In contrast to existing methods,[8-10] in the approach considered here no terms in the initial Klein-Gordon equation need be omitted for the purpose of reducing the equation to a Schrödinger form.

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