Kinetics of Bose condensation in an interacting Bose gas

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The kinetics of the formation of a Bose condensate in a weakly interacting Bose gas is analyzed. The question of the time at which Bose condensation is reached does not have a universal answer. For problems involving short-range correlation properties, this time is determined by the time required for the formation of a "quasicondensate:" a state in which fluctuations of the absolute value of the order parameter ψ are suppressed. This time is the sum of the evolution time in the region described by the Boltzmann kinetic equation (the kinetic stage) and the evolution time in the coherent region, in which the kinetics is described by the equation for the ψ field. It is shown that the kinetic stage of the process takes most of the time. The time scale for the attainment of Bose condensation is the usual time between particle collisions.

1. INTRODUCTION

The kinetics of the formation of a Bose condensate in a gas with a repulsive interaction between bosons is an interesting problem in the kinetics of phase transitions. If, when the conditions for the formation of a Bose condensate arise, the system is far from equilibrium, and there is no condensate, then the temporal evolution differs substantially from the well-known kinetics of second-order phase transitions. This problem has recently attracted particular interest in connection with the process of Bose condensation in a gas of spin-polarized atomic hydrogen. One reason for this interest is that the system itself has a finite lifetime, particularly if its density is high (Refs. 1 and 2, for example). A second reason for the interest is that it has been found[3] that the probabilities for inelastic processes, in particular, the recombination rate, change substantially when a Bose condensate appears in the system. This effect opens up an interesting possibility for the experimental observation of Bose condensation. Although there is no true Bose condensate for $T \neq 0$ in the two-dimensional case, it was shown in Ref. 4 that the change in the probability for the inelastic processes persists below the point of the Kosterlitz-Thouless transition, by virtue of the specific properties of a two-dimensional quasicondensate, i.e., a condensate with a fluctuating phase.

The time over which a Bose condensate is formed has been studied in several places (e.g., Refs. 5–7 and 9). Levich and Yakhot[5] have asserted that if there is no condensate at the initial time then the time over which the condensate forms as the result of an interaction of gas particles with a heat reservoir is infinite. Examining the kinetics under conditions such that particles interact with each other, Levich and Yakhot[6] also found that the time required to achieve Bose condensation was finite within the framework of the Boltzmann kinetic equation. Levich and Yakhot did understand that the kinetic equation was not applicable at low energies (in the so-called coherent region; more on this below). Furthermore, the question of the validity of the result remained open, because of some strong assumptions made in solving the kinetic equation. For this coherent region, the analysis of the time evolution in Ref. 6 was actually based on the random-phase approximation, which is not suitable for this region.

Snoke and Wolfe[7] undertook a numerical calculation using the kinetic equation. Although this calculation did demonstrate a substantial restructuring of the particle distribution function at low energies when the conditions for Bose condensation obtain, the appearance of a Bose condensate was not detected. There was a good reason for that result. It is not possible to find a true condensate with $\varepsilon = 0$ on the basis of a kinetic equation without either (a) singling out a low-energy interval in which collective effects are predominant, and in which the kinetic equation is not valid (more on this below), or (b) introducing seed condensate in a self-consistent way. Eckern[8] has analyzed kinetic processes in a Bose gas in which there is a Bose condensate.

Another extreme result was recently reported by Stoof,[9] who asserted that the time required for the formation of a Bose condensate was $\sim \hbar/T_c$, where T_c is the temperature of Bose condensation. That result looks strange, and it is furthermore physically meaningless, since this time is shorter than both the particle collision time in the gas and the characteristic evolution time $\hbar/n_0\tilde{U}$, in the coherent region in which the kinetic energy of the particles is smaller than their potential interaction ($\tilde{U}$ is the effective particle interaction vertex, and n_0 is the equilibrium density of the condensate).

The question of the condensate formation time in the course of the actual evolution of a gaseous system is not changed in any fundamental way by the interaction with a heat reservoir. Because of the pronounced nonlinearity of the problem in the decisive stage of the temporal evolution, the interaction of particles with each other dominates. This assertion should be valid, in particular, for metastable Bose systems prepared artificially, e.g., excitons and biexcitons (see, for example, Ref. 10, where an attempt was made to achieve Bose condensation experimentally). For systems of this sort, the interaction with excitations of the medium can be important for the overall picture.

Actually, the question of the formation time of a Bose condensate does not have an unambiguous answer. The answer depends strongly on the particular problem under consideration. The ambiguity arises in the final stage of the evolution, in which a substantial fraction of the particles which should go into the condensate are concentrated in the energy

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interval $\varepsilon \lesssim n_0 \tilde{U}$. The state of an interacting Bose gas at temperatures below T_c is known to be characterized by a complex order parameter (Ref. 11, for example). In this final stage of the evolution, which we will call the "coherent" stage, the kinetic energy of the particles becomes smaller than their average potential energy. There is a pronounced mixing of states with different momenta $\mathbf{k}$, and the system should be described by the general equation for a complex classical field (for occupation numbers $n_{\mathbf{k}} \gg 1$). This equation cannot be reduced to an equation for occupation numbers. It is with this coherent stage of the evolution that the formation of new local correlation properties and the formation of a long-range order in the system are associated. The time over which the absolute value of the order parameter (i.e., the condensate density) reaches equilibrium and the phase of this order parameter may in general be very different. Spatial fluctuations of the condensate density die out over times short in comparison with the time scale for relaxation to a spatially uniform phase. The reason is that the change in the energy in the first case is associated with the density change itself. In the second case, in contrast, the energy depends on only the gradient of the phase, and long-wavelength fluctuations die out very slowly. The situation is complicated by the circumstance that vortices may arise in the system in the course of its time evolution. The annihilation of these vortices would impart an additional relaxation time to the problem.

A nonequilibrium, nonuniform spatial distribution of the phase has the consequence that a δ-function distribution of condensate particles and, correspondingly, true long-range order can arise only after a very long (if not infinite) time. After the decay of nonequilibrium density fluctuations, local correlation properties are established in the system. These properties are identical to those which prevail in an equilibrium system with a true condensate. At a length scale on the order of the correlation length

$$r_c = \hbar / (2mn_0\tilde{U})^{1/2} \tag{1.1}$$

(m is the mass of an atom), if all the condensate particles are in the energy interval $\varepsilon \lesssim n_0 \tilde{U}$, the time scale for the attainment of a local equilibrium is given by

$$\tau_{\text{kin}} = \hbar / n_0 \tilde{U} \tag{1.2}$$

(see the following section of this paper). Consequently, for all processes for which the dependence on the presence of a Bose condensate is related to the correlation properties at distances $r \lesssim r_c$, the condensate formation time can actually be assumed to be the finite time τ_{kin} in (1.2) in this energy interval (although this time scale actually corresponds to the formation of a sort of "quasicondensate;" only the absolute value of the order parameter fluctuates). These comments apply in particular to the effect of Bose condensation on inelastic processes in an interacting Bose gas.

In the overall picture of the kinetics, the time τ_{kin} is not the only time scale. Moreover, it is generally not the longest and in this sense not the governing time scale.

Let us assume that a pronounced cooling has initially occurred, as the result of a decrease in the energy in the degrees of freedom with $\varepsilon \gtrsim T$, at a scale such that the final equilibrium temperature is lower than T_c. In this case the excess of particles which should subsequently form a condensate is initially concentrated at high energies, $\sim T$. Since we have $T_c \gg n_0 \tilde{U}$, a flux of particles should form in energy space in this case, directed toward low energies. Although the energy of the particles satisfies $\varepsilon \gg n_0 \tilde{U}$, the evolution process is described by the ordinary kinetic equation for a Bose gas. In this regime, which we will call the "kinetic" regime, there are two intervals along the energy axis. The first—the high-energy interval—is between $\varepsilon \sim T$ and $\varepsilon \sim \varepsilon_*$, where ε_* is the boundary at which the occupation numbers n_ε becomes substantially greater than the equilibrium values. The second interval is $\varepsilon < \varepsilon_*$. Here the collision integral is definitely nonlinear in the occupation numbers. The occupation numbers themselves are determined by the number of excess particles which go into the condensate. The general properties of a nonlinear kinetic equation of this sort have been studied by Zakharov, primarily in application to plasmas (see the review in Ref. 12). We will be drawing on his results below.

As the analysis below shows, the time scale for the passage through the kinetic region is determined in order of magnitude by the reciprocal of the particle collision frequency in the gas:

$$\tau_{\text{kin}} = (n\sigma v_T)^{-1}, \tag{1.3}$$

where σ is the scattering cross section, v_T is the thermal velocity, and n is the total density of the gas.

Comparing the time in (1.3) with the time scale [τ_{kin} from (1.2)] for the decay of nonequilibrium density fluctuations in the coherent regime which arises in the case $\varepsilon \lesssim n_0 \tilde{U}$, we see that the condition

$$\tau_k \gg \tau_{\text{kin}} \tag{1.4}$$

essentially always holds. The kinetic stage of the process is thus the limiting step for the formation time of a quasicondensate (but not for the formation of a long-range order).

The overall scenario for the process by which a Bose condensate forms can be outlined as follows. In the linear region, with $\varepsilon > \varepsilon_*$ (this region is particularly well expressed in the case $n_0 \ll n$), a quasiequilibrium distribution with a time-varying chemical potential arises. This distribution leads to the formation of a density peak, which moves toward small values of ε. In the nonlinear kinetic region, $\varepsilon < \varepsilon_*$, a solution $n_\varepsilon \sim \varepsilon^{-7/6}$ develops. This solution was found by Zakharov[12] for the case of a constant flux of particles in energy space. This energy dependence leads to an elongated energy-space distribution, with a front which reaches the coherent region in a time $\sim \tau_{\text{kin}}$. The time scale for the influx of the rest of the nonequilibrium distribution into the coherent region turns out to be the same. Specific features of the nonlinear kinetic stage have been analyzed previously by one of the present authors.[13]

2. COHERENT REGIME OF THE EVOLUTION

Let us consider the energy interval $\varepsilon \lesssim n_0 \tilde{U}$. We assume that the bulk of the particles which are to form the condensate are in this interval. The large occupation numbers characteristic of this case allow us to replace the operator ψ, the second-quantization operator for the Bose particles, by the amplitude of the boson field, $\psi(\mathbf{r},t)$, which is a c-number, in this energy interval. For the amplitude ψ we have the equation (here and below, $\hbar = 1$)

$$i(\partial\psi/\partial t)=-(1/2m)\Delta\psi+\tilde{U}|\psi|^2\psi. \quad (2.1)$$

In the limit $t\to\infty$, the function ψ should reach its equilibrium value

$$\psi_0=n_0^{1/2}\exp(i\Phi_0-i\mu_0 t), \quad \mu_0=n_0\tilde{U}, \quad (2.2)$$

where Φ_0 is the equilibrium value of the condensate phase. We are ignoring the particles above the condensate in this energy interval, since their total number per unit volume is small in comparison with n_0.

Writing the function ψ in the form

$$\psi=n_0^{1/2}\exp(-i\mu_0 t)f(\mathbf{r},t), \quad (2.3)$$

we can rewrite Eq. (2.1) as

$$i(\partial f/\partial t)=-(1/2m)\Delta f+n_0\tilde{U}\{|f|^2f-f\}.$$

In terms of the dimensionless variables

$$\bar{t}=t/\tau_{\text{kin}},\ \bar{r}=r/r_c \quad (2.4)$$

[see (1.1) and (1.2)], this equation becomes

$$i(\partial f/\partial\bar{t})=-\Delta_{\bar{r}}f+|f|^2f-f. \quad (2.5)$$

As $\bar{t}\to\infty$, we have $f\to\exp(i\Phi_0)$.

It follows from this equation that the evolution on length scales $\sim r_c$ occurs over times $\sim\tau_{\text{kin}}$. With increasing $\bar{t}$, the gradients of the phase and the density are smoothed out. This smoothing is equivalent to a decrease in the scale values of the wave vectors $\mathbf{k}$ and frequencies ω in the Fourier representation of the function f. The excess energy per particle $\sim n_0\tilde{U}$ goes off into the kinetic region. It turns out that the condition $\bar{t}\gg 1$ is by itself a sufficient condition for substantial suppression of the fluctuations in the absolute value of f, i.e., the density.

To demonstrate this point, we introduce a deviation δn from the mean spatial density n_0. Using (2.5), we write separate equations for the absolute value and phase of the function f. After linearization, the equations for the Fourier components of the absolute value and the phase become

$$i\omega_{\mathbf{k}}\tau_c(\delta n_{\mathbf{k}}/n_0)=-2r_c^2k^2\Phi_{\mathbf{k}},$$
$$i\omega_{\mathbf{k}}\tau_c\Phi_{\mathbf{k}}=(\delta n_{\mathbf{k}}/n_0)(1+r_c^2k^2/2). \quad (2.6)$$

This system gives us a spectrum which is the same as the Bogolyubov spectrum,

$$\omega_{\mathbf{k}}^2=(2r_c^2k^2/\tau_c^2)(1+r_c^2k^2/2). \quad (2.7)$$

From the first of these equations we find

$$|\delta n_{\mathbf{k}}/n_0|^2=(4r_c^4k^4/\omega_{\mathbf{k}}^2\tau_c^2)|\Phi_{\mathbf{k}}|^2.$$

For $\bar{t}\gg 1$, the characteristic values k_* of the momentum k becomes smaller than $k_0=r_c^{-1}$. For such values of k, the dispersion law is acoustic,

$$\omega=ck,\quad c=2^{1/2}(r_c/\tau_c),$$

and we have

$$|\delta n_{\mathbf{k}}/n_0|^2=2r_c^2k^2|\Phi_{\mathbf{k}}|^2.$$

Hence

$$\langle(\delta n/n_0)^2\rangle=\sum_{\mathbf{k}}|\delta n_{\mathbf{k}}/n_0|^2=2r_c^2\sum_{\mathbf{k}}k^2|\Phi_{\mathbf{k}}|^2. \quad (2.8)$$

Taking account of the relation $E\sim(k_*^2/2m)n_0V$ for the kinetic energy remaining in the system, and using

$$E\approx(n_0/m)\int d\mathbf{r}(\nabla\Phi)^2=(n_0V/m)\sum_{\mathbf{k}}k^2|\Phi_{\mathbf{k}}|^2,$$

we find from (2.8)

$$\langle(\delta n/n_0)^2\rangle\sim(k_*/k_0)^2\ll 1. \quad (2.9)$$

Consequently, over times determined by τ_{kin}, a state forms in which density fluctuations are suppressed (as in a true condensate). Smearing over a certain finite interval of k persists. It is interesting to show that suppression of density fluctuations is a necessary and sufficient condition for the same change in the probability of inelastic processes to appear as in the presence of a true condensate. With this goal in mind, we consider the example of three-particle recombination. For this process, the probability is determined by the correlation function

$$Z=\langle\hat{\psi}^+(\mathbf{r},t)\hat{\psi}^+(\mathbf{r},t)\hat{\psi}^+(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)\rangle$$

as was shown in Ref. 3. In the case at hand, this correlation function reduces to

$$Z=n_0^3\langle f^{*3}(\mathbf{r},t)f^3(\mathbf{r},t)\rangle=\langle n^3\rangle. \quad (2.10)$$

If fluctuations are suppressed, then

$$Z\approx n_0^3.$$

If fluctuations are instead present, then

$$Z=n_0^3+3n_0\langle(\delta n)^2\rangle+\langle(\delta n)^3\rangle. \quad (2.11)$$

The expression for δn can be written

$$\delta n=n_0(|f|^2-1). \quad (2.12)$$

Consequently,

$$\langle(\delta n)^2\rangle=n_0^2[\langle|f|^4\rangle-2\langle|f|^2\rangle+1],$$
$$\langle(\delta n)^3\rangle=n_0^3[\langle|f|^6\rangle-3\langle|f|^4\rangle+3\langle|f|^2\rangle-1].$$

Near the boundary between the coherent and kinetic regions, where all the field modes corresponding to different wave vectors $\mathbf{k}$ can be regarded as independent, we find the following expression for the binary correlation coefficient of the Fourier components of the field:

$$\langle f_{\mathbf{k}}f_{\mathbf{k}'}\rangle=|f_{\mathbf{k}}|^2\delta_{\mathbf{k}\mathbf{k}'}.$$

Expanding f in a Fourier series, and using an analog of Wick's theorem, we then find

$$\langle|f|^2\rangle=\sum_{\mathbf{k}}|f_{\mathbf{k}}|^2=1,\quad\langle|f|^4\rangle=2,\quad\langle|f|^6\rangle=6,$$

As a result we find

$$Z=6n_0^3.$$

When density fluctuations are suppressed, the probability for three-particle recombination is thus reduced by a factor of 6. This result is precisely the same as that found for the case of a true condensate.[3] From this point of view, a state with a narrow "precondensate" peak ($k_* \ll k_0$) in the particle distribution has all the properties of a true condensate for all such processes; it can thus be called a "quasicondensate." An important point is that the appearance of a true condensate with $k=0$ and the formation of long-range order are limited by the phase relaxation time. This time, which depends on the dimensions of the system, may be vastly longer than τ_c.

3. KINETIC REGION: LINEAR REGIME

Since we have verified that there is a finite quasicondensate formation time, during which the particles go into a coherent region, the time evolution of the particle distribution in the kinetic region, i.e., under the condition $\varepsilon \gg n_0 \widetilde{U}$, becomes an important question. The overall picture here depends strongly on the initial conditions. However, the kinetic time scales turn out to be of comparatively broad applicability. We consider a system at a temperature close to T_c. We assume that cooling has occurred comparatively rapidly at the initial time, primarily because of a decrease in the energy in the degrees of freedom with $\varepsilon \gtrsim T$. If the final temperature, after the attainment of equilibrium, is lower than T_c, then it is easy to see that the excess of particles which are to subsequently form the condensate is initially at high energies. This excess must pass through the entire range of energies in the kinetic region, until it reaches the coherent region. Clearly, a related problem arises in a simple way when additional particles are injected at an energy $\varepsilon \lesssim T$. The estimates derived below also provide answers to the important question of when the cooling is associated with a continuous loss of particles from the tail of the energy distribution.

To make the physical picture more transparent, we consider the case in which the condensate density when equilibrium is reached, n_0, is small in comparison with the total density of particles, n:

$$n_0/n \ll 1. \tag{3.1}$$

In this case we thus have

$$\Delta T/T_c = (T_c - T)/T_c \ll 1,$$

where T is the equilibrium temperature of the gas. We assume

$$n_0/n \gg (na^3)^{1/6}, \tag{3.2}$$

in accordance with the Ginzburg criterion,[11] in order to get outside the fluctuation region.

The kinetic equation for a Bose gas in the spatially uniform case, for a distribution which is isotropic in terms of momentum, has the well-known form (Ref. 12, for example)

$$\dot{n}_1 = -W \int d\varepsilon_2\, d\varepsilon_3\, d\varepsilon_4 \chi\left(\frac{\varepsilon_2}{\varepsilon_1}, \frac{\varepsilon_3}{\varepsilon_1}, \frac{\varepsilon_4}{\varepsilon_1}\right) \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$$
$$\times [n_1 n_2 (1+n_3)(1+n_4) - n_3 n_4 (1+n_1)(1+n_2)] \tag{3.3}$$

($n_i \equiv n_{\varepsilon_i}$), where

$$W = \widetilde{U}^2 m^3 / 4\pi^3 \hbar^7 = 4ma^2/\pi\hbar^3,$$

$$\chi = \frac{4}{\pi} \int \frac{dx}{x^2} \sin(x)$$
$$\times \sin\left[x\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{1/2}\right] \sin\left[x\left(\frac{\varepsilon_3}{\varepsilon_1}\right)^{1/2}\right] \sin\left[x\left(\frac{\varepsilon_4}{\varepsilon_1}\right)^{1/2}\right]. \tag{3.4}$$

We have restored $\hbar$ in (3.4) to exhibit the dimensionality, and we have used the nonrelationship between $\widetilde{U}$ and the scattering length a:

$$\widetilde{U} = 4\pi\hbar^2 a/m. \tag{3.5}$$

The function χ for the entire set of ε_i's satisfying energy conservation can be reduced to the simple form (cf. Ref. 7)

$$\chi = [\min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)/\varepsilon_1]^{1/2}. \tag{3.6}$$

Under condition (3.1), the kinetic region breaks up into two energy intervals. In the first, which extends down the energy scale to some $\varepsilon_* \ll T$, the corrections Δn_ε are small in comparison with the equilibrium distribution function $\bar{n}_\varepsilon$ at the temperature T. We call the kinetic regime corresponding to this region the "linear" regime. Under the condition $\varepsilon \ll \varepsilon_*$, the arrival of excess particles is accompanied by an increase in the distribution function such that the condition $n_\varepsilon \gg \bar{n}_\varepsilon \gg 1$ becomes satisfied. In this region, the collision integral in (3.3) depends on essentially only the distribution of the nonequilibrium particles. A definitely nonlinear regime sets in, in which the time evolution is determined by the interaction between particles.

We begin with a consideration of the linear region. Over most of the corresponding energy interval $\varepsilon \ll T$, the kinetic equation in (3.3), linearized in Δn_ε, takes the following form, where we are using $\bar{n}_\varepsilon \gg 1$:

$$\Delta\dot{n}_1 = -W \int d\varepsilon_2\, d\varepsilon_3\, d\varepsilon_4 \chi\left(\frac{\varepsilon_2}{\varepsilon_1}, \frac{\varepsilon_3}{\varepsilon_1}, \frac{\varepsilon_4}{\varepsilon_1}\right)$$
$$\times\ \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$$
$$\times [\Delta n_1 (\bar{n}_2\bar{n}_3 + \bar{n}_2\bar{n}_4 - \bar{n}_3\bar{n}_4) + \Delta n_2 (\bar{n}_1\bar{n}_3 + \bar{n}_1\bar{n}_4 - \bar{n}_3\bar{n}_4)$$
$$+ \Delta n_3 (\bar{n}_1\bar{n}_2 - \bar{n}_1\bar{n}_4 - \bar{n}_2\bar{n}_4) + \Delta n_4 (\bar{n}_1\bar{n}_2 - \bar{n}_1\bar{n}_3 - \bar{n}_2\bar{n}_3)]$$
$$\equiv \hat{\Omega}[\Delta n_\varepsilon]. \tag{3.7}$$

The solution of this equation under the conditions assumed here should describe a flux of the excess particles in energy space, toward lower energies. In order to determine the nature of the evolution of the distribution in the linear region and the time scale of this evolution, we adopt the approximation that a local quasiequilibrium is reached in the energy interval with the bulk of the excess particles. The particle distribution can be written

$$n_\varepsilon \approx \{\exp[(\varepsilon - \mu(t))/T] - 1\}^{-1} \approx T/(\varepsilon - \mu(t)) \tag{3.8}$$

with a time-dependent chemical potential $\mu(t) > 0$. In general, the temperature may also depend on the time here. However, it is easy to verify that this time dependence can be ignored by virtue of (3.1). Relation (3.8) of course holds only for $\varepsilon > \mu(t)$. Over most of the linear interval, the condition $\mu \ll \varepsilon$ holds, as we will see below. We can thus write

$$\Delta n_\varepsilon = n_\varepsilon - \bar{n}_\varepsilon \approx \mu(t) T/\varepsilon^2. \tag{3.9}$$

This ε dependence corresponds to motion of the front of the $\varepsilon_0(t)$ distribution, with a width determined by the same parameter $\varepsilon_0(t)$. The parameter $\varepsilon_0(t)$ itself is found from the condition

$$\alpha \int_{\varepsilon_0}^{T} \varepsilon^{1/2} \Delta n_\varepsilon \, d\varepsilon = n_0, \tag{3.10}$$

where $\alpha = m^{3/2}/2^{1/2}\pi^2$. The integral is determined by its lower limit and has the value $2T\mu(t)/\varepsilon_0^{1/2}(t)$. Hence

$$\mu(t) = n_0 \varepsilon_0^{1/2}(t)/2\alpha T. \tag{3.11}$$

The boundary of the linear regime, ε_*, determined by the condition $\Delta n_\varepsilon/\bar{n}_\varepsilon \sim 1$, can be found directly:

$$\varepsilon_* \approx T(n_0/n)^2 \tag{3.12}$$

(here we have made use of the circumstance that T is close to T_c, and we have used the relationship between T_c and n). In this case we have

$$\mu(t) = [\varepsilon_* \varepsilon_0(t)]^{1/2} < \varepsilon_0(t). \tag{3.13}$$

The parameter $\mu(t)$ lags behind $\hat{\varepsilon}(t)$ at all times and is comparable to this quantity when the boundary of the linear regime, ε_*, is reached. This circumstance demonstrates that it is legitimate to assume a quasiequilibrium distribution in the form in (3.8), (3.9).

Let us find an expression for the particle flux in energy space. This flux is related to the motion of the boundary $\hat{\varepsilon}(t)$. Using (3.9) and (3.11), we find

$$Q = -\alpha \varepsilon_0^{1/2} \dot{\varepsilon}_0 n_{\varepsilon_0} \approx -{}^1/_2 n_0 (\dot{\varepsilon}_0/\varepsilon_0). \tag{3.14}$$

We are to find the ratio $\dot{\varepsilon}_0/\varepsilon_0$ from the kinetic equation (3.7).

If we substitute $\bar{n}_\varepsilon = T/\varepsilon$ into the linearized version of the collision integral in (3.7), we easily find that the integral operator $\hat{\Omega}$ is a uniform functional of the energy of zeroth order. In other words, it does not change under the replacement $\varepsilon_i \to \lambda \varepsilon_i$. Analysis of the collision integral shows that collisions with $\varepsilon_i \sim \varepsilon_1$ dominate the situation. The collision integral is thus characterized by an *energy-independent* relaxation time τ_{kin}:

$$WT_c^2 \approx 0{,}2\tau_k^{-1}, \tag{3.15}$$

where τ_{kin} is determined by (1.3) with $\sigma = 8\pi a^2$. Here we can assume

$$\dot{\varepsilon}_0/\varepsilon_0 = -\gamma/\tau_{\text{kin}}. \tag{3.16}$$

We can find the constant γ approximately from Eq. (3.7) by examining ε_1 near $\hat{\varepsilon}(t)$, and by using a model in which we take Δn_ε from (3.9) for $\varepsilon > \hat{\varepsilon}(t)$, while for $\varepsilon < \hat{\varepsilon}(t)$ we set $\Delta n_\varepsilon = 0$. Corresponding calculations lead to $\gamma \approx 4$. Approximately the same value of γ is found if we take $\varepsilon_1 > \hat{\varepsilon}(t)$, but in the energy region with most of the excess particles.

Using (3.16), we find that the expression for the particle flux in energy space takes the simple form

$$Q \approx \gamma n_0/2\tau_{\text{kin}}. \tag{3.17}$$

Interestingly, under our assumptions, there is a flux Q (constant at the front) after the spatial distribution (3.9) is formed. This constant flux corresponds to (a) a front amplitude which increases with decreasing $\hat{\varepsilon}(t)$ and (b) a peak width in the particle distribution which decreases.

Using the relation

$$\varepsilon_0(t) \approx T \exp(-\gamma t/\tau_{\text{kin}}),$$

which follows from (3.16), and also using relation (3.12), we find the following result for the time at which the front of the distribution reaches the boundary of the linear region, ε_*:

$$\tau_L \approx (2\tau_{\text{kin}}/\gamma) \ln(n/n_0). \tag{3.18}$$

The time required to cross the linear region is thus determined by the ordinary collision time scale τ_{kin}, enhanced by a factor $\ln(n/n_0)$ at small n_0.

Note that under condition (3.2) the boundary energy satisfies $\varepsilon_* \gg n_0 \tilde{U}$, so the system necessarily passes through the nonlinear kinetic regime.

We should stress that the estimate of the time required to cross the linear interval is comparatively insensitive to the nature of the assumptions which we have made. If the dependence of Δn_ε in the region $\varepsilon \sim \hat{\varepsilon}(t)$ differs from (3.9), then τ_L is again given approximately by (3.18), as can be verified.

4. KINETIC REGION: NONLINEAR REGIME

We now consider the nonlinear region. For this purpose we go back to our original kinetic equation, (3.3). Since the condition $n_\varepsilon \gg 1$ definitely holds in this region, we rewrite the equation as

$$\dot{n}_1 = -W \int d\varepsilon_2 \, d\varepsilon_3 \, d\varepsilon_4 \chi\left(\frac{\varepsilon_2}{\varepsilon_1}, \frac{\varepsilon_3}{\varepsilon_1}, \frac{\varepsilon_4}{\varepsilon_1}\right) \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \times [n_1 n_2 n_3 + n_1 n_2 n_4 - n_1 n_3 n_4 - n_2 n_3 n_4]. \tag{4.1}$$

It can be concluded from the form of the collision integral that the effective kinetic time in this case is determined by

$$\tau_{eff}^{-1}(\varepsilon) \sim W n_\varepsilon^2 \varepsilon^2. \tag{4.2}$$

There is an influx of particles Q into the nonlinear region [see (3.14)]. As Zakharov has shown,[12] the equation $I = 0$ [where I is the collision integral in (4.1)] has the solution

$$n_\varepsilon = A/\varepsilon^{7/6}, \tag{4.3}$$

which corresponds to a steady-state particle flux. Substituting this solution into (4.2), we find

$$\tau_{eff}^{-1}(\varepsilon) \sim WA^2/\varepsilon^{1/3}, \tag{4.4}$$

The meaning here is enhancement of the collision processes with decreasing energy in both the incoming and outgoing terms. It is reasonable to assume here that the incoming flux behind the $\tilde{\varepsilon}(t)$ front has a Zakharov distribution, (4.3). The comparatively narrow particle distribution (of width $\sim \varepsilon_* \ll T$) at the boundary between the linear and nonlinear regions corresponds to the circumstance that most of the excess particles reach the nonlinear region in times shorter than τ_L. Once the distribution (4.3) has been established for these particles, the quantity A, which depends on the time in our case, can be found from the condition

$$\alpha \int_{\tilde{\varepsilon}_0}^{\varepsilon_*} \varepsilon^{1/2} d\varepsilon [A(t)/\varepsilon^{7/6}] \approx n_0. \tag{4.5}$$

Hence for $\tilde{\varepsilon}_0(t)$ much lower than ε_* we have

$$A(t) \approx (n_0/3\alpha)/(\varepsilon_*^{1/3} - \tilde{\varepsilon}_0^{1/3}(t)),$$

Since $\varepsilon_* \gg n_0 \widetilde{U}$, $A(t)$ varies only slightly until the boundary with the coherent region is reached. Note that the restructuring from the distribution (3.9) to (4.3) puts a substantial fraction of the particles in the tail of the distribution $(\varepsilon \sim \varepsilon_*)$.

Substituting the value found for A into (4.4), we can easily verify that

$$\tau_{eff}^{-1}(\varepsilon) \sim \tau_{\rm kin}^{-1}(\varepsilon_*/\varepsilon)^{1/3}.$$

Writing the kinetic equation (4.1) in the conventional form $\dot{n}_\varepsilon = I_+ - I_-$, we have $I_\pm \sim n_\varepsilon/\tau_{\rm eff}(\varepsilon)$, while we have $\dot{n}_\varepsilon \sim n_\varepsilon/\tau_{\rm kin}$. Consequently, the solution in (4.3), which corresponds to the condition $I \approx 0$, is of a self-consistent nature in the time-varying case under consideration here.

The value of the flux associated with the motion of the front is determined by an expression like (3.14):

$$Q = -3\alpha A (d\varepsilon_0^{1/3}(t)/dt). \tag{4.6}$$

Hence, under the assumption of a constant flux, we have (cf. Ref. 13)

$$\hat{\varepsilon}(t) \approx \varepsilon_*(1 - t/t_0)^3, \tag{4.7}$$

where

$$t_0 = n_0/Q \tag{4.8}$$

(t_0 is reckoned from the beginning of the nonlinear regime). The quantity t_0 determines the time at which the distribution front arrives in the coherent region. Using the value of the flux Q entering from the linear region [see (3.14)], we immediately conclude $t_0 \sim \tau_{\rm kin}$.

In the initial stage, the buildup of particles in the coherent region is described by

$$n_c \approx Q(t - t_0). \tag{4.9}$$

To describe the behavior of the particle density n_c in the coherent region after a long time, we need to consider the kinetic equation, allowing for the quasicondensate which arises. At the same time, we can conclude simply from expression (4.9), with (3.17) for the flux Q, that the time scale for the buildup of the quasicondensate from the nonlinear kinetic region is on the order of $\tau_{\rm kin}$. This point can be verified by examining the final stage of the buildup. Since we have $\tau_{\rm eff}^{-1}(\varepsilon) \sim \varepsilon^{-1/3}$, it is clear that this stage is associated with the direct arrival of particles with energies $\varepsilon \sim \varepsilon_*$ in the coherent region. An approximate equation for the buildup of the quasicondensate follows from (4.4):

$$\dot{n}_c \approx n_c W A^2/\varepsilon_*^{1/3}. \tag{4.10}$$

Here A is again found from normalization condition (4.5), but in this case n_0 should be replaced by the density of the excess particles $n_0 - n_c$ which remain outside the coherent region. It follows immediately from (4.10) that the time scale for the buildup of the quasicondensate in the final stage is determined by $\sim \tau_{\rm kin}$.

We thus reach the conclusion that the crossing of the nonlinear kinetic region and the arrival of most of the excess particles in the coherent region occur over a time τ_N whose scale is determined by $\tau_{\rm kin}$. Since we have $\tau_k \gg \tau_{\rm kin}$, all the excess particles are in the quasicondensate for essentially a time $\sim \tau_{\rm kin}$.

We have a comment here. Near the upper boundary ε_* of the nonlinear region the reciprocal of the effective time $\tau_{\rm eff}(\varepsilon)$ in (4.4) is on the order of $\tau_{\rm kin}$, if we use expression (3.12) for ε_* $(T \approx T_c)$. The meaning here is that the time required to reshape the distribution in the tail of the nonlinear region and the actual time required for the crossing of the nonlinear region by the front, $\hat{\varepsilon}(t)$, are determined by the same scale value $\tau_{\rm kin}$.

The results found in this section of the paper are essentially unrelated to the inequality $n_0 \ll n$. As n_0 increases, the boundary ε_* moves into the high-energy region, and the flux Q increases [see (3.14) or (3.17)]. The time scale τ_N is again determined by $\tau_{\rm kin}$. If $n_0 \sim n$ holds, then we have $\varepsilon_* \sim T$, and it becomes meaningless to distinguish a linear regime of the kinetic region. The picture of the nonlinear regime drawn above covers the entire range of kinetic energies, while the scale value τ_N is retained.

5. FINAL COMMENTS

Several general conclusions can be drawn from the results derived above. The time scale for the formation of a quasicondensate, i.e., for the formation of a narrow peak in the particle distribution near $\varepsilon = 0$, in which density fluctuations are suppressed, is finite. Although the width of the peak is finite, because of phase fluctuations, the quasicondensate has the same local correlation properties as those of a true condensate. This comment applies in particular to inelastic processes which depend on the presence of a condensate. The limiting time for the formation of a quasicondensate is $\tau_{\rm kin}$ in (1.3), which is the ordinary time scale between collisions in a gas. This time scale determines the time needed by the excess particles which are formed at $\varepsilon \sim T$, and which should form a condensate at equilibrium, to cross the energy interval between T and the boundary of the coherent region, $n_0 \widetilde{U}$. This time is in general the sum of times corresponding to two regimes. The first of these regimes, which can be clearly distinguished in the case $n_0 \ll n$, corresponds to energies in which the nonequilibrium part of the distribution function is small in comparison with the equilibrium part. The corresponding kinetic time τ_L is given by expression (3.18). Under the condition $n_0 \ll n$, this time is greater than $\tau_{\rm kin}$ by a factor of $\ln(n/n_0)$. In the second regime, in which a highly nonlinear kinetic situation is established by virtue of the condition $n_\varepsilon \gg \bar{n}_\varepsilon \gg 1$, the front of the distribution reaches the coherent region, again over a time $\sim \tau_{\rm kin}$. The time scale for the transition of the bulk of the particles from the nonlinear kinetic region into the coherent region is on the same order of magnitude. The particles which are to form the condensate thus reach the coherent region over a total time characterized by $\tau_{\rm kin}$.

If the bulk of the excess particles were found to lie in the coherent region simultaneously, the quasicondensate formation time would be characterized by τ_c in (1.2). This time is short in comparison with τ_{kin}. This circumstance supports the assertion that the overall process is determined by the time τ_{kin}. The formation of a true condensate with a long-range order in a macroscopic system may require times long in comparison with τ_c in the coherent region. In this case the time scale of the phase relaxation may be greater than τ_k, and in this sense it may become the limiting time.

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