The Virasoro algebra in integrable hierarchies and the method of matrix models

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The action of the Virasoro algebra on hierarchies of nonlinear integrable equations, and also the structure and consequences of Virasoro constraints on these hierarchies, are studied. It is proposed that a broad class of hierarchies, restricted by Virasoro constraints, can be defined in terms of dressing operators hidden in the structure of integrable systems. The Virasoro-algebra representation constructed on the dressing operators displays a number of analogies with structures in conformal field theory. The formulation of the Virasoro constraints that stems from this representation makes it possible to translate into the language of integrable systems a number of concepts from the method of the "matrix models" that describe nonperturbative quantum gravity, and, in particular, to realize a "hierarchical" version of the double scaling limit. From the Virasoro constraints written in terms of the dressing operators generalized loop equations are derived, and this makes it possible to do calculations on a reconstruction of the field-theoretical description. It is also indicated how hierarchies restricted by Virasoro constraints are related to nonlocal evolution equations. The reduction of the Kadomtsev–Petviashvili (KP) hierarchy, subject to Virasoro constraints, to generalized Korteweg–deVries (KdV) hierarchies is implemented, and the corresponding representation of the Virasoro algebra on these hierarchies is found both in the language of scalar differential operators and in the matrix formalism of Drinfel'd and Sokolov. The string equation in the matrix formalism does not replicate the structure of the scalar string equation. The symmetry algebras of the KP and N-KdV hierarchies restricted by Virasoro constraints are calculated: A relationship is established with algebras from the family $W_\infty(J)$ of infinite W-algebras. The method of dressing operators also makes it possible to introduce Virasoro superconstraints on the super-KP hierarchy that are consistent with all (even and odd) super-KP flows.

1. INTRODUCTION

A surprising feature of exactly integrable equations (besides, of course, the integrability itself) is the extremely wide diversity of their physical applications.[1)] Recently there has been a further addition to the latter: Integrable equations arise in nonperturbative two-dimensional quantum gravity, in a rather nontrivial manner as "renormalization group" evolutions of the coupling constants. Because there are infinitely many of the latter, infinite hierarchies[1–3] of integrable equations arise.

The manner in which nonperturbative two-dimensional quantum gravity and the matter interacting with it generate integrable hierarchies was first understood with the aid of matrix models.[4–6] The method of matrix models has made it possible to show that the resulting solutions of the integrable hierarchies are subject to so-called Virasoro constraints—dominant-weight conditions in relation to a certain Virasoro algebra acting on functions of the coupling constants. We recall, for those who are not specialists in quantum field theory, that the Virasoro algebra is one of the fundamental constructions of two-dimensional physics, and is at the same time a nontrivial object of study for mathematicians (because of the felicitous combination, in the infinite-dimensional situation, of representation theory and differential geometry). The study of dynamical realizations of the Virasoro algebra forms the content of conformal field theory.

The Virasoro constraints on integrable hierarchies have paramount significance, since they make it possible to single out the partition function of matrix models from among all the τ-functions,[6–8] and contain, in essence, the nonperturbative dynamics of the model. In the present article we first study the somewhat more general question of how, in general, the Virasoro algebra acts on integrable hierarchies, and then go over to the Virasoro constraints in the form of the requirement of invariance under this action. As a result of our analysis of the structure of the Virasoro constraints the possibility arises of regarding the Virasoro constraints on arbitrary integrable hierarchies as the first principle of a nonperturbative description of a certain class of models of quantum gravity.

The fact that the Virasoro algebra should act in some way on integrable hierarchies is already clear from the construction of Krichever,[9] according to which Riemann surfaces furnish solutions of integrable hierarchies, while the Virasoro algebra undoubtedly acts on Riemann surfaces.[10] We shall consider as an example the Kadomtsev–Petviashvili (KP) hierarchy[1] or its various reductions to generalized (modified) Korteweg–deVries [(m)KdV] hierarchies,[3] describable in terms of (scalar) pseudodifferential (ψ Diff) operators. The action of the Virasoro algebra of Ref. 10 is then carried over on to ψ Diff operators in accordance with the diagram

$$\begin{array}{ccc} \text{Riemann surface} & \longrightarrow & \psi\ \text{Diff operator} \\ \text{BMS (Ref. 10)}\ \Big\downarrow & & \Big\downarrow\ \text{Ref. 11} \\ \text{Riemann surface} & \longrightarrow & \psi\ \text{Diff operator.} \end{array} \qquad (1.1)$$

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This is a diagram "on the mass shell," since the horizontal arrows give solutions, and therefore the right vertical arrow carries solutions over into solutions. We shall see, however, that the latter mapping can also be defined "off the mass shell," i.e., without regard to any particular solutions of the hierarchy.

In other integrable hierarchies the ψ Diff operators are replaced by another (as a rule, infinite-dimensional) Lie algebra $\mathfrak{g}$ (e.g., an algebra of $\infty \times \infty$ matrices). Such an algebra $\mathfrak{g}$ is used in the construction of the Lax representation (the Lax operators are elements of $\mathfrak{g}$), and, on the other hand, plays the role of the phase space of the hierarchy, since a feature of the Hamiltonian interpretation of integrable systems is the circumstance that it is precisely the Lax representations that acquire a Hamiltonian meaning (for example, Gel'fand–Dikiĭ structures on algebras of (pseudo) differential operators).

The description of a hierarchy in the language of flows in phase space is in a certain sense supplementary to the description in terms of the τ-function and Hirota bilinear relations. An advantage of the former is the familiar simplifications that arise from the fact that with the Lax operators one can associate a hierarchy wave function that is, in essence, a linear object. The correspondence with the description in the language of the τ-function is achieved in several steps: The wave function is constructed from the τ-function, and knowledge of the wave function makes it possible to find the dressing operator, which, in its turn, is used to construct the Lax operator:

$$\tau\text{-function} \mapsto \text{wave function} \mapsto \text{dressing operator} \mapsto \text{Lax operator}. \qquad (1.2)$$

The idea of the approach that we are developing is to trace the Virasoro algebra along the above arrows, and then to note that the Virasoro generators that are obtained as a result can be represented in a sufficiently universal (in the sense of lack of dependence on the specific hierarchy) manner, making it possible to postulate these generators on a whole series of integrable hierarchies, including those for which the scheme (1.2) is unknown or not so simple.

Recalling also that the solutions of the integrable equations are given by Riemann surfaces (with auxiliary Krichever data on them), we obtain the chain of mappings

$$\begin{matrix}\text{Riemann}\\ \text{surface}\end{matrix} \mapsto \tau\text{-function} \mapsto \begin{matrix}\text{wave}\\ \text{function}\end{matrix} \mapsto \begin{matrix}\text{dressing}\\ \text{operator}\end{matrix} \mapsto \begin{matrix}\text{Lax}\\ \text{operator}\end{matrix}. \qquad (1.3)$$

Riemann surfaces of various generi are thus described in a completely unified manner, and this is the reason behind a number of attempts to reformulate two-dimensional field theory in the language of integrable equations. As an example of a 2-geometric object that has an extremely natural description in terms of integrable hierarchies, we shall consider the coordinate that appears in the set of Krichever data. From the point of view of "two-dimensional" field theory, this is in fact the coordinate on which the operator insertions depend, while in the language of the hierarchy it is the eigenvalue of the Lax operator:

$$Q\psi(t, z) = z\psi(t, z), \quad Q \in \mathfrak{g} \qquad (1.4)$$

(here ψ is the wave function). Following the coordinate, we should like to "transfer from the right- to the left-hand side of Eq. (1.4)" the energy-momentum tensor. The strategy, as already stated, will consist precisely in the use of the mappings (1.3). As the starting conformal theory we consider the bc-theory of spin J (Ref. 12). Then the first result can be formulated as follows:

On the phase space[2)] there exists a family of vector fields $\hat{\mathfrak{T}}(u)$ that depend on the formal parameter u and

a) satisfy the Virasoro algebra (without a central extension)

$$[[\hat{\mathfrak{T}}(u), \hat{\mathfrak{T}}(v)]] = \frac{1}{v}\frac{\partial}{\partial v}\delta(u, v)\cdot\hat{\mathfrak{T}}(u) - \frac{1}{u}\frac{\partial}{\partial u}\delta(u, v)\cdot\hat{\mathfrak{T}}(v), \qquad (1.5)$$

where

$$\delta(u, v) = \sum_{n\in\mathbb{Z}} \frac{u^n}{v^n}$$

is a formal δ-function (and $[[,]]$ denotes a commutator of vector fields on $\mathfrak{g}$, and not simply a commutator of elements from $\mathfrak{g}$);

b) are tangent to the space of solutions of the hierarchy;

c) with a constraint on the Krichever locus, make the following diagram commutative:

$$\begin{array}{ccc} \text{Riemann surface} & \longrightarrow & \text{Lax operator} \\ \Big\downarrow T(u) & & \Big\downarrow \hat{\mathfrak{T}}(u) \\ \text{Riemann surface} & \longrightarrow & \text{Lax operator,} \end{array} \qquad (1.6)$$

where by the Riemann surface we mean in reality the Krichever data, and the left vertical arrow is given by the action (constructed in Ref. 10) of the vector field $\Sigma_{n\in Z} u^{-n-2} z^{n+1}\partial/\partial z$ (we recall that u is a formal parameter).

Furthermore, as we shall see, the structure of $\hat{\mathfrak{T}}(u)$ replicates the structure of the energy-momentum tensor $(1-J)\partial b\cdot c - Jb\cdot\partial c$ of the bc-theory itself, although the ψ Diff analogs of the fields b and c are in no way fermions. (The trick that "corrects the statistics" involves an analog of the operation $\cdot$ in the middle.) This result is extremely general, and is related to general properties of integrable hierarchies—more specifically, to their r-matrix structure.[13] We recall that the latter is approximately equivalent to specifying, for the Lie algebra $\mathfrak{g}$, its decomposition as a sum of two subalgebras:[3)] $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ (Ref. 13). In the r-matrix formalism, the Virasoro algebra acts on the Lax operators via the differentiation ad$\mathfrak{T}$ of the Lie algebra $\mathfrak{g}$, where $\mathfrak{T}$ has a universal form in terms of the dressing operators, the r-matrix, the character of the subalgebra $\mathfrak{a}$, and certain other data pertaining to the algebra of differentiations (Der$\mathfrak{g}$). In the present paper, which has a more physical orientation, we shall not describe this general construction, but confine ourselves to a number of meaningful examples of importance in applications.

The study of the structure of the action of the Virasoro algebra on integrable hierarchies has an important application to the nonperturbative description of two-dimensional quantum gravity and the matter interacting with it—an area familiar under the general name of "matrix models."[6–8,14,15] Knowledge of the explicit form of the generating expression $\mathfrak{T}(u)$ for the Virasoro generators is used in the imposition of the Virasoro constraints on integrable hierarchies. As is well known, it is precisely hierarchies restricted by Virasoro constraints that arise from matrix models: The partition function of a matrix model (up to the double scaling limit) is given by the τ-function of a certain integrable hierarchy:[7]

$$\tau(t)=\int (dM)\exp\Big(-\mathrm{Tr}\sum_{r\geq 1} t_r M^r\Big), \tag{1.7}$$

where the integration is performed over the space of all (say, Hermitian) matrices. The times (t) are the coupling constants of the theory (more precisely, of the discretization of the theory). Shifting M in (1.7) by $M\mapsto M+\varepsilon M^{n+1}$ $(n\geqslant -1)$, we discover[8] that the τ-function is annihilated by the $(n\geqslant -1)$-generators of the Virasoro algebra that are of the type written out below in Eq. (2.16):

$$\mathscr{L}_n\tau(t)=0. \tag{1.8}$$

Again transferring these constraints on to the phase space, we obtain practically universal[13] constraints on the dressing operators of the integrable hierarchy. For example, for the Toda hierarchy, which plays an important role both in matrix models and in our analysis below, the Virasoro constraints have the form[16]

$$\mathfrak{L}_n\equiv\Big(W\Big\{[J(n+1)+\hat{p}]\Lambda^n+\sum_{r\geq 1} r x_r\Lambda^{r+n}\Big\}W^{-1}\Big)_-=0, \tag{1.9}$$

where $n\geqslant -1$. The details of the notation pertaining to the Toda hierarchy will be explained in the appropriate place, but meanwhile we shall attempt to understand what the Virasoro constraints (1.9) could mean for an arbitrary integrable hierarchy. The left-hand side of Eq. (1.9) is an element of a Lie algebra $\mathfrak{g}$, and $(\ldots)_-$ is then a projection on to one of the subalgebras appearing in the decomposition $\mathfrak{g}=\mathfrak{a}+\mathfrak{b}$ (Ref. 13). Furthermore, $W\ldots W^{-1}$ is the adjoint action by the dressing operator, and Λ is the character (one-point coadjoint orbit) of one of the subalgebras. Finally, $\hat{p}$ is chosen in such a way that

$$\Lambda\hat{p}-\hat{p}\Lambda=\Lambda. \tag{1.10}$$

We have enough of these general properties to verify the fulfillment of the commutation relations of the Virasoro algebra for the vector fields $\hat{\mathfrak{L}}_n$ associated with the left-hand sides in (1.9) (although, of course, the definition of both the operator $\hat{p}$ and expressions of the type $\hat{p}\Lambda^n$ in the general case imposes requirements on the Lie algebra $\mathfrak{g}$).

Therefore, Virasoro constraints in a form of the type (1.9) can be postulated simply for a rather wide class of integrable systems, without raising the question of the existence of an intrinsically matrix variant of the theory. The correspondence with a field-theoretical description can then be sought as follows. Knowledge of the explicit algebraic structure of the Virasoro constraints makes it possible, as we shall see, to construct their generating function. The latter, after a Laplace transformation, can be rewritten in the form of a relation that has the structure of recursion/loop equations.[17,18,6,5] Recursion equations are one of the formulations of two-dimensional topological theories, i.e., theories interacting with gravity. The generalized recursion equations obtained from the integrable hierarchies can therefore be used to restore the field-theoretical description. Although the latter problem already lies beyond the scope of this paper, we should like to draw attention to the correspondence

$$\begin{pmatrix}\text{theory interacting with}\\ \text{two-dimensional gravity}\end{pmatrix} \leftrightarrow\begin{pmatrix}\text{integrable hierarchies, restricted}\\ \text{by Virasoro constraints}\end{pmatrix}. \tag{1.11}$$

Our aim in this paper is to develop a formalism for the description of the right-hand side of (1.11).

The content of the paper is as follows. In Sec. 2 we start from the KP hierarchy and, after introducing the basic notation, recall the results of Ref. 11 on Virasoro-type action. The latter demonstrates analogies with the formulation on the world sheet familiar from conformal field theory, and these analogies, which are useful for the following, are discussed in Sec. 3 (among them is the "hierarchical" version of the boson–fermion correspondence).

Another example of a hierarchy with Virasoro generators of the type (1.9) besides the KP hierarchy is the important case of the (two-dimensional) Toda lattice hierarchy.[2] The necessary notation is introduced in Sec. 4, and in Sec. 5 we calculate the action of the Virasoro algebra on the corresponding spaces of $\infty\times\infty$ matrices. The results, in fact, structurally repeat those obtained in Sec. 2, and therefore the correspondences noted in Sec. 3 could be reproduced almost word-for-word in application to the Toda hierarchy (the reader, of course, will be spared this).

An important example of the reduction of the Toda hierarchy is provided by the N-periodic Toda hierarchies. In Sec. 6 we elucidate which of the generators of the Virasoro algebra can be restricted to the N-periodic case. In addition, as is well known, an N-periodic Toda hierarchy admits a formulation related to the current algebra $sl(N)$. The action of the Virasoro algebra is also carried over to this case, and this has required a certain special realization of the commutation relations (1.10). The spirit of this $N\times N$ formulation consists in the fact that it clarifies the group-theoretical nature of the periodic Toda hierarchy, and makes it possible, at the cost of only slight effort, to replace $sl(N)$ by other Kac–Moody algebras and thereby to construct in a systematic way the generalized Toda hierarchies corresponding to these algebras.

In the next sections we specifically consider applications to matrix models, i.e., to integrable hierarchies restricted by Virasoro constraints. In Sec. 7 we investigate the question of the correspondence of the Virasoro constraints that arise at the discrete level (i.e., for lattice hierarchies[19] of the Toda type)[20] to the "continuum" Virasoro constraints from Ref. 6.

It turns out that the two principal examples of hierarchies restricted by Virasoro constraints—the KP hierarchy and the Toda hierarchy—are related by a certain scaling limit. This limit is taken extremely naturally in the language of dressing operators, and carries certain combinations of "discrete" Virasoro constraints into their continuous variant. In this way, the scaling limit proposed in Sec. 7 turns out to be the "hierarchical" version of the so-called double scaling limit that forms the basis of the interpretation of the matrix models.[4,5,20]

Next, in Sec. 8, we study the Virasoro-constrained KP hierarchy that is obtained. It is shown there that the analog (obtained in Sec. 3) of the bosonized representation for the energy-momentum tensor makes it possible to interpret the Virasoro constraints as an (operator) version of the recursion/loop equation that arises in the context of topological field theories.[6,17] Here, it is pertinent to stress once again the possibility of generalizations of the construction of this equation to a broad class of integrable hierarchies that admit an r-matrix formulation. We also note a connection with the nonlocal integrable equations considered recently in Ref. 21.

The next step in the study of the recursion relations obtainable in the context of integrable hierarchies is the reduction (performed in Sec. 9) to generalized N-KdV hierarchies;[1] the latter, strictly, have also been noted in matrix models. We elucidate how and in what sense the KP hierarchy subject to Virasoro constraints is reduced to generalized N-KdV hierarchies, and what restrictions on the latter arise in the process. Here we investigate the complete symmetry algebra of the hierarchies that are subject to Virasoro constraints. This question has been considered recently in a number of papers,[22,23] with somewhat contradictory conclusions. The advantage of our approach, using dressing operators, is based on knowledge of the explicit forms of the Virasoro generators, which not only makes the conclusions very transparent but also opens up possibilities for further generalizations. As regards the symmetry algebra, for the KP hierarchy we have ascertained that it coincides with a Borel subalgebra of the recently constructed algebra $W_\infty(J)$ (Ref. 24). For N-reduced hierarchies we also discuss the appearance of the current algebra $sl(N)$ in the language of dressing operators.[25]

There is also an $sl(N)$ formulation for the N-reduced KdV hierarchies. This formulation constitutes the content of the fundamental work of Drinfel'd and Sokolov,[3] the "physical" part of which is, in particular, the theory of W-algebras.[4)] In Sec. 10 the Virasoro generators and constraints are "raised" from scalar generators and constraints to the corresponding matrix differential operators. This makes it possible to clarify the meaning and formulation of the matrix string equation, which, contrary to some expectations, turns out to be different from the scalar string equation.

Finally, in Sec. 11, we consider the supersymmetric KP hierarchy. For supermatrix models a naive matrix formulation has been found to be inadequate,[26] but, of course, it is indubitable that both supersymmetric theories interacting with supergravity and supersymmetric hierarchies themselves exist. In this situation it is all the more natural to use the formalism of dressing operators. As we shall see, this formalism makes it possible to introduce Virasoro constraints that are consistent with all (even and odd) equations of the super-KP hierarchy.

2. ACTION OF THE VIRASORO ALGEBRA ON THE KP HIERARCHY

We shall work with a KP hierarchy describable in terms of pseudodifferential operators.[1] (We note in passing that there exists an alternative description in the spirit of the $n=\infty$ KdV hierarchy, i.e., in terms of $\infty\times\infty$ matrices.[27])

We recall that the KP hierarchy is formulated as an infinite system of pairwise-commuting evolution equations

$$\frac{\partial K}{\partial t_r}=-(KD^rK^{-1})_-K, \quad r\geqslant 1 \tag{2.1}$$

on the coefficients $w_n(x\equiv t_1,t_2,t_3,...)$ of a ψ Diff operator (more precisely, of a ψ Diff symbol) K of the form

$$K=1+\sum_{n\geqslant 1} w_nD^{-n}. \tag{2.2}$$

We use the notation

$$D=\frac{\partial}{\partial x} \tag{2.3}$$

and identify $x\equiv t_1$ in accordance with Eq. (2.1) for $r=1$:

$$\frac{\partial K}{\partial t_1}=-(KDK^{-1})K+(KDK^{-1})_+K=-KD+DK=\frac{\partial K}{\partial x}. \tag{2.4}$$

The operator K will also be called a dressing operator.

The commutativity of the flows determined by Eqs. (2.1) is ensured by the classical Yang–Baxter equation, which is satisfied by the "r matrix"

$$r:\psi\,\mathrm{Diff}\to\psi\,\mathrm{Diff}:A\mapsto A_- \tag{2.5}$$

is the projector on to the integral part. (Correspondingly, the standard notation $(...)_+$ indicates that the purely differential part of the pseudodifferential operator is singled out.)

We introduce the "matrix-model potential"

$$\xi(t,z)=\sum_{r\geqslant 1} t_rz^r \tag{2.6}$$

[henceforth, t in the argument of a function will denote the set $(t_1,t_2,...)$]. The wave function and conjugate wave function are defined by the equalities

$$\psi(t,z)=Ke^{\xi(t,z)}, \quad \psi^*(t,z)=K^{*-1}e^{-\xi(t,z)}, \tag{2.7}$$

where K^* is formally conjugate to K. It is obvious that ψ satisfies Eq. (1.4), in which

$$Q=KDK^{-1}. \tag{2.8}$$

We also define w and w^* by the formulas

$$\psi(t,z)=e^{\xi(t,z)}w(t,z), \quad \psi^*(t,z)=e^{-\xi(t,z)}w^*(t,z). \tag{2.9}$$

Like ψ and ψ^*, w and w^* will also be called wave functions. Obviously [compare with Eq. (2.2)],

$$w(t,z)=1+\sum_{n\geqslant 1} w_n(t)z^{-n}.$$

A basic property of the wave functions is their relation to the τ-function:

$$\psi(t,z)=\frac{\Gamma(t,z)\tau(t)}{\tau(t)}=\frac{\tau(t-[z^{-1}])}{\tau(t)}e^{\xi(t,z)}, \tag{2.10}$$

$$\psi^*(t,z)=\frac{\Gamma^*(t,z)\tau(t)}{\tau(t)}=\frac{\tau(t+[z^{-1}])}{\tau(t)}e^{-\xi(t,z)}, \tag{2.11}$$

where

$$\Gamma(t,z)=e^{\xi(t,z)}\exp\left\{-\sum_{r\geqslant 1}\frac{1}{r}z^{-r}\frac{\partial}{\partial t_r}\right\},$$

$$\Gamma^*(t,z)=e^{-\xi(t,z)}\exp\left\{\sum_{r\geqslant 1}\frac{1}{r}z^{-r}\frac{\partial}{\partial t_r}\right\} \tag{2.12}$$

and, correspondingly,

$$t\pm[z^{-1}]=(t_1\pm z^{-1},\ t_2\pm{}^1\!/_2z^{-2},\ t_3\pm{}^1\!/_3z^{-3},\ \ldots). \tag{2.13}$$

We shall need the following combination of the mappings (1.3) and (1.4):

$$\begin{array}{ccccc} \text{Riemann surface} & \mapsto & \tau\text{-function} & \mapsto\psi & \text{Diff operator} \\ \downarrow T(u) & & \downarrow \mathcal{T}(u) & & \downarrow \mathfrak{T}(u) \\ \text{Riemann surface} & \mapsto & \tau\text{-function} & \mapsto\psi & \text{Diff operator} \end{array} \tag{2.14}$$

(this diagram is projectively commutative). By a Riemann surface we shall mean a set of Krichever data in which the linear bundle is the bundle of J-differentials and the trivialization is determined by a parameter a_0 [see Ref. 11, in which $a_0 = N + \frac{1}{2}$ and $J \leftrightarrow (1 - J)$. Then

$$\mathcal{T}(u) = \sum_{p\in\mathbb{Z}} u^{-p-2}\mathcal{L}_p \tag{2.15}$$

with the usual expressions for the generators of the Virasoro algebra:

$$\mathcal{L}_{p>0} = \frac{1}{2}\sum_{k=1}^{p-1} \frac{\partial}{\partial t_{p-k}}\frac{\partial}{\partial t_k} + \sum_{k\geq 1} kt_k\frac{\partial}{\partial t_{p+k}} + \left(a_0 + \left(J - \frac{1}{2}\right)p\right)\frac{\partial}{\partial t_p},$$

$$\mathcal{L}_0 = \sum_{k\geq 1} kt_k\frac{\partial}{\partial t_k} + \frac{1}{2}a_0^2 - \frac{1}{24}, \tag{2.16}$$

$$\mathcal{L}_{p<0} = \sum_{k\geq 1} (k-p)t_{k-p}\frac{\partial}{\partial t_k} + \frac{1}{2}\sum_{k=1}^{-p-1} k(-p-k)t_k t_{-p-k} + \left(a_0 + \left(J - \frac{1}{2}\right)p\right)(-p)t_{-p},$$

which form the algebra

$$[\mathcal{L}_p, \mathcal{L}_q] = (p-q)\mathcal{L}_{p+q} + \delta_{p+q,0}(-p^3)\left(J^2 - J + \frac{1}{6}\right), \tag{2.17}$$

from which one can see, in particular, the role that is played by the parameter J. [By the shift $\mathcal{L}_0 \mapsto \mathcal{L}_0 - \frac{1}{2}(J^2 - J + \frac{1}{6})$ in (2.17) one reproduces the standard central term $-\delta_{p+q,0}(p^3 - p)(J^2 - J + \frac{1}{6})$.]

Thus, we deform the τ-function by means of the Virasoro generators:

$$\tau(t) \mapsto \tau(t) + \delta\tau(t) = \tau(t) + \mathcal{T}(u)\tau(t) \tag{2.18}$$

and apply Eqs. (2.7)–(2.10) with the new τ-function. As a result we obtain a new dressing operator $K - \mathfrak{T}(u)K$, where[11] (see also Ref. 28 for $J = 1$)

$$\mathfrak{T}(u) = (1-J)\frac{\partial\psi(t,u)}{\partial u}\circ D^{-1}\circ\psi^*(t,u) - J\psi(t,u)\circ D^{-1}\circ\frac{\partial\psi^*(t,u)}{\partial u} \tag{2.19}$$

(and, for simplicity, we have chosen $a_0 = -\frac{1}{2}$). Upon transfer of the operator D^{-1} to the right there appears an infinite "tail" of negative powers of differentiation, in accordance with the formula

$$D^{-1}\circ f(x) = f(x)D^{-1} + \sum_{n=1}^{\infty}(-1)^n\partial^n f D^{-n-1}. \tag{2.20}$$

In the action of the operator $\mathfrak{T}(u)$ on arbitrary functions of K one must regard $\mathfrak{T}(u)K$ as the components X^i of a vector field $X^i\partial/\partial x_i$. In other words, we define a vector field as the differentiation of a ring of functions of K:

$$\hat{\mathfrak{T}}(u)\cdot F(K) = \frac{d}{d\varepsilon}F(K + \varepsilon\mathfrak{T}(u)K)\big|_{\varepsilon=0}. \tag{2.21}$$

The functions ψ and ψ^* in (2.19) are, of course, functions of K through Eq. (2.7).

We can now forget about the solution from which we started, and take the derivation of $\mathfrak{T}(u)$ as simply a way of guessing the form of Eq. (2.19) [and of proving the assertion (c) from the Introduction]. Equations (1.5) follow as a result of a simple calculation,[11] as does the assertion (b). The latter is valid irrespective of whether the operator K is a solution of the KP hierarchy.

Starting from the variation $\delta K = -\mathfrak{T}(u)K$, it is easy to determine the variation of the wave function: Eqs. (2.7) give

$$\delta\psi(t,z) = \hat{\mathfrak{T}}(u)\cdot\psi(t,z) = -(1-J)\omega(z,u,t)\frac{\partial\psi(t,u)}{\partial u} + J\frac{\partial\omega(z,u,t)}{\partial u}\psi(t,u). \tag{2.22}$$

$$\omega(z,u;t) = \partial^{-1}(\psi(t,z)\psi^*(t,u)) = \int^x \psi(x', t_{\geq 2}, z)\psi^*(x', t_{\geq 2}, u)dx'. \tag{2.23}$$

These formulas have been reproduced in a more geometrical context in Ref. 29.

3. CONFORMAL THEORY ON INTEGRABLE HIERARCHIES?

In this section we shall discuss certain consequences of the formulas (2.19), (2.22), and (2.23) obtained, and also formal aspects of them that are important for what follows.

Equation (2.19) displays a remarkable similarity to the energy-momentum tensor of a bc-system of spin J (Ref. 12):

$$T = -Jb\cdot\partial c + (1-J)\partial b\cdot c. \tag{3.1}$$

The fact that, unlike (3.1), the right-hand side of (2.19) contains only bosonic quantities "is compensated" by the presence of the operator D^{-1} between the two wave functions.

Moreover, Eq. (2.22) can be regarded as the abstract version of the usual operator product

$$T(u)b(z) = \frac{Jb(u)}{(u-z)^2} + \frac{(1-J)\partial b(u)}{(u-z)}.$$

since $\omega(z,u;t)$ is the analog of the Cauchy kernel off the equation of motion, i.e., 'off shell": In fact, "on shell," i.e., for a given algebro-geometric solution,[5)] when z and u become genuine coordinates on the Riemann surface, we have

$$\omega(z,u;t) = \frac{1}{u-z} + \ldots, \tag{3.2}$$

where ... denotes terms that are regular at $z = u$.

Another important object in bc-theory is the ghost current $j = -bc$. From the formulas of Ref. 11 it follows that its KP analog has the form

$$\mathfrak{j}(u)=\psi(t,u)\circ D^{-1}\circ\psi^*(t,u). \tag{3.3}$$

This can also be written in the form

$$\mathfrak{j}(u)=\left(K\frac{1}{u}\delta(u,D)K^{-1}\right)_- =\frac{1}{u}\delta(u,Q)_- \tag{3.4}$$

[where Q is defined in Eq. (2.8)].

Next, the ψ Diff analog of one further fundamental operator product

$$j(u)b(z)=\frac{b(u)}{u-z},$$

namely,

$$\hat{\mathfrak{j}}(u)\psi(t,z)=\omega(z,u)\psi(t,u) \tag{3.5}$$

also follows rapidly from Eq. (3.3).

The observed similarity with the structures of conformal field theory leads us to pose the question of the ψ Diff description of *bosonization* in terms of the ψ Diff analog of a scalar field. In fact, whereas Eq. (2.19) suggests "fermionized" analogies, it can be rewritten identically in the form

$$\mathfrak{T}(u)=\frac{1}{2}\left(KP\frac{1}{u}\delta(u,D)K^{-1}+K\frac{1}{u}\delta(u,D)PK^{-1}\right)_- -\frac{q}{2}\frac{\partial}{\partial u}\left(\frac{1}{u}\delta(u,Q)_-\right),\quad q=2J-1, \tag{3.6}$$

where the operator

$$P=\sum_{r\geq 1} rt_rD^{r-1} \tag{3.7}$$

represents, on ψ Diff operators, the derivative with respect to the spectral parameter (and therefore is the principal player in the Douglas equations[5]). We shall call Eq. (3.6) the "bosonized" form of the energy-momentum tensor on the hierarchy. It is convenient to use it in the calculation of the commutators of $\hat{\mathfrak{T}}(u)$ with other vector fields. Indeed, for any vector fields [as in Eq. (2.21)] $\hat{\mathfrak{a}}$ and $\hat{\mathfrak{b}}$ of the form

$$\hat{\mathfrak{a}}=\mathfrak{a}K\frac{\delta}{\delta K},\quad \hat{\mathfrak{b}}=\mathfrak{b}K\frac{\delta}{\delta K},\quad \mathfrak{a}=(KAK^{-1})_-,\quad \mathfrak{b}=(KBK^{-1})_- \tag{3.8}$$

we find

$$\begin{aligned}\hat{\mathfrak{a}}\hat{\mathfrak{b}}K&=\hat{\mathfrak{a}}[(KBK^{-1})_-K]\\&=((KAK^{-1})_-KBK^{-1})_-K-(KBK^{-1}(KAK^{-1})_-)_-K\\&\quad+(KBK^{-1})_-(KAK^{-1})_-K\\&=((KAK^{-1})_-KBK^{-1})_-K-(KBAK^{-1})_-K\\&\quad+((KBK^{-1})_-KAK^{-1})_-K,\end{aligned}$$

whence follows the useful relation

$$[[\hat{\mathfrak{a}},\hat{\mathfrak{b}}]]K=(K[A,B]K^{-1})_-K. \tag{3.9}$$

Using this for commutation of the energy-momentum tensor $\hat{\mathfrak{T}}(u)$ and the current $i(\hat{v})$, we substitute, respectively,

$$A=\left(-J\frac{1}{u}\frac{\partial}{\partial u}+\frac{1}{u}\frac{\partial\xi(t,u)}{\partial u}\right)\delta(u,D),\qquad B=\frac{1}{v}\delta(v,D), \tag{3.10}$$

so that

$$\begin{aligned}[A,B]&=\frac{1}{uv}[x,\delta(v,D)]\delta(u,D)=\frac{1}{uD}\frac{\partial}{\partial v}\delta(v,D)\delta(u,D)\\&=\frac{1}{u^2}\frac{\partial}{\partial v}\delta(v,D)\delta(u,D)\end{aligned} \tag{3.11}$$

and

$$[[\hat{\mathfrak{T}}(u),\hat{\mathfrak{j}}(v)]]=\frac{1}{u}\frac{\partial}{\partial v}\delta(u,v)\hat{\mathfrak{j}}(u). \tag{3.12}$$

Like (1.5), Eq. (3.12) is fulfilled "off the mass shell," and duplicates the standard commutation relations.

We note, however, that although the energy-momentum tensor written in the form (3.6) resembles the well known construction

$$T=\frac{1}{2}jj-\frac{q}{2}\partial j, \tag{3.13}$$

it is nevertheless not expressed precisely by this formula in terms of the current $\hat{\mathfrak{j}}$ from (3.4), since all modes of the current $\hat{\mathfrak{j}}$ are mutually [[,]]-commuting, so that the "compensating" presence of the operator P is necessary. In an analogous way, the appearance of P is unavoidable in the construction of a "bosonized" representation for the wave functions ψ and ψ^*.

We recall that the analogs of ψ and ψ^* on the complex plane, i.e., the fields b and c, are expressed in terms of the current j by

$$b=e^{-\varphi},\quad c=e^{\varphi},\quad \partial\varphi=j. \tag{3.14}$$

However, even on Riemann surfaces of the leading generi, these simple formulas are modified because of the fact that the field φ is no longer uniquely determined by the current j as $\varphi(x)\sim\int^x j$, since the integral executes jumps around the homologies. The Baker–Akhiezer mechanism copes with this ambiguity by giving to the naive expression

$$b(u)c(v)\propto\exp\int_u^v j \tag{3.15}$$

an invariant meaning when it is introduced into the background of a certain operator $\mathscr{B}$, the basic ingredient of which is the operator θ-function that depends on the b-periods of the operator current j (Ref. 31):

$$\mathscr{B}\propto\theta\left(\ldots+\oint_b j\right). \tag{3.16}$$

Upon normal ordering of the product (3.15) and (3.16) the argument of the θ-function is shifted as follows: $\oint_b j\mapsto\oint_b j+\mathbf{v}-\mathbf{u}$ (where $\mathbf{v}-\mathbf{u}$ denotes a Jacobi mapping), thereby supporting the Baker–Akhiezer structure $(\exp\int_u^v j)\theta(\mathbf{v}-\mathbf{u}+\oint_b j)$.

The moral of this recollection is that bosonization presupposes the presence of objects of two kinds: a background operator $\mathscr{B}$, representing the Riemann surface as a whole with the minimum number of operator insertions on it, and additional insertions. The former correspond to the operators Q (or K), while the latter correspond to the vector fields on the space of these operators.

The elementary neutral insertion $b(u)c(v)$ is described by the vector field

$$\hat{\mathfrak{D}}(u,v)=\mathfrak{D}(u,v)K\frac{\delta}{\delta K},$$

$$\mathfrak{D}(u,v)=\left(Ke^{(u-v)P}\frac{1}{v}\delta(v,D)K^{-1}\right)_{-}. \tag{3.17}$$

This expression, which appeals for a "bosonic" interpretation, can also be rewritten in a "fermionized" form: By direct application of the Campbell–Hausdorff formula we find

$$e^{zP}\delta(v,D)=e^{\xi(t,v+z)-\xi(t,v)}\delta(v,D) \tag{3.18}$$

and, furthermore,

$$Ke^{zP}\delta(v,D)=\sum_{n\geq0}w_n(t)D^{-n}\circ\exp\times\left\{xz+\sum_{r\geq1}t_r((v+z)^r-v^r)\right\}\delta(v,D)$$

$$=e^{\xi(t,v+z)-\xi(t,v)}\sum_{n\geq0}w_n(t)(D+z)^{-n}\delta(v,D)$$

$$=\psi(t,v+z)e^{-\xi(t,v)}\delta(v,D). \tag{3.19}$$

Applying the analogous arguments to $\delta(v,D)K^{-1}$, we finally obtain

$$\mathfrak{D}(v+z,v)=\left(Ke^{zP}\frac{1}{v}\delta(v,D)K^{-1}\right)_{-}$$

$$=\psi(t,v+z)e^{-\xi(t,v)}\frac{1}{D-v}e^{\xi(t,v)}\psi^{*}(t,v)$$

$$=\psi(t,v+z)\circ D^{-1}\circ\psi^{*}(t,v), \tag{3.20}$$

which is the "fermionized" version of (3.19) and shows that $\mathfrak{D}(u,v)$ indeed describes the insertion of $b(u)c(v)$ into the background of the operator $Q=KDK^{-1}$.

It is perhaps worth mentioning again that it is not necessary to assume that the equations of the KP hierarchy are fulfilled along times t_r for $r\geqslant2$; the constructions given above pertain simply to the space of ψ Diff operators with respect to x.

It is also instructive to calculate the behavior of the "bc insertion" (3.19) under the action of the Virasoro algebra; commuting $\hat{\mathfrak{T}}(z)$ with $\mathfrak{D}(u,v)$, we obtain two groups of terms, one of which contains $\delta(z,u)$ while the other contains $\delta(z,v)$. The terms of the first group have the form

$$\left[\frac{1}{z}\frac{\partial\ln\psi(t,z)}{\partial z}\delta(z,u)-J\frac{\partial}{\partial z}\left(\frac{1}{z}\delta(z,u)\right)\right]\mathfrak{D}(u,v), \tag{3.21}$$

which can be rewritten as

$$\left[\delta(z,u)\frac{1}{u}\frac{\partial\psi(t,u)}{\partial u}-J\frac{\partial}{\partial z}\left(\frac{1}{z}\delta(z,u)\right)\psi(t,u)\right]\circ\times D^{-1}\circ\psi^{*}(t,v), \tag{3.22}$$

thereby attesting that the variation reduces to a natural variation of the wave function $\psi(t,u)$ appearing in (3.20). For the $\delta(z,v)$ terms the situation is analogous. The square bracket in (3.22) does not coincide with the right-hand side of Eq. (2.22), since in (3.22) we have calculated the commutator $[[\hat{\mathfrak{T}}(z),\hat{\mathfrak{D}}(u,v)]]$, while in (2.22) we have applied $\hat{\mathfrak{T}}(z)$ to a *function* on the phase space.

4. THE TODA HIERARCHY—DEFINITIONS[2]

We shall apply the strategy of Secs. 1 and 2 to calculate the action of the Virasoro algebra on the phase space of the two-dimensional Toda hierarchy. First of all, we recall some basic definitions. A reader familiar with the formalism of Ref. 2 can proceed to the next section. (We note, however, that our W is the $\hat{W}$ from Ref. 2.)

The dressing operators are represented by $\mathbb{Z}\times\mathbb{Z}$ matrices, which act on vectors that can be written in the form $\Sigma_{s\in\Pi}v(s)|s\rangle$ [in reality, the construction is doubled, since the operators carry superscripts (∞) and (0)]. We define the operators ($\mathbb{Z}\times\mathbb{Z}$ matrices) $\hat{p}$ and Λ by

$$\hat{p}|s\rangle=s|s\rangle,\ \Lambda|s\rangle=|s-1\rangle\Rightarrow[\Lambda,\hat{p}]=\Lambda. \tag{4.1}$$

Let $|u_\lambda\rangle$ denote the eigenvectors of Λ:

$$\Lambda|u_\lambda\rangle=\lambda|u_\lambda\rangle. \tag{4.2}$$

The operator $\hat{p}$ is assumed to be Hermitian, and Λ is assumed to be unitary with respect to the scalar produce $\langle s'|s\rangle=\delta_{s,s'}$. We easily find that $\langle s|u_\lambda\rangle=\lambda^s\langle0|u_\lambda\rangle=\lambda^s$. We define the dressing matrix $W^{(\infty)}$ as

$$W^{(\infty)}=\sum_s|s\rangle\langle s|w^{(\infty)}(s;x,y;\Lambda), \tag{4.3}$$

where $w^{(\infty)}(s;x,y;\lambda)$ is expressed as follows in terms of the τ-function:

$$w^{(\infty)}(s;x,y;\lambda)\equiv\sum_{j=0}^{\infty}w_j^{(\infty)}(s;x,y)\lambda^{-j}=\frac{\tau(s;x-[\lambda^{-1}],y)}{\tau(s;x,y)}. \tag{4.4}$$

We normalize $w_0\equiv1$. From the definitions there follow directly the useful relations

$$\nabla(\mu,x)\ln\tau(s,x)=\left(\mu\frac{\partial}{\partial\mu}-\nabla(\mu,x)\right)\ln w^{(\infty)}(s,\mu),$$

$$w^{(\infty)}(s,\mu)=\mu^{-s}\langle s|W^{(\infty)}|u_\mu\rangle, \tag{4.5}$$

where we have denoted

$$\nabla(\mu,x)\equiv\sum_{r\geq1}\mu^{-r}\frac{\partial}{\partial x_r}.$$

Next we introduce

$$(W^{(\infty)})^{-1}=\sum_s w^{(\infty)*}(s;x,y;\Lambda)|s\rangle\langle s|, \tag{4.6}$$

where the conjugate wave function $w^{(\infty)*}$ is found to be equal to[2]

$$w^{(\infty)*}(s;x,y;\lambda)=\frac{\tau(s;x+[\lambda^{-1}],y)}{\tau(s;x,y)}. \tag{4.7}$$

The Lax operator L is defined as

$$L=W^{(\infty)}\Lambda(W^{(\infty)})^{-1}. \qquad (4.8)$$

We shall also need the "complete" wave functions[2]

$$\psi^{(\infty)}(s;\ x,\ y;\ \mu)=w^{(\infty)}(s;\ x,\ y;\ \mu)\mu^{s}e^{\xi(x,\mu)}, \qquad (4.9)$$

$$\psi^{(\infty)*}(s;\ x,\ y;\ \mu)=w^{(\infty)*}(s;\ x,\ y;\ \mu)\mu^{-s}e^{-\xi(x,\mu)}. \qquad (4.10)$$

The associated linear problem with eigenvalue λ can be written by introducing the vector

$$\Psi(\lambda)=\sum_{s}|s\rangle\psi^{(\infty)}(s;\lambda), \qquad (4.11)$$

for which

$$L\Psi(\lambda)=\lambda\Psi(\lambda). \qquad (4.12)$$

This equation is transformed into an identity if we note that, from knowledge of $W^{(\infty)}$, we can reconstruct $\Psi(\lambda)$ as follows:

$$\Psi(\lambda)=W^{(\infty)}e^{\xi(x,\Lambda)}|u_\lambda\rangle. \qquad (4.13)$$

The second dressing matrix ($W^{(0)}$) of the two-dimensional Toda hierarchy is constructed as

$$W^{(0)}=\sum_{s}|s\rangle\langle s|w^{(0)}(s;x,y;\Lambda), \qquad (4.14)$$

$$w^{(0)}(s;x,y;\lambda)=\frac{\tau(s+1;x,y-[\lambda])}{\tau(s;x,y)} \qquad (4.15)$$

and the x-half of the equations of the Toda hierarchy is written in the form

$$\frac{\partial W^{(\infty)}}{\partial x_r}=-(L^r)_-W^{(\infty)}, \qquad (4.16)$$

$$\frac{\partial W^{(0)}}{\partial x_r}=(L^r)_+W^{(0)}, \qquad (4.17)$$

where $(\ldots)_{\mp}$ denotes that one takes the strictly lower-triangular part (nonstrictly upper-triangular part, respectively) of the matrix.

We shall not be interested in the dependence of our functions and operators on the y-times of the two-dimensional Toda hierarchy, and shall not introduce the corresponding second set of generators of the Virasoro algebra. Accordingly, we shall frequently omit the dependence on the times y.

5. THE TODA HIERARCHY—ACTION OF THE VIRASORO ALGEBRA

We now apply the generators of the Virasoro algebra to the τ-function of the Toda hierarchy and use Eqs. (4.1)–(4.15) for the systematic elimination of the derivatives $\partial/\partial x_r$ in favor of the hierarchy flows (4.16) and (4.17). Under the action of the Virasoro generators the τ-function varies as

$$\tilde{\tau}(s)=\tau(s)+\sum_{k\in\mathbb{Z}}\mu^{-k-2}\mathcal{L}_k\tau(s)\equiv\tau(s)+\mathcal{T}(\mu)\tau(s), \qquad (5.1)$$

where the generators of the Virasoro algebra are given by Eqs. (2.16) with the replacement $t_r\mapsto x_r$, and the operator a_0 is related to $\hat{p}$ by

$$a_0=\hat{p}+{}^1/_2q, \qquad (5.2)$$

where q has not yet been fixed.

5.1. The $^{(\infty)}$-part

For a time we omit the superscript (∞). Under the variation (5.1) the wave function $w\equiv w^{(\infty)}$ changes as follows:

$$\tilde{w}(s,\lambda)=\frac{\tilde{\tau}(s;x-[\lambda^{-1}])}{\tilde{\tau}(s;x)}=\frac{\Gamma(x;\lambda)\tilde{\tau}(s;x)}{\tilde{\tau}(s;x)}$$
$$=w(s;x;\lambda)-w(s;x;\lambda)\frac{\mathcal{T}(\mu)\tau(s;x)}{\tau(s;x)}$$
$$+\frac{\Gamma(x;\lambda)\mathcal{T}(\mu)\tau(s;x)}{\tau(s;x)},$$

where the energy-momentum tensor $\mathcal{T}(\mu)$ is equal to

$$\mathcal{T}(\mu)=\mu^{-2}\Big\{\frac{1}{2}\nabla(\mu,x)\nabla(\mu,x)$$
$$+\mu\frac{\partial\xi(x,\mu)}{\partial\mu}\nabla(\mu,x)+a_0\nabla(\mu,x)$$
$$-\Big(J-\frac{1}{2}\Big)\mu\frac{\partial}{\partial\mu}\nabla(\mu,x)+\frac{1}{2}\Big[\mu\frac{\partial\xi(x,\mu)}{\partial\mu}\Big]^2$$
$$+\mu\frac{\partial\xi(x,\mu)}{\partial\mu}a_0-\Big(J-\frac{1}{2}\Big)\mu\frac{\partial}{\partial\mu}\Big[\mu\frac{\partial\xi(x,\mu)}{\partial\mu}\Big]+\frac{1}{2}a_0^2-\frac{1}{24}\Big\}.$$

Here we have used the notation

$$\xi(x,\mu)=\sum_{r\geq1}x_r\mu^r,\qquad\nabla(\mu,x)=\sum_{r\geq1}\mu^{-r}\frac{\partial}{\partial x_r},$$
$$\Gamma(x,\lambda)=\exp\Big\{-\sum_{r\geq1}\frac{1}{r}\lambda^{-r}\frac{\partial}{\partial x_r}\Big\}.$$

Calculating now $\bar{\Gamma}(x,\lambda)\mathcal{T}(\mu)\bar{\Gamma}(x,\lambda)^{-1}$ we find the variation δW from the relation

$$\mu^2\delta W|u_\lambda\rangle=\mu^2\sum_{s}|s\rangle\delta w(s;\lambda)\lambda^s$$
$$=\sum_{s}(\nabla(\mu,x)\ln\tau(s,x)+a_0)(\nabla(\mu,x)-F(\mu,\lambda))w(s;\lambda)|s\rangle\lambda^s$$
$$+\Big[-\Big(J-\frac{1}{2}\Big)\mu\frac{\partial}{\partial\mu}+\mu\frac{\partial\xi(x,\mu)}{\partial\mu}+\frac{1}{2}(\nabla(\mu,x)-F(\mu,\lambda))\Big]$$
$$\times(\nabla(\mu,x)-F(\mu,\lambda))W|u_\lambda\rangle$$
$$+\sum_{s}\Big(\frac{1}{2}a_0^2-\frac{1}{24}\Big)w(s;\lambda)|s\rangle\lambda^s, \qquad (5.3)$$

where we have denoted

$$F(\mu,\lambda)=\sum_{r\geq1}(\mu/\lambda)^r. \qquad (5.4)$$

Use of the equations of the Toda hierarchy gives

$$(\nabla(\mu,x)-F(\mu,\lambda))W|u_\lambda\rangle=(\mathcal{P}(\mu)_-W-WF(\mu,\Lambda))|u_\lambda\rangle$$
$$=\delta(\mu,L)_-W|u_\lambda\rangle, \qquad (5.5)$$

where

$$\mathscr{P}(\mu)=\sum_{n\geq 0} L^n\mu^{-n}. \qquad (5.6)$$

Substituting (5.5) into (5.3), we consider first the second term in the right-hand side of (5.3). The operator in the square brackets in it is applied to the expression that we have just calculated in (5.5). Using again the equations of the hierarchy, and also the identity

$$\mu\frac{\partial}{\partial\mu}\delta(\mu,L)_-=\delta(\mu,L)_-+F(\mu,L)^2-(\mathscr{P}(\mu)^2)_-, \qquad (5.7)$$

by straightforward exercises in a game involving arrangement of $(\ldots)_+$ and $(\ldots)_-$ we bring the second term in (5.3) to the form

$$\Big[(\mathscr{P}(\mu)_-\delta(\mu,L))_- -\frac{1}{2}\delta(\mu,L)_- +\Big(J\mu\frac{\partial}{\partial\mu}-\mu\frac{\partial\xi(x,\mu)}{\partial\mu}\Big)\delta(\mu,L)_-\Big]W|u_\lambda\rangle. \qquad (5.8)$$

We now consider the first term in (5.3), in which the summation over s is performed not so directly. First of all, we use Eqs. (4.5) in order to represent the first term in (5.3) in the form

$$-\sum_s\Big[\Big(\mu\frac{\partial}{\partial\mu}-\nabla(\mu,x)\Big)\ln w(s;\mu)+a_0\Big] \times|s\rangle\langle s|\delta(\mu,L)_-W|u_\lambda\rangle. \qquad (5.9)$$

It is also not difficult to convince oneself that

$$\delta(\mu,L)=\sum_{s',s''}|s''\rangle w(s'',\mu)\langle s''|\delta(\mu,\Lambda)|s'\rangle w^*(s'+1;\mu)\langle s'|. \qquad (5.10)$$

(It is helpful to note that the δ-function in the right-hand side of (5.10) is the projector

$$\delta(\mu,\Lambda)=|u_\mu\rangle\langle u_\mu|,$$

as is confirmed by the formal manipulations

$$|u_\lambda\rangle\langle u_\lambda|=\sum_{s,r}\lambda^s|s\rangle\langle r|\lambda^{-r}=\sum_{p,r}\lambda^p|r+p\rangle\langle r| =\sum_{p,r}\lambda^p\Lambda^{-p}|r\rangle\langle r| =\sum_{p\in\mathbb{Z}}\lambda^p\Lambda^{-p}).$$

The operation of taking the lower-triangular part commutes with multiplication by a diagonal matrix, so that (5.9) takes the form

$$-\sum_{s,s'}\Big(\mu\frac{\partial}{\partial\mu}-\nabla(\mu,x)+a_0\Big)w(s;\mu)|s\rangle\langle s|\delta(\mu,\Lambda)_-|s'\rangle \times w^*(s'+1;\mu)\langle s'|W|u_\lambda\rangle. \qquad (5.11)$$

Finally, we make use here of Eqs. (4.5) and recall the equations of motion (4.16):

$$\sum_s|s\rangle\nabla(\mu,x)w(s,\mu)\langle s|\delta(\mu,\Lambda)_-=(\nabla(\mu,x)W\delta(\mu,\Lambda))_- =-(\mathscr{P}(\mu)_-W\delta(\mu,\Lambda))_-. \qquad (5.12)$$

Then for (5.11) we obtain the expression

$$-((\mathscr{P}(\mu)_-+{}^1/_2q)\delta(\mu,L))_-W|u_\lambda\rangle -\sum_{s,s'}|s\rangle\Big(\mu\frac{\partial}{\partial\mu}+s\Big)w(s;\mu) \times\langle s|\delta(\mu,\Lambda)_-|s'\rangle w^*(s'+1;\mu)\langle s'|W|u_\lambda\rangle. \qquad (5.13)$$

Collecting (5.8) and (5.13) together, we observe pleasing cancellations, after which the result is found to be

$$\mu^2\delta W^{(\infty)}=-\sum_{s,s'}|s\rangle\Big[\Big(\mu\frac{\partial}{\partial\mu}+s\Big)w(s;\mu)\Big]\langle s|\delta(\mu,\Lambda)_-|s'\rangle \times w^*(s'+1;\mu)\langle s'|W^{(\infty)}+ \times\Big(J\mu\frac{\partial}{\partial\mu}-\mu\frac{\partial\xi(x,\mu)}{\partial\mu}-\frac{1}{2}q-\frac{1}{2}\Big)\delta(\mu,L)_-W^{(\infty)} +\Big(\frac{1}{2}\hat{p}^2+\frac{1}{2}q\hat{p}+\frac{1}{8}q^2-\frac{1}{24}\Big)W^{(\infty)}\equiv-\mu^2\mathfrak{T}(\mu)W^{(\infty)}. \qquad (5.14)$$

The last step consists in expressing the functions $w^{(\infty)}$ and $w^{(\infty)*}$ in (5.14) in terms of the complete wave functions (4.9) and (4.10). Here we replace $\langle s|\mu^{-s}$ by $\langle s|\mu^{-\hat{p}}$, and do the same for $\mu^{s'+1}|s'\rangle$, and also use the following identity, which follows from (4.1):

$$\mu^{-\hat{p}}\delta(\mu,\Lambda)_-\mu^{\hat{p}+1}=\mu\sum_{r\geq 1}\Lambda^{-r}. \qquad (5.15)$$

Then

$$\mathfrak{T}(\mu)=\sum_s\sum_{r\geq 1}|s\rangle\Big[(1-J)\frac{\partial\psi(s;\mu)}{\partial\mu}\psi^*(s-r+1;\mu) -J\psi(s;\mu)\frac{\partial\psi^*(s-r+1;\mu)}{\partial\mu}\Big] \times\langle s-r|-\mu^{-1}\Big(J-\frac{1}{2}q-\frac{1}{2}\Big)\sum_s\sum_{r\geq 1}|s\rangle \times\psi(s;\mu)\psi^*(s-r+1;\mu)\langle s-r|-\mu^{-2} \times\Big(\frac{1}{2}\hat{p}^2+\frac{1}{2}q\hat{p}+\frac{1}{8}q^2-\frac{1}{24}\Big). \qquad (5.16)$$

As can be seen, it is natural to choose q to be equal to

$$q=2J-1, \qquad (5.17)$$

which coincides with the value of the background charge. Thus, the energy-momentum tensor on the Toda hierarchy is given by the following (nonstrictly) lower-triangular matrix:

$$\mathfrak{T}(\mu)=\sum_s\sum_{r\geq 1}|s\rangle\Big[(1-J)\frac{\partial\psi^{(\infty)}(s;\mu)}{\partial\mu}\psi^{(\infty)*}(s-r+1;\mu) -J\psi^{(\infty)}(s;\mu)\frac{\partial\psi^{(\infty)*}(s-r+1;\mu)}{\partial\mu}\Big] \times\langle s-r|-\frac{1}{2}\mu^{-2}\Big(\hat{p}^2+q\hat{p}+J^2-J+\frac{1}{6}\Big). \qquad (5.18)$$

The appearance in $\mathfrak{L}_0$ of the additional term proportional to $J^2 - J + 1/6$ has already been discussed in Sec. 2.

The equations (5.18) are the direct analog for the Toda hierarchy of our earlier result (2.19). In fact, if we forget about the y-times of the Toda hierarchy we can obtain from it the KP hierarchy in matrix form (as in Ref. 27). More precisely, there arises an infinite set of KP-hierarchy versions, shifted with respect to each other by a Schlesinger transformation [the latter effects a shift along the discrete "zero time" s (Ref. 29)]. The energy-momentum tensor (5.18) then corresponds to that for the KP hierarchy from Eq. (2.19). In Sec. 7 we obtain a completely different correspondence between the Toda hierarchy, subject to Virasoro constraints, and *one* version of the KP hierarchy, also subject to Virasoro constraints; here the discrete parameter s goes over into the continuous first time $x \equiv t_1$ of the KP hierarchy.

5.2. The $^{(0)}$-part

It remains to elucidate what happens with the second dressing matrix $W^{(0)}$. As before, we vary the τ-function, as indicated in (5.1). We now write Eqs. (4.13) in the form

$$w^{(0)}(s;x,y;\lambda)=\frac{\Gamma(y,\lambda^{-1})e^{q}\tau(s;x,y)}{\tau(s;x,y)},$$

where

$$e^{q}\tau(s;x,y)=\tau(s+1;x,y),\qquad e^{q}a_0e^{-q}=a_0-1,$$
$$\Gamma(x,\lambda^{-1})=\exp\Big\{-\sum_{r\geq1}\frac{1}{r}\lambda^{r}\frac{\partial}{\partial x_r}\Big\}.$$

This time, $\bar\Gamma(x,\lambda^{-1})$ commutes with $\mathcal{T}(\mu)\equiv\mathcal{T}(\mu;x)$; then

$$e^{q}\mathcal{T}(\mu)e^{-q}=\mathcal{T}(\mu)+a_0+\frac{1}{2}\nabla(\mu,x)+\mu\frac{\partial\xi(x,\mu)}{\partial\mu}. \qquad (5.19)$$

Hence,

$$\begin{aligned}w^{(0)}(s;x,y;\lambda)&=w^{(0)}(s;x,y;\lambda)\\&+\Big[\frac{1}{2}\nabla(\mu,x)+\nabla(\mu,x)\ln\tau(s)+\mu\frac{\partial\xi(x,\mu)}{\partial\mu}\\&-\Big(J-\frac{1}{2}\Big)\mu\frac{\partial}{\partial\mu}+a_0+1\Big]\nabla(\mu,x)w^{(0)}(s;x,y;\lambda)\\&+\Big(a_0+\nabla(\mu,x)\ln\tau(s)+\mu\frac{\partial\xi(x,\mu)}{\partial\mu}\Big)w^{(0)}(s;x,y;\lambda).\end{aligned} \qquad (5.20)$$

From the hierarchy equations (4.17) we now have

$$\nabla(\mu,x)W^{(0)}(x,y)=(\mathcal{P}(\mu)_+-1)W^{(0)}(x,y). \qquad (5.21)$$

Here it is very convenient to replace $\mathcal{S}(\mu)_+$ by $\delta(\mu,L)_+$. We then obtain

$$\begin{aligned}&\mu^2\delta W^{(0)}(x,y)|u_\lambda\rangle\\&=\sum_s\Big[\frac{1}{2}\nabla(\mu,x)+\nabla(\mu,x)\ln\tau(s)+\mu\frac{\partial\xi(x,\mu)}{\partial\mu}\\&-\Big(J-\frac{1}{2}\Big)\mu\frac{\partial}{\partial\mu}+a_0+1\Big]|s\rangle\langle s|\delta(\mu,L)_+-1)W^{(0)}|u_\lambda\rangle\\&+\sum_s\Big(a_0+\nabla(\mu,x)\ln\tau(s)+\mu\frac{\partial\xi(x,\mu)}{\partial\mu}\Big)|s\rangle w^{(0)}(s;x,y;\lambda)\lambda^s\end{aligned} \qquad (5.22)$$

and, after a number of cancellations,

$$\begin{aligned}\delta W^{(0)}|u_\lambda\rangle&=-\frac{1}{2}\mu^{-2}W^{(0)}|u_\lambda\rangle\\&+\sum_s\sum_{r\geq1}|s\rangle\Big[(1-J)\frac{\partial\psi^{(\infty)}(s;\mu)}{\partial\mu}\psi^{(\infty)*}(s+r+1;\mu)\\&-J\psi^{(\infty)}(s;\mu)\frac{\partial\psi^{(\infty)*}(s+r+1;\mu)}{\partial\mu}\Big]\langle s+r|W^{(0)}|u_\lambda\rangle\\&+\mu^{-1}\Big(-J+\frac{1}{2}q+\frac{1}{2}\Big)\\&\times\sum_s\sum_{r\geq1}|s\rangle\psi(s;\mu)\psi^*(s+r+1;\mu)\langle s+r|W^{(0)}|u_\lambda\rangle,\end{aligned} \qquad (5.23)$$

whence follows the structure of the energy-momentum tensor acting on $W^{(0)}$. As before, $q=2J-1$ is found to be the most natural choice, and, as before, we observe a generalized structure of the type $(1-J)\partial b\cdot c-Jb\cdot\partial c$.

6. THE N-PERIODIC TODA HIERARCHY

Of interest in itself, and also useful for what follows (see Sec. 10), is the N-periodic reduction of the action (calculated in Sec. 5) of the Virasoro algebra. We recall[2] that the N-periodic Toda hierarchy arises when the objects introduced in Sec. 4 are additionally subjected to the conditions

$$W^{(\infty)}\Lambda^{N}(W^{(\infty)})^{-1}=\Lambda^{N},\quad W^{(0)}\Lambda^{-N}(W^{(0)})^{-1}=\Lambda^{-N}. \qquad (6.1)$$

These conditions "freeze" the evolutions along the times x_{Ni}, y_{Ni} $(i\geqslant1)$, so that these times can be omitted.

Is the action of the Virasoro algebra limited to the submanifold of those matrices W that satisfy (6.1)? As we shall see now, only some of the Virasoro generators preserve the conditions (6.1). For this we shall learn how to single out individual generators from the energy-momentum tensor (5.18). (We again concentrate on the $^{(\infty)}$-sector; for the $^{(0)}$-sector the treatment is analogous.) The expansion

$$\mathfrak{T}(z)=\sum_{n\in\mathbb{Z}}\mathfrak{L}_n z^{-n-2} \qquad (6.2)$$

is easy to affect explicitly, by inserting into (5.14) Eq. (5.10) and the identity

$$W^{(\infty)}p\delta(\mu,\Lambda)=\sum|s\rangle\Big[\Big(s+\mu\frac{\partial}{\partial\mu}\Big)w^{(\infty)}(s,\mu)\Big]\langle s|\delta(\mu,\Lambda). \qquad (6.3)$$

As a result, we find

$$\mathfrak{L}_n=\Big(W^{(\infty)}\Big\{[J(n+1)+p]\Lambda^{n}+\sum_{r\geq1}rx_r\Lambda^{r+n}\Big\}\Big(W^{(\infty)}\Big)^{-1}\Big). \qquad (6.4)$$

We now calculate the commutators of the vector fields $\hat{\mathfrak{L}}_n$ that correspond [according to (3.8) and (3.9), with obvious changes for application to the Toda hierarchy] to the operators (matrices) $\mathfrak{T}_n$ with the vector field $\hat{\mathscr{C}}_N$ corresponding to the constraint

$$\mathscr{C}_N \equiv (W^{(\infty)}\Lambda^N(W^{(\infty)})^{-1})^-, \tag{6.5}$$

which follows from (6.1). We find

$$[[\hat{\mathfrak{L}}_n, \hat{\mathscr{C}}_N]] = -N\hat{\mathfrak{l}}_{N+n},$$

where the expression

$$\hat{\mathfrak{l}}_{N+n} = (W^{(\infty)}\Lambda^{N+n}(W^{(\infty)})^{-1})_-$$

should thus vanish. This, however, freezes the evolution along the times x_{N+n} and, in this sense, is not compatible with the equations of the Toda hierarchy. Only in the case when $n = Nj$ ($j\in\mathbb{Z}$) do no additional restrictions arise. Thus, only the generators

$$\mathfrak{L}_{(j)} = \frac{1}{N}\mathfrak{L}_{Nj} \tag{6.6}$$

can act on the N-periodic Toda hierarchy. The normalization in (6.6) has been changed in such a way that the generators $\mathfrak{L}_{(\cdot)}$ form their own Virasoro algebra.

Turning to the explicit form for the generators $\mathfrak{L}_{Nj}$, we note that, as can be seen from (6.4), the term proportional to J is equal to

$$J(Nj+1)(W^{(\infty)}\Lambda^{Nj}(W^{(\infty)})^{-1})_- = \begin{cases} 0, & j\geqslant 0, \\ J(Nj+1)\Lambda^{Nj}, & j<0 \end{cases}$$

and does not give a contribution to the variation of the operator L (4.8) and so can be omitted. Thus, the Virasoro generators on the N-periodic Toda hierarchy have the form

$$\mathfrak{L}_{(j)} = \frac{1}{N}\Big(W^{(\infty)}\Big(\hat{p} + \sum_{\substack{r\geqslant 1 \\ r\neq 0 \bmod N}} r x_r \Lambda^r\Big)\Lambda^{Nj}(W^{(\infty)})^{-1}\Big)_- . \tag{6.7}$$

The N-periodic Toda hierarchy admits an alternative description in terms of the current algebra $\widetilde{sl}(n) \equiv sl(n,\mathbb{C}[\zeta,\zeta^{-1}])$, which is extremely important for the geometrical interpretation and generalizations of this hierarchy.[32,3] The connection with the $\infty\times\infty$ matrix formalism is furnished by a homomorphism[2]

$$\widetilde{sl}(N) \to sl(\infty), \tag{6.8a}$$

such that

$$\sum_{n\in\mathbb{Z}} A_n\zeta^n \mapsto \begin{pmatrix} \cdot & \cdot & & & \\ \cdot & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & A_{-1} & A_0 & \cdot \\ & & & \cdot & \cdot & \cdot \end{pmatrix}, \tag{6.8b}$$

where, obviously, $A_n \in sl(N)$. In particular, the matrix Λ introduced in (4.1) turns out to be the image, under (6.8), of the element

$$\Lambda_N(\zeta) = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \cdot & \cdot & \\ 0 & & & \cdot & 1 \\ \zeta & 0 & & & 0 \end{pmatrix} \in sl(N), \quad \Lambda_N(\zeta)^N = \zeta\cdot 1. \tag{6.9}$$

The key point is that, by virtue of the conditions (6.1), all the ingredients of the N-periodic Toda hierarchy lie in the image of the homomorphism (6.8). Therefore, in particular, there exists an element $L(\zeta)\in SL(N)$ of the form

$$L(\zeta) = W_N^{(\infty)}(x,y;\zeta)\Lambda_N(\zeta)W_N^{(\infty)}(x,y;\zeta)^{-1}, \tag{6.10}$$

$$L(\zeta) = \Lambda_N(\zeta) + \begin{pmatrix} * & & 0 \\ * & * & \\ & & \cdot \\ * & & \cdot & * \end{pmatrix} \tag{6.11}$$

with a certain $W_N^{(\infty)}(x,y;\zeta)\in SL(N)$ such that its image under (6.8) is $L = W^{(\infty)}\Lambda W^{(\infty)^{-1}}$. Here, if

$$W_N^{(\infty)}(x,y;\zeta) = \sum_{j\geqslant 0}\sum_{s=0}^{N-1} |s\rangle w_j(s)\langle s|\Lambda_N(\zeta)^{-j}, \tag{6.12}$$

then

$$W^{(\infty)}(x,y) = \sum_{k\in\mathbb{Z}}\sum_{j\geqslant 0}\sum_{s=0}^{N-1} |s+Nk\rangle w_j(s)\langle s+Nk|\Lambda^{-j}.$$

In this formulation it is natural to call the N-periodic Toda hierarchy the $sl(N)$ Toda hierarchy.

However, it is obvious that the operator $\hat{p}$ in (6.7) does not lie in the image of the homomorphism (6.8). Nevertheless, as already emphasized above, the Virasoro commutation relations for the generators of the form (6.7) are based on the formal commutation relations between the operators $\hat{p}$ and Λ. This defines the following "reduced version" of the operator $\hat{p}$:

$$N p_N = -N\zeta\frac{\partial}{\partial\zeta} + \begin{pmatrix} 1 & & 0 \\ & 2 & \\ & & \cdot \\ 0 & & & N \end{pmatrix} - N\sum h_i\Lambda_N(\zeta)^{-i}, \tag{6.13}$$

where h_i are arbitrary constants. Thus, the derivative with respect to the spectral parameter, which was introduced *ad hoc* in Ref. 35, appears here for the first time.

The Virasoro generators (6.7) can now be realized by the following operators, acting from the left on the current matrix $W_N^{(\infty)}(\zeta)\in SL(N)$:

$$\mathfrak{L}_{(j)} = \Bigg(W_N^{(\infty)}(\zeta)\Bigg\{-\zeta\frac{\partial}{\partial\zeta} + \frac{1}{N}\begin{pmatrix} 1 & & 0 \\ & 2 & \\ & & \cdot \\ 0 & & & N \end{pmatrix} + \sum_{a=1}^{N-1}\Big[\sum_{i\geqslant 0}\Big(i+\frac{a}{N}\Big)x_{Ni+a}\zeta^i\Big]\Lambda_N(\zeta)^a\Bigg\}\zeta^j W_N^{(\infty)}(\zeta)^{-1}\Bigg)_- . \tag{6.14}$$

It is important to note that $(\ldots)_-$ can now denote either of two operations (cf. Ref. 3): (*i*) singling out of the negative powers of the matrix $\Lambda_N(\zeta)$, in accordance with the fact that any matrix can be represented in the form

$$\mathscr{R} = \sum_{i\leqslant M}\sum_{s=0}^{N-1} |s\rangle r_i(s)\langle s|\Lambda_N(\zeta)^i,$$

$$\text{so that}\quad \mathscr{R}_- = \sum_{i<0}\sum_{s=0}^{N-1} |s\rangle r_i(s)\langle s|\Lambda_N(\zeta)^i, \tag{6.15}$$

or (*ii*) singling out of the negative powers of the spectral parameter ζ. The generators (and the flows of the hierarchy) obtained in these two ways turn out to be gauge-equivalent [the gauge group consists of matrices of the form (6.15) with $M=0$].

7. CONTINUUM LIMIT OF THE TODA HIERARCHY RESTRICTED BY VIRASORO CONSTRAINTS

We intend now to apply the technique developed to the analysis of problems that arise in "matrix models," which are a nonperturbative method of description of two-dimensional quantum gravity and the matter interacting with it. To be precise, we shall proceed to an analysis of the Virasoro constraints on integrable hierarchies and to the study of hierarchies restricted by Virasoro constraints. As was noted in the Introduction, the Virasoro constraints on the partition function of a matrix model are fulfilled, as Ward identities, even before the continuum limit,[8] when the τ-function is represented in the form of a matrix integral. In this section we shall start from the Virasoro constraints on discrete hierarchies and consider their scaling limit.

Namely, starting from the constraints (1.8)

$$\mathscr{L}_n\tau(x,y)=0,\quad n\geqslant 0, \tag{7.1}$$

on the Toda hierarchy, we first of all carry them over to the phase space. Then, according to (6.4), we obtain the *constraints*

$$\mathfrak{L}_n\equiv\Big(W^{(\infty)}\Big\{[J(n+1)+p]\Lambda^n+\sum_{r\geqslant 1}rx_r\Lambda^{r+n}\Big\}W^{(\infty)-1}\Big)_-=0,$$

$$n\geqslant 0. \tag{7.2}$$

We recall that the expansion of $\mathfrak{T}(\mu)$ in modes follows easily from the "bosonized" form of the energy-momentum tensor on the phase space. From an abstract point of view, the role of J in (7.2) reduces to taking the order of the operators $\hat{p}$ and Λ^n into account.

Our aim is to elucidate how the constraints (7.2) can be related to the "continuum" Virasoro constraints from Refs. 6. The correspondence will be established in two stages. As the first step we shall realize the scaling limit of the Toda hierarchy restricted by Virasoro constraints. (We stress that the entire hierarchy is subjected as a whole to the scaling.[6)]) This scaling is none other than the "hierarchical" variant of the double scaling limit that lies at the basis of the method of matrix models.[4,5] After the limit is taken, a KP hierarchy restricted by Virasoro constraints arises. Here, the Virasoro-algebra generators turn out to be exactly the same as those that we had in Secs. 2 and 3. In the second step we perform the reduction of the resulting Virasoro-constrained KP hierarchy to the generalized KdV hierarchies that, in essence, were discovered in Refs. 5 (see Sec. 9).

Since we shall consider below only the $^{(\infty)}$-part, we omit the superscript (∞) in the notation.

7.1. Redefinitions

We shall start from a certain notational modification that is convenient for the subsequent scaling. First of all, in view of the commutativity of the flows of the hierarchy there is freedom in the choice of linear combinations of flows, and hence of the corresponding times. We introduce new times $\tilde{x}_r$ ($r\geqslant 1$) that are more convenient for performing the scaling. In general, a redefinition of the times has been employed in the literature (in particular, in Ref. 33) to establish the connection between two formulations of the one-dimensional Toda hierarchy, so that the use of one or other choice of the times depends on the desire to simplify the formulation of one or other group of properties of the hierarchy. In the situation under consideration, we set

$$x_r=\frac{1}{r}\sum_{s=r}^{\infty}\binom{s}{r}(-1)^{s+r}(s+1)\tilde{x}_s,\quad r\geqslant 1, \tag{7.3}$$

where $\binom{a}{b}$ are binomial coefficients and the $\tilde{x}_s$, after the scaling, become the new times. Furthermore, there is also freedom in the composition of the linear combinations of the constraints (7.2) themselves. For $p\geqslant 0$ we define

$$\bar{\mathfrak{L}}_{p-1}=\sum_{n=0}^{p}\binom{p}{n}\mathfrak{L}_n(-1)^{n+p}. \tag{7.4}$$

If the $(\ldots)_-$ projection were absent in (7.2), it would not be difficult to calculate the sums that arise when (7.3) and (7.2) are substituted into (7.4). However, the presence of the $(\ldots)_-$ projection is extremely important, and this somewhat complicates the calculations that we have to perform. We shall start from a reinterpretation of the operator Λ. Namely, we shall regard the vectors $\Sigma_s|s\rangle v(s)$ (see Sec. 4) as functions

$$s\mapsto v(s), \tag{7.5}$$

defined on the integers $\mathbb{Z}$. Then Λ can be formally identified with the operator e^{∂}, where ∂, also formally, is $\partial/\partial s$. Thus,

$$(e^{\partial}v)(s)=v(s+1). \tag{7.6}$$

The operator $\hat{p}$ introduced in (4.1) can now be transformed into multiplication by the argument:

$$(\hat{p}v)(s)=sv(s). \tag{7.7}$$

Abusing the notation, we can write s instead of $\hat{p}$. The dressing operator and its inverse now look like

$$W=\sum_{j\geqslant 0}w_j(\cdot)e^{-j\partial},\qquad W^{-1}=\sum_{i\geqslant 0}e^{-i\partial}\circ w_i^*(\cdot). \tag{7.8}$$

where $w_j(\cdot)$ are functions $s\mapsto w_j(s)$ acting by dot multiplication.

7.2. The scaling

We shall apply scaling in order to transform s into a continuous variable. We set

$$s=\frac{t_1}{\varepsilon},\quad \partial=\varepsilon D,\quad D\equiv\frac{\partial}{\partial t_1},\quad \varepsilon\to 0. \tag{7.9}$$

It may be assumed that ε has the dimensions of length, so that t_1 has the same dimensions. The operator Λ takes the form

$$\Lambda=e^{\varepsilon D}. \tag{7.10}$$

We shall also change the scale of the constraints $\mathfrak{L}$ introduced above, replacing them by

$$\tilde{\mathfrak{L}}_{p-1}=\sum_{n=0}^{p}\binom{p}{n}\mathfrak{L}_n(-1)^{n+p}\varepsilon^{-p+1},\quad p\geqslant 0. \tag{7.11}$$

The algebra of the vector fields $\hat{\tilde{\mathfrak{L}}}$ associated with the operators $\tilde{\mathfrak{L}}$ has the form

$$[[\hat{\tilde{\mathfrak{L}}}_p, \hat{\tilde{\mathfrak{L}}}_q]] = (p-q)\sum_{r\geqslant 0}\hat{\tilde{\mathfrak{L}}}_r \varepsilon^{r-p-q}\begin{pmatrix} 1 \\ r-p-q \end{pmatrix}$$

$$= (p-q)(\hat{\tilde{\mathfrak{L}}}_{p+q} + \varepsilon\hat{\tilde{\mathfrak{L}}}_{p+q+1}). \qquad (7.12)$$

It would be premature, however, to conclude from this that in the limit $\varepsilon\to 0$ the commutation relations of the Virasoro algebra arise, since this limit presupposes certain infinite changes of scale and, in doing so, touches upon the construction of the mapping $\mathfrak{L}_n \to \hat{\mathfrak{L}}_n$ and even of the space itself on which the vector fields $\hat{\mathfrak{L}}_n$ act. In other words, the bracket in the left-hand side of (7.12) carries a certain dependence on ε when the scaling limit is taken. Nevertheless, of course, Eq. (7.12) is a favorable sign.

The main content of the continuum limit is the following scaling ansatz for the coefficients [introduced in (4.3) and (4.4)] of the operator $W \equiv W^{(\infty)}$:

$$w_j = \sum_{l\geqslant 1}\varepsilon^l \begin{pmatrix} j+l-1 \\ l-1 \end{pmatrix} k_l, \quad j\geqslant 1. \qquad (7.13)$$

The k_i will be found below to be the coefficient functions of the KP operator K, which are denoted by w_i in (2.2). From (7.13) it follows that

$$W = k_0 + \sum_{l\geqslant 1} k_l \varepsilon^l (1-e^{-\varepsilon D})^{-l}, \qquad (7.14)$$

where

$$k_0 = 1 - \sum_{l\geqslant 1} k_l \varepsilon^l. \qquad (7.15)$$

Analogously, for certain k^*_m we set

$$w_i^* = \sum_{m\geqslant 1}\varepsilon^m \begin{pmatrix} i+m-1 \\ m-1 \end{pmatrix} k_m^*, \quad i\geqslant 1. \qquad (7.16)$$

Finally, let $x_r = t_{r+1}/\varepsilon^{r+1}$ for $r\geqslant 1$, so that

$$x_r = \frac{1}{r}\sum_{s=r}^{\infty}\begin{pmatrix} s \\ r \end{pmatrix}(-1)^{s+r}(s+1)\frac{t_{s+1}}{\varepsilon^{s+1}}, \quad r\geqslant 1, \qquad (7.17\text{a})$$

(in the same way, as a supplement to (7.17a), we also have

$$s = \sum_{r\geqslant 1}(-1)^{r+1} r t_r \varepsilon^{-r}. \qquad (7.17\text{b})$$

In addition, we shall have to redefine the time t_1 as follows:

$$t_1 \mapsto t_1 - \sum_{r\geqslant 2} r t_r (-1)^r \varepsilon^{-r+1}.$$

We shall assume that in the limit $\varepsilon\to 0$ all the k_l and k^*_m are finite functions of the new times. We then obtain

$$W \to K = 1 + \sum_{l\geqslant 1} k_l D^{-l}, \quad W^{-1} \to \tilde{K} = 1 + \sum_{m\geqslant 1} D^{-m}\circ k_m^*, \qquad (7.18)$$

and the bilinear relation $WW^{-1} = 1$ in the limit $\varepsilon\to 0$ gives the bilinear relation $K\tilde{K} = 1$ for the ψ Diff operators, whence

$$\tilde{K} = K^{-1}. \qquad (7.19)$$

We now proceed to elucidate what happens to the constraints (7.11) as a result of scaling. The answer is written in Eq. (7.38) below. As already noted, a delicate aspect is the allowance for the $(\ldots)_-$ projection onto the lower-triangular matrices. After the scaling, for the KP hierarchy Virasoro constraints arise in which $(\ldots)_-$ denotes an entirely different operation on the ψ Diff operators, namely, projection onto the integral part. The two projections appear to be in no way related, and so we shall start from direct use of the definition of the matrix operations $(\ldots)_-$ in the formulas for the Toda hierarchy. Here, we shall go over to a gauge (see Ref. 33) in which $(\ldots)_-$ includes the diagonal part as well. Thus, we rewrite the equality (7.2) as

$$\sum_{\substack{i,j\geqslant 0\\ i+j\geqslant n}} w_j\left(J(n+1) + \frac{t_1}{\varepsilon} - j\right)e^{-j\varepsilon D}e^{n\varepsilon D}e^{-i\varepsilon D}\circ w_i^*$$

$$+ \sum_{r\geqslant 1} r x_r \sum_{\substack{i,j\geqslant 0\\ i+j\geqslant n+r}} w_j e^{-(j+i-r-n)\varepsilon D}\circ w_i^* = 0 \qquad (7.20)$$

and into the left-hand side of (7.20) substitute (7.13) and (7.16), and also (7.17) for the times x, after which we go over to the constraints $\tilde{\mathfrak{L}}$ defined by Eqs. (7.11). The number of summation symbols grows in this procedure, and one should exercise the usual care in the order of the summation; all the finite sums are calculated first, and the summation of the infinite series in powers of $e^{\varepsilon D}$ is performed last of all.

We shall consider a typical contribution to $\tilde{\mathfrak{L}}_{p-1}$, proportional to $(r+1)t_{r+1}$ with $r\geqslant 1$. After all the substitutions enumerated above, the factor multiplying $(r+1)t_{r+1}$ turns out to be equal to

$$\sum_{n=0}^{p}(-1)^{n+p}\begin{pmatrix} p \\ n \end{pmatrix}\sum_{s=1}^{r}(-1)^{s+r}\begin{pmatrix} r \\ s \end{pmatrix}\varepsilon^{-r-1}$$

$$\times \sum_{\substack{i,j\geqslant 0\\ i+j\geqslant n+s}}\ \sum_{l\geqslant 0}\sum_{m\geqslant 0}\varepsilon^{l+m}$$

$$\times k_l\begin{pmatrix} j+l-1 \\ l-1 \end{pmatrix}e^{-j\varepsilon D}e^{(n+s)\varepsilon D}e^{-i\varepsilon D}\circ\begin{pmatrix} i+m-1 \\ m-1 \end{pmatrix} k_m^*,$$

$$(7.21)$$

where the condition $i+j\geqslant n+s$ is what singles out the $(\ldots)_-$ part.

First, we shall perform the summation over all i and j for which $i+j=b$, with a temporarily fixed b. Here we need the identity

$$\sum_{\substack{i,j\geqslant 0\\ i+j=b}}\begin{pmatrix} j+l-1 \\ l-1 \end{pmatrix}\begin{pmatrix} i+m-1 \\ m-1 \end{pmatrix} = \begin{pmatrix} l+m+b-1 \\ l+m-1 \end{pmatrix}. \qquad (7.22)$$

In an analogous way we perform the summation over values of n and s satisfying the condition $n+s=a$:

$$\sum_{s=0}^{r}\sum_{n=0}^{p}(-1)^{n+p}\begin{pmatrix} p \\ n \end{pmatrix}(-1)^{s+r}\begin{pmatrix} r \\ s \end{pmatrix}f(n+s)$$

$$= \sum_{a=0}^{p+r}\begin{pmatrix} p+r \\ a \end{pmatrix}(-1)^{r+p+a}f(a). \qquad (7.23)$$

Thus, Eq. (7.21) takes the form of the difference of two expressions: first,

$$(-1)^{r+p}\varepsilon^{-p-r}\sum_{l,m\geqslant 0}\varepsilon^{l+m}k_l\sum_{k\geqslant 0}e^{-k\varepsilon D}\times\sum_{a=0}^{p+r}(-1)^a\binom{p+r}{a}\binom{l+m-1+k+a}{l+m-1}k_m^{*}, \quad (7.24)$$

in which we have replaced indices as $b-a=k$, and, second,

$$(-1)^{r+p}\varepsilon^{-p-r}\sum_{l,m\geqslant 0}\varepsilon^{l+m}k_l\sum_{k\geqslant 0}e^{-k\varepsilon D}\times\sum_{n=0}^{p}(-1)^n\binom{p}{n}\binom{l+m-1+k+n}{l+m-1}k_m^{*}. \quad (7.25)$$

The central point is that the alternating sum over a of the two binomial coefficients in (7.24) is equal to

$$(-1)^{p+r}\binom{l+m-l-c}{l+m-1-p-r}$$

and is nonvanishing only for $l+m-1\geqslant p+r$. In fact, the condition for this to vanish follows from the identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{q+m+k}{m}=0 \quad \text{for} \quad m\leqslant n.$$

In this way, the range of the summation over l and m is restricted by the condition

$$l+m\geqslant p+r+1. \quad (7.26)$$

Completing the calculation does not present any difficulty now: The sum over k in Eq. (7.24) is brought to the form

$$\frac{1}{(l+m-1-p-r)!}\sum_{k\geqslant 0}\frac{(l+m-1+k)!}{(p+r+k)!}e^{-k\varepsilon D}$$
$$=\frac{1}{(l+m-1-p-r)!}\frac{(l+m-1)!}{(p+r-1)!}(1-e^{-\varepsilon D})^{p+r-l-m}\sum_{k=0}^{l+m-p-r-1}(-1)^k\times\binom{l+m-p-r-1}{k}\frac{e^{-k\varepsilon D}}{p+r+k}. \quad (7.27)$$

The finite sum over k does not give zeros or poles for $\varepsilon\to 0$, and so can be replaced by its $\varepsilon\to 0$ limit, equal to

$$\sum_{k=0}^{l+m-p-r-1}(-1)^k\binom{l+m-p-r-1}{k}\frac{1}{p+r+k}=\frac{(l+m-p-r-1)!}{(p+r)(p+r+1)\dots(l+m-1)}. \quad (7.28)$$

This gives directly for (7.27) the answer

$$(1-e^{-\varepsilon D})^{p+r-l-m}, \quad (7.29)$$

so that the expression (7.24) takes the form

$$\sum_{\substack{l,m\geqslant 0\\ l+m\geqslant p+r+1}}\varepsilon^{-p-r+l+m}k_l(1-e^{-\varepsilon D})^{p+r-l-m}k_m^{*}.$$

Here, finally, the $\varepsilon\to 0$ limit is simply equal to

$$\sum_{\substack{l,m\geqslant 0\\ l+m\geqslant p+r+1}}k_lD^{p+r-l-m}\circ k_m^{*}=(KD^rD^pK^{-1})_-, \quad (7.30)$$

where $(\dots)_-$ now refers to the algebra of the ψ Diff operators and denotes the projection onto the purely integral part. We recall that we have calculated the coefficient multiplying $(r+1)t_{r+1}$. We have thereby obtained a contribution to the $(p-1)$-th Virasoro constraint in the continuum theory, equal to

$$\Big(K\sum_{r\geqslant 2}rt_rD^{r-1}D^pK^{-1}\Big)_-. \quad (7.31)$$

In the equality (7.2) it remains to consider the term containing the operator $\hat{p}$, and also to take into account the contribution made by the expression (7.25). We start from the term containing $\hat{p}\Lambda^n$. As before, we substitute (7.13), (7.16), and (7.11) into the sum over i and j in (7.20). We first sum over i and j with a fixed value of $i+j=a$:

$$\sum_{\substack{i,j\geqslant 0\\ i+j=a}}\binom{l-1+j}{l-1}(Jn+J+s-j)\binom{m-1+i}{m-1}$$
$$=(Jn+J+s)\binom{l+m-1+a}{l+m-1}-l\binom{l+m-1+a}{l+m}. \quad (7.32)$$

As the next step, in accordance with the definition of the constraints $\widetilde{\mathfrak{L}}$, we must perform the summation over n. We note that the $(\dots)_-$ projection denotes $a\geqslant n$. Setting $a=n+k$, from the last equation we obtain

$$\sum_{k=0}^{\infty}\varepsilon^{-p+1}(-1)^p\sum_{n=0}^{p}(-1)^n\binom{p}{n}\Big\{Jn\binom{l+m-1+k+n}{l+m-1}+(J+s)\binom{l+m-1+k+n}{l+m-1}-l\binom{l+m-1+k+n}{l+m}\Big\}e^{-k\varepsilon D}$$
$$=\sum_{k=0}^{\infty}\varepsilon^{-p+1}\Big\{Jp\binom{l+m+k}{l+m-p}+(J+s)\binom{l+m-1+k}{l+m-1-p}-l\binom{l+m-1+k}{l+m-p}\Big\}e^{-k\varepsilon D}. \quad (7.33)$$

As before, the value of $l+m$ is found to be bounded by the relation $l+m\geqslant p$ or $p+1$. The series in k is summed as above, and the leading singularity as $\varepsilon\to 0$ arises from

$$(Jp-l)\frac{\varepsilon^{-p+1}}{(1-e^{-\varepsilon D})^{l+m-p+1}}+s\frac{\varepsilon^{-p+1}}{(1-e^{-\varepsilon D})^{l+m-p}}. \quad (7.34)$$

We recall now that we shall also have to subtract the contribution of the terms (7.25) (appearing in the sum $\Sigma_{r\geqslant 1}(r+1)t_{r+1}\dots$). Repeating the steps (7.27)–(7.29), we obtain

$$-\sum_{r\geqslant 1}(r+1)t_{r+1}(-1)^r\frac{\varepsilon^{-p-r}}{(1-e^{-\varepsilon D})^{l+m-p}}, \quad (7.35)$$

which it is necessary to add to (7.34). But when the expression (7.17b) for s is substituted into (7.34), the sum (7.35) cancels with all terms in (7.17b) except the first. To sum up, the contribution to $\widetilde{\mathfrak{L}}_{p-1}$ is found to be equal to

$$\sum_{\substack{l,m\\ l+m\geqslant p}} (Jp-l+t_1D)k_lD^{-l-m+p-1}\circ k_m^{*}$$

$$=\sum_{l,m}(k_lD^{-l}(Jp+t_1D)D^{p-1}D^{-m}\circ k_m^{*})_-$$

$$=(K(Jp+t_1D)D^{p-1}\circ K^{-1})_-. \qquad (7.36)$$

Combining Eqs. (7.31) and (7.36), we arrive at the set of Virasoro constraints

$$\mathfrak{L}_{p-1}^{(KP)}\equiv\left(K\left(Jp+\sum_{r\geqslant 1}rt_rD^r\right)D^{p-1}K^{-1}\right)_-=0, \quad p-1\geqslant -1. \qquad (7.37)$$

These conditions are indeed Virasoro constraints: the Virasoro algebra is fulfilled for the vector fields (on the space of the operators K) associated with the left-hand sides of (7.37) [and not only for $n=p-1\geqslant -1$, but also for the generators for $p\in\mathbb{Z}$ that are defined by the left-hand side of (7.37)]. Moreover, the Virasoro generators obtained coincide in essence with the ones that we had in Secs. 2 and 3, i.e., with those that follow from the "bosonized" representation for the energy-momentum tensor (3.6). The latter statement, however, needs to be made somewhat more precise. The most general form of the Virasoro generators on the KP hierarchy[11] includes an arbitrariness associated with the choice of a_0 in Eq. (2.16):

$$\mathfrak{L}_n^{(KP)}=(K(Jn+N+PD)D^nK^{-1})_- \qquad (7.38)$$

[the operator P was defined in (3.7)]. Here, $N\equiv a_0+\frac{1}{2}$, and the geometric meaning of this parameter is explained in Ref. 11. The choice of one or other value of N is unimportant. In Sec. 2 we set $N=0$ for simplicity. As a result of the scaling limit, the Virasoro generators (7.38) with $n\geqslant -1$ and $N=J$ were obtained.

In the next two sections the resulting constraints $\mathfrak{L}_n^{(KP)}=0$ $(n\geqslant -1)$ are investigated in more detail.

8. VIRASORO CONSTRAINTS ON THE KP HIERARCHY AND TOPOLOGICAL THEORIES

The Virasoro-constrained KP hierarchy can be defined directly by imposing the constraints

$$\mathscr{L}_r\tau(t)=0, \quad r\geqslant -1 \qquad (8.1)$$

on the τ-function. In accordance with the results of Secs. 2 and 3, this can be rewritten in the form of the constraints

$$\mathfrak{L}_r\equiv(K(J(r+1)D^r+PD^{r+1})K^{-1})_-=0, \quad r\geqslant -1 \qquad (8.2)$$

on the dressing operator. For us it is important that these very constraints have been obtained by means of the scaling limit of the Virasoro-constrained Toda hierarchy. In particular, the origination of the Virasoro constraints (8.2) from matrix models has thereby been established. We shall consider them in more detail.

The constraints (8.2) can be summed to give the following generating expression:

$$(K(P+lJ)e^{Dl}K^{-1})_-=0. \qquad (8.3)$$

This equality is an operator relation of the form of a loop equation or recursion equation for the correlation functions,[17,18] and l plays the role of a loop parameter—the length.[5,16,18,34] (The reduction to the generalized KdV equations is performed in a similar way to the reduction considered in Sec. 9.) As regards consistency between Eqs. (8.2) or (8.3) and the KP hierarchy, by applying the derivatives $\partial/\partial t_r$ to Eqs. (8.5) we see that they all vanish by virtue of the equations of the KP hierarchy.

One can view the constraints (8.3) in approximately the same way as one views the constraints that single out the generalized KdV hierarchies[3] from the KP hierarchy. In the latter case one imposes the constraint $(KD^NK^{-1})_-=0$, i.e., the operator $A=KD^NK^{-1}$ should be a purely differential operator. This operator parametrizes the independent degrees of freedom, and, in this sense, "relaxes" the constraints. In our case we also introduce a differential operator A and rewrite the Virasoro constraints in the form

$$K(x)\circ\left(x+lJ+\sum_{s\geqslant 2}st_sD^{s-1}\right)=A\circ K(x+l). \qquad (8.4)$$

An obvious possibility here is

$$A=P+Jl, \qquad (8.5)$$

and hence there appears the requirement

$$[K,P+Jl]=0. \qquad (8.6)$$

More generally, suppose that for some concrete model the times t_r are nonzero only for $r\leqslant N+1$. Then the order of the operator A is equal to N $(A=\Sigma_{i=0}^N a_iD^i)$, and Eq. (8.4) is transformed into extremely restrictive conditions that state that the argument x of the ψ Diff operator K is translated by means of differential operators of order N. Namely, for K as in Eq. (2.2), we obtain

$$\sum_{j=m+1}^{N+1} jt_jw_{j-m-1}(x)=\sum_{i=m}^{N}a_i\sum_{j=m}^{i}\binom{i}{j}\partial^{i-j}w_{j-m}(x+l),$$

$$1\leqslant m\leqslant N, \qquad (8.7)$$

$$\sum_{j=1}^{N+1}jt_jw_{j-1}(x)+Jl=\sum_{i=m}^{N}a_i\sum_{j=1}^{i}\binom{i}{j}\partial^{i-j}w_j(x+l), \qquad (8.8a)$$

$$\sum_{j=1}^{N+1}jt_jw_j(x)+Jlw_1(x)=\sum_{i=0}^{N}a_i\sum_{j=0}^{i}\binom{i}{j}\partial^{i-j}w_{j+1}(x+l), \qquad (8.8b)$$

$$\sum_{j=1}^{N+1}jt_jw_{j-m-1}(x)+Jlw_{-m}(x)+mw_{-m-1}(x)$$

$$=\sum_{i=0}^{N}a_i\sum_{j=0}^{i}\binom{i}{j}\partial^{i-j}w_{j-m}(x+l), \quad m\leqslant -1. \qquad (8.9)$$

The coefficients a_i are easily determined from (8.7) and (8.8) in terms of the first n coefficient functions in K as

$$(N+1)t_{N+1}=a_N, \qquad (8.10)$$

$$Nt_N+(N+1)t_{N+1}w_1(x)=a_{N-1}+a_Nw_1(x+l),$$
$$(N-1)t_{N-1}+Nt_Nw_1(x)+(N+1)t_{N+1}w_2(x) \qquad (8.11)$$

$$= a_{N-2}+a_{N-1}w_1(x+l)+a_N\left\{\binom{N}{N-1}\partial w_1(x+l)+w_2(x+l)\right\}$$

(8.12)

etc. The remaining conditions, given by Eqs. (8.9) with the values of a_i substituted into them, impose serious restrictions on the dependence of the operator K on the times of the hierarchy (and on x).

We note the following interesting system of nonlocal evolution equations that is satisfied by the operator $A = A(x,l;D) \in \mathrm{Diff}$:

$$\frac{\partial A}{\partial t_r} = Q(x)_+{}^r A - AQ(x+l)_+{}^r. \qquad (8.13)$$

Similar nonlocal hierarchies have been studied recently in Ref. 21, in which it was suggested that they be viewed as the result of "quantization" of the spectral parameter in ordinary (local) integrable hierarchies. A similar ideology ("quantum" Riemann surfaces) was also developed earlier in Ref. 35, also in connection with Virasoro constraints on integrable equations. Expanding the operator A in powers of l, we find that its modes A_r, which are differential operators of order $n(r) = 1 + r(n(1) - 1)$, also satisfy the Virasoro algebra.

Equation (8.13) can evidently be placed at the basis of an alternative formulation of the Virasoro-constrained KP hierarchy. Apart from its direct relationship with "nonlocal hierarchies," Eq. (8.3) can be rewritten in the form

$$\left(\sum_{r\geqslant 1} rt_r\frac{\partial^{r-1}}{\partial l^{r-1}} + lJ\right)(Ke^{Dl}K^{-1})_- = -(K'e^{Dl}K^{-1})_-, \qquad (8.14)$$

where the operator

$$K' = \frac{\partial K}{\partial D} = -\sum_{r\geqslant 1} rw_r D^{-r-1} \qquad (8.15)$$

has arisen because of the noncommutativity of K and x. Equation (8.14) is a prototype of the loop equations for the case of an infinite number of "primary fields." In view of their infinite number, we may also expect that the "higher" loop equations will be fulfilled. Indeed, we shall derive them as a consequence of the gigantic symmetry of the Virasoro-constrained KP hierarchy.

It is not difficult to derive the complete algebra of the symmetry admitted by the Virasoro-constrained KP hierarchy now that we have at our disposal the explicit form of the Virasoro generators. The latter have the structure $\mathfrak{L}_p = (Kl_pK^{-1})_-$, which may be expressed by saying that $\mathfrak{L}_p$ is the dressing of the operator l_p. We now consider the expression

$$(Kl_pl_qK^{-1})_- = ((Kl_pK^{-1})(Kl_qK^{-1}))_-.$$

When p and q label generators that appear in the set of Virasoro constraints, both factors in the right-hand side are found to be of the pure $(\ldots)_+$ type, and so the overall $(\ldots)_-$ gives zero. Continuing in the same spirit, we obtain

$$\mathfrak{L}_{p_1\ldots p_s} \equiv (Kl_{p_1}\ldots l_{p_s}K^{-1})^- = 0, \quad p_i \geqslant -1, \quad s \geqslant 0. \qquad (8.16)$$

We recall once again that the commutator algebra of the vector fields associated with the generators $\mathfrak{L}$ is isomorphous to the "bare" algebra of the operators $l_{p_1},\ldots,l_{p_s}$ [see (3.9)]. Thus, the algebra generated by the Virasoro constraints on the KP hierarchy turns out to be simply the associative algebra generated by the generators

$$l_n = J(n+1)D^n + PD^{n+1}, \quad n \geqslant -1. \qquad (8.17)$$

Its structure becomes comprehensible if we consider first the subalgebra $sl(2)$ generated by the generators $\mathfrak{L}_{\pm 1}$ and $\mathfrak{L}_0$. In the representation (8.17) they have a Casimir operator equal to $\lambda = J - J^2$, and thereby generate, by means of (8.16), the algebra[36]

$$B_\lambda = Usl(2)/I_\lambda, \quad \lambda = J - J^2, \qquad (8.18)$$

which is defined as the factor algebra of the universal wrapping algebra $sl(2)$ with respect to the ideal generated by the relation

$$(Casimir) = \lambda.$$

Such algebras B_λ are of interest in themselves: They have appeared, in particular, in the study of massless higher-spin fields in 2 + 1 dimensions. The generators B_λ lie inside the wedge $-s+1 \leqslant n \leqslant s-1$ (Ref. 36). As regards the Virasoro conditions, it is obvious that even the generator l_2 "protrudes" out of the wedge. On the other hand, the addition of just l_2 leads to the appearance of all the other $l_{p>0}$. Thus, we obtain a "Borel" subalgebra of the algebra $W_\infty(J)$. The latter is the bosonic part (more precisely, half the bosonic part) of the super-$W_\infty(J)$ algebra constructed recently in Ref. 24. A "Borel" subalgebra implies that we retain only the generators $l_n^{(s)}$ for which $n \geqslant -s+1$.

The system of W_∞ constraints can be written compactly in a form analogous to (8.3). We note first of all that we can get rid of the dependence on J by including in the dressing the conjugation $e^{JlD}\ldots e^{-JlD}$, thereby reducing everything to the case $J = 0$: Indeed, Eq. (8.3) can be rewritten in the form

$$(Ke^{JlD}Pe^{lD}e^{-JlD}K^{-1})_- = 0.$$

But in the case $J = 0$ we find that the W_∞ constraints have a very simple form:

$$l_m^{(s+1)} \equiv (KP^sD^{s+m}K^{-1})_- = 0, \quad s \geqslant 1, \quad m \geqslant -s$$

and can be summed to give the generating expression

$$(K(e^{zP}-1)e^{lD}K^{-1})_- = 0 \qquad (8.19)$$

or, equivalently,

$$\exp\left\{\sum_{r\geqslant 2} t_r\left[\left(\frac{\partial}{\partial l}+z\right)^r - \frac{\partial^r}{\partial l^r}\right]\right\}(Ke^{zx}e^{lD}K^{-1})_- = (Ke^{lD}K^{-1})_-$$

(8.20a)

or, finally,

$$\exp\left\{\sum_{r\geqslant 1} t_r\left[\left(\frac{\partial}{\partial l}+z\right)^r - \frac{\partial^r}{\partial l^r}\right]\right\}(K[D+z]e^{lD}K^{-1})_-$$

$$= (Ke^{lD}K^{-1})_-. \qquad (8.20b)$$

where $K[D+z]$ denotes the same as K but with D replaced by $D+z$; z here is a formal parameter. The equations we have written out are a complete system of W_∞ constraints on the KP hierarchy, and, at the same time, turn out to be the prototype of the leading loop equations. As was noted above, these equations pertain to the situation with an infinite number of "primary fields." Theories with a finite number N of primary fields follow as a result of N-reduction of the Virasoro-constrained KP hierarchy.

9. SYMMETRIES AND REDUCTIONS OF THE VIRASORO-CONSTRAINED KP HIERARCHY

9.1. N-reduction of the Virasoro generators

We recall that the N-reduction of the KP hierarchy itself, without Virasoro constraints, is achieved by requiring that the Nth power of the Lax operator be a purely differential operator:

$$Q^N \equiv L \in \mathrm{Diff} \quad (\Rightarrow Q^{Nk} \in \mathrm{Diff}, \quad k \geqslant 1). \tag{9.1}$$

Then, in the standard way, the evolutions along the times t_{Nk} ($k \geqslant 1$) become trivial, and these times themselves can be set equal to zero. It is convenient to rename the remaining time as $t_{a,i} = t_{Ni+a}$ ($i \geqslant 1$, $a = 1, ..., N-1$).

We shall examine first whether the action of the Virasoro algebra is reduced in the framework of this scheme, and then discuss the possibility in imposing Virasoro constraints. Thus, the Virasoro generators are given by the left-hand sides of Eq. (8.2), in which r runs over all integer values ($r \in \mathbb{Z}$). Now, however, the operator P is expressed by the formula

$$P = \sum_{a=1}^{N-1} \sum_{i \geqslant 0} (Ni+a) t_{a,i} D^{Ni+a-1}. \tag{9.2}$$

By virtue of (9.1), not all powers of the differentiation D give a contribution to the dressed operator P: Writing r in the form $r = Nj + \beta$ ($\beta = (0,b) = 0,1,...,N-1$), we obtain

$$\mathfrak{L}_{Nj} = \Big(K \sum_{a,i} (Ni+a) t_{a,i} D^{N(i+j)+a} K^{-1}\Big)_- + \begin{cases} 0, & j \geqslant 0, \\ JNjKD^{Nj}K^{-1}, & j < 0, \end{cases} \tag{9.3a}$$

$$\mathfrak{L}_{Nj+b} = J(Nj+b)(KD^{Nj+b}K^{-1})_- + \Big(K \sum_{\substack{a,i \\ a+b \neq 0 \bmod N}} (Ni+a) t_{a,i} D^{N(i+j)+a+b} K^{-1}\Big)_- . \tag{9.3b}$$

[We note that for $b = N-1$ the term containing $t_1 \equiv x$ is always present in the right-hand side of (9.3b):

$$(KxD^{N(j+1)}K^{-1})_- \neq 0,$$

since K does not commute with x.]

Up to now we have only rewritten the formulas for the generators, and there are no grounds to expect that the action of the Virasoro algebra on the KP hierarchy will be consistent with the N-KdV reduction. The consistency is verified most simply by commuting the vector fields associated with the left-hand sides of (9.1) (see Sec. 3) with $\hat{\mathfrak{L}}_r$. As already mentioned more than once, these commutators are determined by the "bare" commutators [cf. (3.9)]. Thus, we convince ourselves that consistency of (9.3b) with the N-KdV reduction requires the conditions

$$(KD^{N(k+j)+b}K^{-1})_- = 0, \tag{9.4}$$

which freeze the evolution along the times t_{bj+k}. The only Virasoro generators that are consistent with the reduction turn out to be those written out in (9.3a). Furthermore, the J-dependent part of these generators for negative j transforms K as $K \mapsto K + JNjKD^{Nj}$ ($j<0$), which is part of the allowed "gauge" freedom and can be omitted (in other words, such transformations do not affect either the Lax operator Q, in view of the fact that $\delta Q = JNjK[D^{Nj}, D]K^{-1} = 0$, or, as will be shown below, the τ-function). After a change of normalization, these generators form their own Virasoro algebra. Thus, the "inner" N-KdV generators are

$$\mathfrak{L}_{(j)} = \frac{1}{N}\Big(K \sum_{a,i} (Ni+a) t_{a,i} D^{N(i+j)+a} K^{-1}\Big)_- . \tag{9.5}$$

Thus, the Virasoro-constrained N-KdV hierarchies are obtained by imposing the condition (9.1) on the KP hierarchy and requiring that the generators (9.5) for $j \geqslant -1$ vanish. The relationship to the *Virasoro-constrained* KP hierarchy is as follows: Starting from the KP hierarchy restricted by Virasoro constraints $\mathfrak{L}_{>-1}$, we first weaken these constraints to $\mathfrak{L}_{Ni}$ ($i \geqslant 0$). This gives

$$\mathfrak{L}_{(j)} = 0, \quad j \geqslant 0.$$

Apart from this, however, *after* the imposition of the constraint (9.1) it turns out to be possible to require in a consistent manner that, as well,

$$\mathfrak{L}_{-N} = 0.$$

This is none other than the KdV generator $\mathfrak{L}_{(-1)}$, so that, finally, we obtain the N-KdV hierarchy restricted by the Virasoro constraints

$$\mathfrak{L}_{(j)} = 0, \quad j \geqslant -1. \tag{9.6}$$

The analysis of the associative algebra generated by the operators $\mathfrak{L}_{(j)}$ proceeds in the N-reduced case in a similar manner to the analysis considered above for the KP hierarchy. As we have just seen, the surviving constraints do not contain J-dependent terms, and so one can assume that, formally, the N-KdV reduction presupposes that $J = 0$, and this, in its turn, determines the zero value of the Casimir invariant for the $sl(2)$ subalgebra generated by the operators $\mathfrak{L}_{(\pm 1)}$ and $\mathfrak{L}_{(0)}$.

For $J = 0$ the associative algebra generated by the constraints takes an especially simple form, and is generated by the operators

$$\begin{cases} P^i D^{Nj+i}, & i \geqslant 1, \quad i \neq 0 \bmod N, \quad j \geqslant -i, \\ PD^{Nj_1+1} \ldots PD^{Nj_s+1}, & -s \leqslant (j_1 + \ldots + j_s) \leqslant -2, \\ P^{Nk}, & k \geqslant 1. \end{cases} \tag{9.7}$$

(Of course, it is after dressing that these expressions strictly become constraints.) In fact, for the product of two "bare" Virasoro generators we have

$$l_{(j_1)} l_{(j_2)} = [P^2 D^{N(j_1+j_2)+2} + (Nj_1+1) l_{(j_1+j_2)}].$$

If both indices j here are non-negative, the second term in the right-hand side is again a Virasoro generator that vanishes after dressing. Continuing in the same spirit, we successively obtain $P^i D^{N(j_1 + \ldots + j_i) + i}$, until the operator

$$P^N D^{N(j_1 + \ldots + j_N) + N}$$

appears. Writing the corresponding dressed constraint in the form

$$(KP^N K^{-1} L^{(j_1+\ldots j_N+1)})_-=0,$$

where L is a differential operator, we finally obtain

$$(KP^N K^{-1})_-=0 \quad (\Rightarrow (KP^{Ni}K^{-1})_-=0, \quad i\geqslant 1), \tag{9.8}$$

which is what gives the third line in (9.7). If, however, a sufficient number of indices j are equal to -1, so that $Nj+i<0$, we obtain the second line in (9.7).

The constraints can be summed to give a generating expression analogous to (8.19):

$$(K[\exp(zPD^{1-N})-1]\exp(lD^N)K^{-1})_-=0. \tag{9.9}$$

(If we preserve the interpretation of l as a length, we should correspondingly redefine the dimensions assigned to the times of the hierarchy.) Expanding (9.9) in powers of z, we obtain a set of loop equations, and thereby establish a connection with the field-theoretical description. We note also that the fact that in the "kernel" $\exp lD^N$ the differentiation D is present only in the form D^N singles out in the sum

$$PD^{1-N}=\sum_{a,i}(Ni+a)t_{a,i}D^{N(i-1)+a}$$

the first $N-1$ terms, which correspond to primary fields. On the other hand, by defining (in a similar manner to that used by Dijkgraaf, Verlinde, and Verlinde in Ref. 17, but slightly differently) the derivative $(\partial/\partial l)^{1/N}$ by means of

$$\left(\frac{\partial}{\partial l}\right)^{1/N}\exp(lD^N)=D\exp(lD^N)$$

we can rewrite Eq. (9.9) in the form

$$\left\{\exp\sum_{a,i}{}^{*}t_{a,i}\left[\left(\frac{\partial}{\partial l}+Nz\right)^{i+a/N}-\left(\frac{\partial}{\partial l}\right)^{i+a/N}\right]\right\}$$

$$\times\{K[\exp(zxD^{1-N})-1]$$

$$\times\exp(lD^N)K^{-1}\}_-=0, \tag{9.10}$$

where the term with $i=0$ and $a=1$ is absent from the sum Σ^*.

We note the "symmetry" (or, rather, even the duality) between Eqs. (9.6) and (9.15), which is emphasized by the commutation relation

$$[D,P]=1,$$

so that it appears to be rather natural to forget about the initial motivation (the study of the Virasoro constraints in the form in which they have arisen, up to now, in matrix models) and study the diverse possibilities of imposing on the integrable hierarchies constraints that are constructed by dressing expressions composed of P and D.

In Secs. 2 and 3 we noted formal analogies between the description of the Virasoro algebra on integrable hierarchies and the structures of conformal field theory [compare Eqs. (2.19) and (3.1)]. In the N-KdV case the "energy-momentum tensor" also repeats the "generalized structure (3.1)." To be precise, we take the Virasoro generators $\mathfrak{L}_{(j)}$, where j runs over all integer values, and, setting $\zeta=z^N$, "fermionize" the following "energy-momentum tensor" on the KdV hierarchy:

$$\mathfrak{L}^{(N)}(\zeta)(d\zeta)^2=\sum_{j\in\mathbb{Z}}\zeta^{-j-2}\mathfrak{L}_{(j)}(d\zeta)^2$$

$$=N\left(K\sum_{b,j}(Nj+b)t_{b,j}D^{Nj+b}\frac{1}{z^2}\delta(D^N,z^N)K^{-1}\right)_-(dz)^2. \tag{9.11}$$

We recall that $\delta(z,D)$ is the projector on to the subspace of the eigenvectors of the operator D with eigenvalue z. It is then obvious that

$$\delta(D^N,z^N)=\frac{1}{N}\sum_{a=0}^{N-1}\delta(z_a,D),\qquad z_a=e_a z,$$

$$e_a=\exp\left(2\pi\sqrt{-1}\frac{a}{N}\right). \tag{9.12}$$

The right-hand side of (9.11) thereby takes the form [where we omit the two-differential $(dz)^2$]

$$\frac{1}{z^2}\sum_{a=0}^{N-1}\sum_{b,j}(Nj+b)t_{b,j}z_a^{Nj+b}\,w^{(a)}(t,z)\delta(z_a,D)_-w^{(a)*}(t,z)$$

$$+\frac{1}{z^2}\sum_{a=0}^{N-1}z_a\frac{\partial w^{(a)}(t,z)}{\partial z_a}\delta(z_a,D)_-w^{(a)*}(t,z). \tag{9.13}$$

Here we have introduced the wave functions [cf. Eqs. (2.7)]

$$\psi^{(a)}(t,z)=Ke^{\xi^{(a)}(t,z)}\equiv w^{(a)}(t,z)e^{\xi^{(a)}(t,z)},$$

$$\xi^{(a)}(t,z)=\sum_{j,b}t_{b,j}z_a^{Nj+b} \tag{9.14}$$

(and analogously for the conjugate wave functions). In reality, they must be regarded as functions of a parameter $E\in\mathbb{C}P^1$. We recall that the spectral parameter of the N-KdV hierarchy lives on a complex curve specified in $\mathbb{C}^2\ni(z,E)$ by the equation $z^N=P(E)$, where P is a polynomial. The projection on to $\mathbb{C}P^1\ni E$ is an N-sheet covering, as a result of which new wave functions $\psi^{(a)}(t,E)$ arise.

Calculating explicitly now the $(\ldots)_-$ part of the δ-operators in (9.11), we obtain the N-KdV "energy-momentum tensor":

$$\mathfrak{T}^{(N)}(E)=\sum_{c=0}^{N-1}e_c\frac{\partial\psi^{(c)}}{\partial z}\circ D^{-1}\circ\psi^{(c)*}=\sum_{c=0}^{N-1}e_c{}^2\frac{\partial\psi^{(c)}}{\partial z_c}\circ D^{-1}\circ\psi^{(c)*}. \tag{9.15}$$

The similarity with two-dimensional field theories is thereby preserved as before: As in conformal field theory on $\mathbb{Z}_N$ curves, the energy-momentum tensor is given by a sum over sheets of a covering on $\mathbb{C}P^1$ (Ref. 37).

This also explains, in particular, the $\mathbb{Z}_N$ twist (see Refs. 6, 22, and 23) of the Virasoro generators that act on the τ-function in the form of differentiations with respect to the times, as in (2.16). In fact, the variation of the τ-function is restored from the variation of the wave function in the usual way, starting from res $K=-\partial\log\tau$, whence

$$\delta\partial\ln\tau=\mathrm{res}\,\delta K=\mathrm{res}\,\mathfrak{T}(z)K=\mathrm{res}\,\mathfrak{T}(z). \tag{9.16}$$

The residue of the operator $\mathfrak{T}(z)$ follows rapidly from its "fermionized" representation (9.15). To the combination of wave functions which then arises we apply the formula[38]

$$\frac{\tau(t+[z^{-1}]-[u^{-1}])}{\tau(t)}=(u-z)e^{\xi(t,z)-\xi(t,u)}\partial^{-1}(\psi(t,u)\psi^*(t,z)) \tag{9.17}$$

(which follows without difficulty as a result of applying the vertex operator $\exp\Sigma_{r\geq1}\frac{1}{r}(z^{-r}-u^{-r})\partial/\partial t_r$ to a bilinear identity for the KP hierarchy[1] and calculating the integral as a sum of residues). From (9.17) we see that the expression of interest to us is equal to

$$\frac{\partial\psi(t,z)}{\partial z}\psi^*(t,z)=\partial\left\{\frac{1}{2}\frac{1}{\tau(t)}\frac{1}{z}\nabla(t,z)\frac{1}{z}\nabla(t,z)\tau(t)\right.$$
$$+\frac{1}{\tau(t)}\frac{\partial\xi(t,z)}{\partial z}\frac{1}{z}\nabla(t,z)\tau(t)$$
$$\left.+\frac{1}{2}\frac{1}{\tau(t)}\frac{1}{z}\nabla(t,z)\tau(t)+\frac{1}{2}\left[\frac{\partial\xi(t,z)}{\partial z}\right]^2\right\}. \tag{9.18}$$

This expression, which applies in essence to the KP hierarchy, does not yet give the answer, since we still have to perform the summation over c after the replacement $z\mapsto z_c$ in accordance with (9.15). For this we recall that ξ and ∇ are defined by the formulas

$$\xi(t,z_c)=\sum_{a,i}(Ni+a)t_{a,i}z^{Ni+a-1}\exp\left[\frac{2\pi\sqrt{-1}}{N}c(a-1)\right].$$

$$\frac{1}{z_c}\nabla(t,z_c)=\sum_{a,i}z^{-Ni-a-1}\exp\left[-\frac{2\pi\sqrt{-1}}{N}c(a+1)\right]\frac{\partial}{\partial t_{a,i}}.$$

The sum over c with weight $e_c^2=\exp(2\pi\sqrt{-1}/N)2c$ plays the role of the projector on to the singlet in the group of the Nth roots of unity, so that from (9.18) we obtain

$$\sum_{c=0}^{N-1}e_c{}^2\frac{\partial\psi(t,z_c)}{\partial z_c}\psi^*(t,z_c)$$
$$=N\partial\left\{\frac{1}{2}\sum_{a=1}^{N-1}\sum_j(Nj+a)(N(j+1)-a)t_{a,i}t_{N-a,j}z^{N(i+j+1)-2}\right.$$
$$+\frac{1}{2}\frac{1}{\tau(t)}\sum_{a=1}^{N-1}\sum_{i,j}z^{-N(i+j+1)-2}\frac{\partial^2\tau(t)}{\partial t_{a,i}\partial t_{N-a,j}}$$
$$\left.+\frac{1}{\tau(t)}\sum_{a=1}^{N-1}\sum_{i,j}(Ni+a)t_{a,i}z^{N(i-j)-2}\frac{\partial\tau(t)}{\partial t_{a,j}}\right\}. \tag{9.19}$$

Next, we shall relate the singling out of the modes $\mathscr{L}_{(n)}$ from the energy-momentum tensor acting on the τ-function to the expansion of the KdV hierarchy in powers of the spectral parameter $\zeta=z^N$. Restoring the two-differential $(dz)^2$ that appeared as a factor in the right-hand side of (9.24) and then expressing it in terms of $(d\zeta)^2$ and again omitting $(d\zeta)^2$, we expand in powers of ζ and thus arrive at the virasoro generators

$$\mathscr{L}_{(n)}=\frac{1}{N}\frac{1}{2}\sum_{a=1}^{N-1}\sum_{i=0}^{n-1}\frac{\partial^2}{\partial t_{a,i}\partial t_{N-a,n-i-1}}$$
$$+\frac{1}{N}\sum_{a=1}^{N-1}\sum_{i\geq0}(Ni+a)t_{a,i}\frac{\partial}{\partial t_{a,i+n}},\quad n>0,$$

$$\mathscr{L}_{(0)}=\frac{1}{N}\sum_{a=1}^{N-1}\sum_{i\geq0}(Ni+a)t_{a,i}\frac{\partial}{\partial t_{a,i}}, \tag{9.20}$$

$$\mathscr{L}_{(n)}=\frac{1}{N}\frac{1}{2}\sum_{a=1}^{N-1}\sum_{i=0}^{-n-1}(Ni+a)(-N(i+n)-a)t_{a,i}t_{N-a,-i-n-1}$$
$$+\frac{1}{N}\sum_{a=1}^{N-1}\sum_{i\geq-n}(Ni+a)t_{a,i}\frac{\partial}{\partial t_{a,i+n}},\quad n<0.$$

These operators, acting on the τ-function of the N-KdV hierarchy, have been written in inner terms of this hierarchy. We have introduced them as a result of N-reduction from the KP generators (2.16). We note, however, that the prescription for this derivation follows automatically from the preceding analysis of the Virasoro generators in terms of the dressing operators.

When we attempt to include in the analysis the J-dependent part of the generators (9.3a), we obtain a zero result at the point at which we sum over c [see (9.19)]:

$$\sum_{c=0}^{N-1}e_c{}^2\frac{\partial}{\partial z_c}\psi(t,z_c)\psi^*(t,z_c)=0, \tag{9.21}$$

which confirms the unimportance of the J-dependent part of (9.3a).

9.2. The $sl(N)$ currents on the space of the dressing operators

The above formulas, which display remarkable analogies with theories on the world sheet, suggest the following construction, which repeats the structure of currents with values in the Kac–Moody algebra:[39]

$$\mathfrak{J}^{ab}(E)=\psi^{(a)}(t,E)\circ D^{-1}\circ\psi^{(b)*}(t,E). \tag{9.22}$$

The associated vector fields $\hat{\mathfrak{T}}^{ab}$ do indeed satisfy the current algebra $sl(N)$, as is most simply verified by first rewriting the "currents" (9.22) in the form

$$\mathfrak{J}^{ab}=\left(K\exp[(z_a-z_b)P]\frac{1}{z_b}\delta(z_b,D)K^{-1}\right)_-. \tag{9.23}$$

Using (3.9), we find

$$[[\hat{\mathfrak{J}}^{ab}(E),\hat{\mathfrak{J}}^{cd}(E')]]$$
$$=\delta^{bc}\frac{1}{z}\delta(z,z')\hat{\mathfrak{J}}^{ad}(E)-\delta^{ad}\frac{1}{z}\delta(z,z')\hat{\mathfrak{J}}^{cb}(E) \tag{9.24}$$

(the appearance of the Kronecker symbols reflects the fact that two points can coincide only if they are on the same sheet of the Riemann surface). The commutation relations (9.24) are supplemented by a tracelessness condition of the type (9.21).

The $sl(N)$ Kac–Moody symmetry obtained in this way can turn out to be extremely important in the context of W-gravity and can reflect hidden symmetry of the latter.[40] But in application specifically to constrained integrable hierarchies as well, the presence of the symmetry under the current algebra makes it possible to introduce hierarchies subject to the corresponding constraints. To be precise, as well as satisfying the [[,]] commutation relations (9.24), the currents also commute in the standard way with the "energy-momentum tensor" (9.15) [see (3.21)], so that it is possible to impose dominant-weight constraints with respect to the Kac–Moody algebra and the Virasoro algebra simultaneously. This possibility is illustrated by the general procedure that we are proposing: to reject the *matrix* formulation of the matrix models and, as the initial object for the description of nonperturbative quantum gravity, to consider integrable hierarchies subject to the appropriate constraints.

The origin of the $sl(N)$ currents is also made clear from the point of view of the reduction from the KP hierarchy. The symmetries[28] of the latter are represented by the vector fields $\hat{\mathfrak{D}}(u,v)$, where $\mathfrak{D}(u,v)$ is given by Eq. (3.20). In the reduction to the N-KdV hierarchy, of course, one must verify that these symmetries are consistent with the constraint (9.1). The verification again reduces to the calculation of the commutator of the corresponding vector fields, which is found to be proportional to the expression

$$\left(Ke^{(u-v)P}(u^{Ni}-v^{Ni})\frac{1}{v}\delta(v,D)K^{-1}\right)_-,$$

so that one requires that

$$u=v\exp\left(k\frac{2\pi\sqrt{-1}}{N}\right),\qquad k\in\{0,1,\ldots,N-1\},$$

and the "bilocal operators" (3.21) are thereby restricted to $\mathfrak{D}(z_a,z_b)$, in agreement with (9.23).

We note that the "hierarchical" analog of non-Abelian bosonization[39] would be the representation of the "currents" $\mathfrak{T}^{ab}$ in the matrix formalism of Drinfel'd and Sokolov,[3] in which, of course, it is natural to expect the appearance of the $sl(N)$ current algebra.

10. $N\times N$ FORMULATION OF THE VIRASORO CONSTRAINTS AND OF THE STRING EQUATION

10.1. Virasoro generators and constraints on the $sl(N)$-KdV hierarchy

Besides that used in Sec. 9, the N-KdV hierarchies also have another, extremely important formulation[3] in terms of *first*-order differential operators with matrix coefficients. This formulation is the basis both for generalizations to arbitrary semisimple Lie and Kac–Moody algebras and for displaying the geometrical nature of the equations under consideration (Hamiltonian reduction) and establishing the connection with other physically important structures—primarily, with W-algebras (see footnote 4).

It is important, therefore, to construct a "raising" of the action of a Virasoro algebra from scalar to matrix differential operators. This will make it possible, in particular, to apply the apparatus of Ref. 3 to extend the construction that gives the action of the Virasoro algebra to generalized (m)KdV hierarchies associated with Kac-Moody algebras other than $\widetilde{sl}(N)$. (We recall that the "simply" N-KdV hierarchy with which we have worked up to now turns out to be associated with the current algebra $sl(N)$, and so we shall now call it the $\widetilde{sl}(N)$-KdV hierarchy.) As we have already seen, the construction that we have developed of Virasoro generators in terms of dressing operators permits us to make advances in the analysis of hierarchies restricted by Virasoro constraints. It is obvious that the Virasoro-constrained $\widetilde{G}$-KdV hierarchies obtained from the matrix formulation by replacing the algebra $sl(N)$ by $\widetilde{G}$ will describe the interaction with gravity of the corresponding minimal models from their ADE classification.[41] (We note in this connection the work of Yen.[23])

To construct the Virasoro generators in the $sl(N)$ formalism, we shall use the experience gained in Sec. 6, in which the Virasoro generators on another hierarchy associated with the current algebra $sl(N)$ were found. Although, from the standpoint of Sec. 7, the $sl(N)$-Toda hierarchy belongs to the "discrete" hierarchies, while the N-KdV hierarchy is undoubtedly "continuous," which is important for us now is only the formal structure of the Virasoro generators on hierarchies associated with the $sl(N)$ current algebra. It turns out that this structure can be copied from the $sl(N)$-Toda to the $sl(N)$-KdV hierarchy in such a way that both consistency with the flows and the Virasoro commutation relations will hold.[7)]

We recall that the equations of the $sl(N)$-KdV hierarchy are imposed on the differential operator

$$\mathscr{L}=\frac{\partial}{\partial x}+L \tag{10.1}$$

where L has the form (6.11). There exists a matrix W of the form

$$W=1+\sum_{j\geq1}\sum_{s=0}^{N-1}|s\rangle w_j(s)\langle s|\Lambda_N(\zeta)^{-j}, \tag{10.2}$$

such that

$$L=W\left(\frac{\partial}{\partial x}+L_0\right)W^{-1},\qquad L_0=\Lambda_N(\zeta)+\sum_{i\geq0}f_i\Lambda_N(\zeta)^{-i}. \tag{10.3}$$

where the f_i are scalar functions. Transformations of the form (10.2) form a group of gauge transformations.

The flows of the $sl(N)$-KdV hierarchy are induced on gauge-equivalence classes by the following equations:

$$\frac{d\mathscr{L}}{dt_{a,i}}=-[(\zeta^iA^a)_-,\mathscr{L}],\qquad A=W\Lambda_N(\zeta)W^{-1}. \tag{10.4}$$

Here, to within the gauge equivalence, it does not matter which of the two possibilities—negative powers of $\Lambda_N(\zeta)$ or negative powers of ζ—is chosen as $(\ldots)_-$.

The times $t_{a,i}$ are labeled in the same way as in the scalar variant: $a\in\{1,2,\ldots,N-1\}$ $(i\geqslant0)$. It is important, however, to note that we do not now identify the lowest time $t_{1,0}$ with the variable x. The flow corresponding to this time "is trivialized" [cf. (2.4)] only on the gauge-equivalence classes, but not directly in terms of the matrices with which we are working.

The flows (10.4) are induced by the following evolution equations on the dressing operators:

$$\frac{dW}{dt_{a,i}}=-(W\zeta^i\Lambda_N(\zeta)^aW^{-1})_-W. \tag{10.5}$$

Introducing

$$U=WC,$$

where C is an ordered exponential—the solution of the equation

$$\left(\frac{\partial}{\partial x}+L_0\right)C=0,$$

we achieve fulfillment of the equality

$$\mathscr{L}U=0,$$

after which

$$\Upsilon = U \exp \sum_{a,i} t_{a,i} \zeta^i \Lambda_N(\zeta)^a \tag{10.6}$$

also satisfies the condition

$$\mathscr{L}\Upsilon = 0. \tag{10.7}$$

This is one of the equations of a "generalized linear system" (cf. Ref. 35). Besides (10.7), we also have

$$\frac{d\Upsilon}{dt_{a,i}} = -(W\zeta^i \Lambda_N(\zeta)^a W^{-1})_- \Upsilon. \tag{10.8}$$

The Virasoro transformations have the form

$$\delta_j W = \mathfrak{L}_j W, \Rightarrow \delta_j \mathscr{L} = [\mathfrak{L}_j, \mathscr{L}], \tag{10.9}$$

where

$$\mathfrak{L}_{(j)} = \Big(W\Big\{ -\zeta\frac{\partial}{\partial\zeta} + \hat{\pi} + \sum_{a=1}^{N-1} \Big[\sum_{i\geqslant 0} \Big(i + \frac{a}{N}\Big) t_{a,i}\zeta^i \Big] \Lambda_N(\zeta)^a \Big\} \zeta^j W^{-1} \Big)_-, \tag{10.10}$$

$$\hat{\pi} = \frac{1}{N}\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & N \end{pmatrix} + \dots$$

[cf. (6.13)]. The condition for the transformations (10.9), in which $\mathfrak{L}_j = (W l_j W^{-1})_-$, to be consistent with the flows (10.10) reduces, in the standard way, to

$$\Big[\Big(W\Big(\frac{\partial l_j}{\partial t_{a,i}} + [\zeta^i \Lambda^a, l_j] \Big) W^{-1} \Big), \mathscr{L} \Big] = 0,$$

which is fulfilled for (10.10). Just below, we shall show that in the "reduction"[3] to scalar (pseudo)differential operators the Virasoro generators (10.10) are consistent with those which we had in Sec. 9.

The explicit form of the Virasoro *constraints* now follows, obviously, from (10.10). Equating $\mathfrak{L}_j$ to zero for $j \geqslant -1$, we arrive at the unified equation

$$\frac{\partial W^{-1}}{\partial\zeta} = W^{-1}A_+ + \Big[\hat{\pi} + 1 + \sum_{a,i} \Big(i + \frac{a}{N}\Big) t_{a,i}\zeta^i \Lambda_N(\zeta)^a \Big] \zeta^{-1} W^{-1}, \tag{10.11}$$

where A_+ is a certain $(\dots)_+$ matrix.[8] Here the definition of $(\dots)_+$ in the sense of powers of the spectral parameter ζ turns out to be very advantageous.

Rewriting (10.11) for the matrix Ψ introduced in (10.6), we have

$$-\frac{\partial\Upsilon}{\partial\zeta} = A_+\Upsilon + \Upsilon(\hat{\pi}+1)\zeta^{-1}. \tag{10.12}$$

Thus, Eq. (10.12) gives a refined variant of one of the equations of the "generalized linear system" proposed for the formulation of the string equation in Ref. 35. It should be noted that we have obtained (10.12) by a direct derivation from the Virasoro constraints, whereas in Ref. 35, strictly speaking, only the contrary implication was asserted. In addition, the more invariant approach that we have developed yields automatically the generalizations from $sl(2)$ to $sl(N)$ and next, with the use of the technique of Ref. 3, to other algebras.

More important, however, is the following remark: Not only is the use in Ref. 35 of the canonical form of the Lax operator for the generalizations $sl(2) \to sl(N)$ incorrect in view of the fact that his canonical form is not preserved during evolution along the flows of the hierarchy (see Ref. 3), but the very formulation of the string equation as the condition $[\mathscr{P}, \mathscr{Q}] = 1$, where $\mathscr{P}$ and $\mathscr{Q}$ are matrix differential operators, is not equivalent to the scalar string equation $[P,Q] = 1$. We shall demonstrate this in the next subsection.

10.2. Correspondence with the scalar formalism

First of all we shall elucidate the question of whether the Virasoro generators (9.10) and (10.5) correspond to each other in any sense upon standard "reduction" from the matrix to the scalar formalism.

We shall recall the correspondence between the matrix description and the scalar description.[3] Suppose, as before, that ψ Diff denotes a ring of pseudodifferential operators of the form

$$A = \sum a_i D^i, \quad D = \frac{\partial}{\partial x}.$$

Suppose also that $\mathscr{F}$ denotes the space of vectors that are columns of height N, the elements of which are formal Laurent series in ζ and simultaneously functions of x. A given operator $\mathscr{L}$ of the form (10.1) makes it possible to define the following action of ψ Diff on $\mathscr{F}$, transforming $\mathscr{F}$ into an algebra over the ψ Diff ring:

$$\underset{\mathscr{L}}{\cdot} : \psi\text{Diff} \times \mathscr{F} \mapsto \mathscr{F},$$

$$A \underset{\mathscr{L}}{\cdot} \eta = \sum a_i \mathscr{L}^i \eta, \tag{10.13}$$

where, in the right-hand side, we have in mind matrix multiplication and the standard action of the operator $D = \partial/\partial x$ on a function of x. Because of the fact that $[\mathscr{L}, f] = \partial f$ for a scalar function $f(x)$, we have

$$D \underset{\mathscr{L}}{\cdot} (f\eta) = (f\eta) \underset{\mathscr{L}}{\cdot} \eta + (\partial f) \qquad (D \circ f) \underset{\mathscr{L}}{\cdot} \eta,$$

whence $A, B \in \psi$Diff

$$(AB) \underset{\mathscr{L}}{\cdot} \eta = A \underset{\mathscr{L}}{\cdot} (B \underset{\mathscr{L}}{\cdot} \eta), \tag{10.14}$$

so that the action of (10.13) does indeed make $\mathscr{F}$ an algebra over ψ Diff (Ref. 3).

The correspondence between the matrix operator $\mathscr{L}$ and the scalar operator L [see (9.1)] is specified by the formula

$$L \underset{\mathscr{L}}{\cdot} \eta_0 = \zeta\eta_0, \quad \eta_0 \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \tag{10.15}$$

from which L is determined uniquely in terms of the given $\mathscr{L}$ and is found to be an operator of the form

$$L=D^N+\sum_{i=0}^{N-1} u_iD^i,$$

where the u_i are certain functions of x.

To verify the consistency of the scalar and matrix Virasoro generators, we shall find the total variation of the left-hand side of (10.15) under the simultaneous action of generators of both types. Thus, we shall vary both the "argument" L and the operation $\dot{\mathscr{L}}$ itself. More precisely, as we shall see, in view of the choices of signs made earlier, the transformations to be made consistent are now the simultaneous transformations

$$\delta_j L=[\mathfrak{L}_j^{sc}, L], \quad \delta_j\mathfrak{L}=-[\mathfrak{L}_j^{m}, \mathscr{L}], \qquad (10.16)$$

where the superscripts indicate the scalar and matrix Virasoro generators (9.5) and (10.10), respectively. Thus, we calculate

$$\delta_j (L\underset{\mathscr{L}}{\cdot}\eta_0)=(\mathfrak{L}_j^{sc}L)\underset{\mathscr{L}}{\cdot}\eta_0-(L\mathfrak{L}_j^{sc})\underset{\mathscr{L}}{\cdot}\eta_0$$
$$-\Big\{[\mathfrak{L}_j^{m}, \mathscr{L}^N]+\sum_{i=0}^{N-1} u_i[\mathfrak{L}_j^{m}, \mathscr{L}^i]\Big\}\eta_0,$$

where the u_i are the same coefficients of the unperturbed operator L as above. Taking into account that $[\mathfrak{L}_j^m, u_i]$, and also using (10.15) again, and, of course, (10.14), we obtain

$$\delta_j (L\underset{\mathscr{L}}{\cdot}\eta_0)=\mathfrak{L}_j^{sc}\underset{\mathscr{L}}{\cdot}\zeta\eta_0-\mathfrak{L}_j^{m}\zeta\eta_0$$
$$-L\underset{\mathscr{L}}{\cdot}(\mathfrak{L}_j^{sc}\underset{\mathscr{L}}{\cdot}\eta_0-\mathfrak{L}_j^{m}\eta_0)=\zeta^{j+1}\eta_0$$
$$+(\zeta-L\underset{\mathscr{L}}{\cdot})\{\mathfrak{L}_j^{sc}\underset{\mathscr{L}}{\cdot}\eta_0-\mathfrak{L}_j^{m}\eta_0\}. \qquad (10.17)$$

We recall now that the generator $\mathfrak{L}_j^{sc}$ appearing here is the $\mathfrak{L}_{(j)}$ from Eq. (9.5). To be more precise, we refrain from identifying the "first" time $t_{1,0}$ with the variable x, and, using also the definition (9.1), we write the scalar Virasoro generator in the form

$$\mathfrak{L}_j^{sc}=\frac{1}{N}(KxD^{Nj+1}K^{-1})_-+\sum_{a=1}^{N-1}\sum_{i\geq 0}\Big(i+\frac{a}{N}\Big)t_{a,i}(L^{i+j+a/N})_-. \qquad (10.18)$$

Strikingly,[3]

$$(L^{i+j+a/N})_-\underset{\mathscr{L}}{\cdot}\eta_0=(W\Lambda_N(\zeta)^{N(i+j)+a}W^{-1})_-\eta_0, \qquad (10.19)$$

where W is defined precisely by the relations (10.13), and $(\ldots)_-$ in the right-hand side singles out the negative powers of $\Lambda_N(\zeta)$. In this way, for the $\Sigma\Sigma$ part of the generators (10.18) we have

$$\sum_{a=1}^{N-1}\sum_{i\geq 0}(\ldots)\underset{\mathscr{L}}{\cdot}\eta_0$$
$$=\Big(W\Big(\sum_{a,i}\Big(i+\frac{a}{N}\Big)t_{a,i}\zeta^i\Lambda_N(\zeta)^a\Big)\zeta^jW^{-1}\Big)_-\eta_0, \qquad (10.20)$$

which coincides with the corresponding part in (10.10). It remains only to calculate the $\dot{\mathscr{L}}$ action of the first term in (10.18).

As shown in Ref. 3, the arrangement of the $(\ldots)_-$'s is "equivariant." It remains, therefore, to find the χ_j from the formula

$$\frac{1}{N}(KxD^{Nj+1}K^{-1})\underset{\mathscr{L}}{\cdot}\eta_0=\chi_j\eta_0. \qquad (10.21)$$

The pseudodifferential operator from the left-hand side satisfies the commutation relation

$$\Big[L^{1/N}, \frac{1}{N}(KxD^{Nj+1}K^{-1})\Big]=\frac{1}{N}L^{j+1/N},$$

so that for χ_j the following equality should be fulfilled:

$$[W\Lambda_N(\zeta)W^{-1}, \chi_j]=\frac{1}{N}W\Lambda_N(\zeta)^{Nj+1}W^{-1},$$

which determines χ_j in the form

$$\chi_j=W\left[-\zeta\frac{\partial}{\partial\zeta}+\frac{1}{N}\begin{pmatrix}1 & & 0\\ & 2 & \\ & & \ddots \\ 0 & & & N\end{pmatrix}+\sum h_i\Lambda_N(\zeta)^{-i}\right]\Lambda_N(\zeta)^{Nj}W^{-1}, \qquad (10.22)$$

where the h_i are arbitrary scalars, as in (6.13).

Thus, combining (10.20) and (10.22), we obtain

$$\mathfrak{L}_j^{sc}\underset{\mathscr{L}}{\cdot}\eta_0=\left(W\left[-\zeta\frac{\partial}{\partial\zeta}+\frac{1}{N}\begin{pmatrix}1 & & 0\\ & 2 & \\ & & \ddots \\ 0 & & & N\end{pmatrix}+\sum_{a,i}\Big(i+\frac{a}{N}\Big)t_{a,i}\zeta^i\Lambda_N(\zeta)^a+\ldots\right]\zeta^jW^{-1}\right)_-\eta_0, \qquad (10.23)$$

where the ellipsis (...) denotes the terms $h_i\Lambda_N(\zeta)^{-i}$ from (10.22). Choosing the latter to coincide (and even to be equal to zero), both in the definition (7.13) of the matrix $\hat{\pi}$ appearing in (10.10), and in (10.22), we see that the expression in the curly brackets in (10.17) vanishes and the variation of the equality (10.15) takes the form

$$\delta_j^{tot}(L\underset{\mathscr{L}}{\cdot}\eta_0-\zeta\eta_0)=0, \qquad (10.24)$$

where δ_j^{tot} includes the variations (10.16) and also the usual ("reparametrization") action of the Virasoro algebra on the complex parameter:

$$\delta_j\zeta=l_j\zeta=\zeta^{j+1}. \qquad (10.25)$$

This result gives an exhaustive answer to the question of the consistency of the Virasoro action on the N-KdV hierarchy in the scalar formalism and in the Drinfel'd-Dokolov formalism.

There is an important difference between the matrix description and the scalar description in regard to the string equation. In the literature (see, in particular, Ref. 35), it has been assumed that the string equation in the matrix formalism is obtained, as in the scalar case, by equating to zero the commutator of two operators, one of which is the Lax operator

(10.1). We recall that, in view of the structure (9.5) of the Virasoro generators, the constraint $\mathfrak{L}_{-1}^{sc} \equiv (\mathscr{P}^{sc})_-$ implies that the operator $\mathscr{P}^{sc}$ is a purely differential operator. Then, trivially,

$$[L, \mathscr{P}^{sc}] = 1 \tag{10.26}$$

where $L = KD^N K^{-1}$ is the Lax operator (also a differential operator!). But in application to the matrix formalism the condition (10.26) implies that

$$[\zeta, \mathscr{P}^m] = 1,$$

where, again, $(\mathscr{P}^m)_- = \mathfrak{L}_{-1}^m$. This equation has a different structure, since, contrary to the scalar case, ζ is not an eigenvalue of the Lax operator $\mathscr{L}$, and, generally speaking, $[\mathscr{L}, \mathscr{P}^m] \neq 1$.

A similar comment also applies to the analogous matrix constructions for other Kac–Moody algebras. In the next section we also encounter, albeit in an entirely different context, a situation in which the string equation requires, as a minimum, further commentary.

11. SUPERSYMMETRIZATION OF THE VIRASORO CONDITIONS

Up to now we have considered models for which the correspondence with the matrix formulation is known. The situation with supersymmetric theories is somewhat different. While the naive matrix formulation can be trivialized,[26] it is undoubtedly true that both supersymmetric theories interacting with supergravity and, of course, supersymmetric hierarchies themselves exist. In this situation it is in fact extremely natural to *postulate* Virasoro constraints on dressing operators as a "first principle," if, or course, this can be done in agreement with the equations of the hierarchy. We note that, precisely because of the (odd) flows of the hierarchy, it has turned out to be impossible to manage with just one analog of the string equation.[42] At the same time, we shall see that the dressing-operator formalism makes it possible to impose in an entirely satisfactory manner Virasoro constraints subordinate to all flows of the super-KP hierarchy.

Thus, we shall introduce the super-KP hierarchy by extending the space of ψ Diff operators to super-ψ Diff operators. We shall describe the version of the supersymmetric KP hierarchy due to Manin and Radul.[43] The Mulase–Rabin variant[44] admits an analogous analysis. Let ξ be the superpartner of the variable x, and let

$$\nabla = \frac{\partial}{\partial \xi} + \xi D, \qquad \nabla^2 = D \equiv \frac{\partial}{\partial x}. \tag{11.1}$$

The dressing operator is now represented as

$$K = 1 + \sum_{n \geqslant 1} w_n \nabla^{-n}. \tag{11.2}$$

where $\{w_n\}$ are functions of x and ξ. Next, we introduce even times $\{t_{2j}, j \geqslant 1\}$ and odd times $\{\tau_{2j+1}, j \geqslant 0\}$. The even times are the ordinary times of the KP hierarchy, relabeled as $t_j \to t_{2j}$ since the odd indices are now used for the odd times.

The flows of the super-KP hierarchy have the form

$$\boldsymbol{\Theta}_m K = -(K \nabla^m K^{-1})_-, \quad m = 1, 2, \ldots, \tag{11.3}$$

where we have introduced the vector fields

$$\boldsymbol{\Theta}_{2j} = \frac{\partial}{\partial t_{2j}}, \quad j \geqslant 1,$$

$$\boldsymbol{\Theta}_{2j+1} = -\frac{\partial}{\partial \tau_{2j+1}} + \sum_{k=0}^{\infty} \tau_{2k+1} \frac{\partial}{\partial t_{2j+2k+2}}, \quad j \geqslant 0 \tag{11.4}$$

and $(\ldots)_-$ denotes taking only the negative powers of the odd differentiation ∇. We note that in the supercase the variable ξ is not identified with any of the odd times τ_{2j+1}. For symmetry of the entire formalism, we also refrain from identifying x with the lowest even time t_2. (Thus, $[D, t_2] = 0$.)

On the basis of our previous experience, it is natural to expect that the super-Virasoro generators on the space of the operators K will be given by left multiplication on a purely $(\ldots)_-$ super-ψ Diff operator of the form $(K \ldots K^{-1})_-$. The "bare" generators can be found be requiring, first, satisfaction of the Virasoro algebra, and, second, consistency with the equations of the hierarchy. Thus, we arrive at the following Neveu–Schwarz generators:

$$\mathfrak{G}_{m-1/2} = (K[(\Pi - (P + 2mJD^{-1})(\overline{\nabla} + \Pi D)] D^m K^{-1})_-, \tag{11.5}$$

where $m \in \mathbb{Z}$ and

$$P = x + \sum_{j \geqslant 1} j t_{2j} D^{j-1} + \xi \sum_{j \geqslant 0} \tau_{2j+1} D^j + \sum_{j \geqslant 0} \sum_{i \geqslant 0} \frac{j-i}{2} \tau_{2j+1} \tau_{2i+1} D^{i+j} - \sum_{j \geqslant 0} j \tau_{2j+1} D^{j-1} \nabla \tag{11.6}$$

is, obviously, the superextension of the operator (3.7), while

$$\Pi = \xi - \sum_{i \geqslant 0} (i+1) \tau_{2i+1} D^i \tag{11.7}$$

is its superpartner. Furthermore, the operator

$$\overline{\nabla} = \frac{\partial}{\partial \xi} - \xi D \tag{11.8}$$

commutes with ∇. If we insist on a rigorous formulation of all the operators as elements of a ring of superdifferential operators in ∇, we ought to express $\overline{\nabla}$ as $\nabla - 2\xi \nabla^2$.

Commuting the vector fields $\widehat{\mathfrak{G}}$ corresponding to the generators $\mathfrak{G}$ with each other, we find the even Virasoro generators

$$\mathfrak{L}_n = \left(K \left[PD + (n+1) J + \frac{n+1}{2} \Pi \overline{\nabla} \right] D^n K^{-1} \right)_-, \quad n \in \mathbb{Z}. \tag{11.9}$$

These, obviously, are the superextension of the generators (8.2) [with a more subtle $(\ldots)_-$ operation!].

It is now not difficult to verify that infinitesimal deformations of the dressing operator that have the form $\delta K = \mathfrak{L}_n K$ and $\mathfrak{G}_{m-(1/2)} K$ are tangent to the manifold of solutions of the super-KP hierarchy.

Thus, the Virasoro-constrained super-KP hierarchy can be defined by the equalities

$$\begin{cases} \mathfrak{G}_{m-1/2} = 0, & m \geqslant 0, \\ \mathfrak{L}_n = 0, & n \geqslant -1. \end{cases} \tag{11.10}$$

The entire previous analysis of the algebraic structure and symmetries of the Virasoro-constrained KP hierarchy can be carried over without difficulty to the supercase, since it is based wholly on the algebraic structure of the "bare" generators, i.e., on the commutation relations between the operators P, D, Π and $\overline{\nabla}$, ∇.

As regards the string equation, we note that, of course, the operator

$$\mathscr{P}=KPDK^{-1},$$

which is a super-*differential* operator by virtue of the constraint $\mathfrak{L}_{-1}=0$, satisfies the equality

$$[Q^2, \mathscr{P}]=1, \qquad (11.11)$$

in which

$$Q=K\nabla K^{-1} \qquad (11.12)$$

is a Lax operator. On the other hand, if we attempt to find the superpartner of this string equation we discover that, although the equality $\mathfrak{G}_{-1/2}=0$ is precisely the condition for the operator

$$\mathscr{B}=K(\Pi-P(\overline{\nabla}+\Pi D))K^{-1},$$

to be a superdifferential operator, nevertheless its commutator with the Lax operator (11.12) is not too comprehensible, and, generally speaking, is not equal to 1. It turns out that not Q, but another operator

$$\mathscr{C}=K(\overline{\nabla}+\Pi D)K^{-1}$$

satisfies the equality

$$\{\mathscr{C}, \mathscr{B}\}=1.$$

In reality, there is no need to insist on the string equation as the first principle in the description of supersymmetric models interacting with gravity. Postulating that the object that is actually needed is a Virasoro-constrained (super)hierarchy, we see that, in the supercase as well, it is natural to define such hierarchies with the use of the more powerful technique of dressing operators. We note at the same time that less important here is the correspondence with the description in the language of the τ-function (which, in fact, would give rise to further complications in the supercase).

An analysis of the super-KP hierarchy and of the Virasoro constraints on it from a somewhat different ("Grassmannian") standpoint has been performed by Schwarz in Ref. 45. There is no doubt that the supervariant of the technique developed in Ref. 46 will make it possible to make the results of Ref. 45 just as explicit as in our above analysis, and the equivalence of the Grassmannian approach of Ref. 45 to the above-described "(super)pseudodifferential" approach to Virasoro superconstraints on supersymmetric hierarchies will thereby be established.

12. CONCLUDING REMARKS

We shall look once again at the Virasoro constraints with which we have worked; in the Introduction these constraints were written out for the toda hierarchy, and so now we let the KP case play the role of the example:

$$(K[J(n+1)D^n+PD^{n+1}]K^{-1})_-, \quad P=\sum_{r\geq 1} rt_rD^{r-1}. \qquad (12.1)$$

The fact that the times of the hierarchy appear explicitly in this only linearly is essentially what determines the relative simplicity with which the continuum limit and N-reduction were implemented above. The simplifications that have arisen are the essence of the dressing method. In fact, we shall consider the "bare" Virasoro generators, i.e.,

$$l_n=J(n+1)D^n+PD^{n-1}. \qquad (12.2)$$

In view of the commutation relation

$$[D, P]=1 \qquad (12.3)$$

this is none other than the usual representation

$$J(n+1)\theta^n+\theta^{n+1}\frac{\partial}{\partial\theta} \qquad (12.4)$$

of the diffeomorphisms of a circle. The corresponding dressing of this simple construction leads to Virasoro generators (on the hierarchies) that are no longer so banal. The dual role [emphasized by the commutation relation (12.3)] of the operators P and D, is related, as was suggested in Ref. 47, to the (p,q) symmetry of the minimal models; in a recent paper (Ref. 48; see also Ref. 45) this has in fact been established.

The approach that we have developed makes it possible to introduce Virasoro-constrained hierarchies without reference to the presence of the τ-function, directly in terms of dressing operators. We had such examples in Secs. 10 and 11. Also, when the continuum limit of discrete hierarchies is taken, and/or reductions are performed, the question of the τ-function corresponding to, say, the reduced hierarchy becomes secondary: First one performs the reduction in the language of the dressing operators, and only then is it possible to seek, if one wishes, the τ-function corresponding to the new hierarchy obtained.

Non-Abelian symmetries of the Virasoro type have been known to experts for some time as "mastersymmetries" of integrable hierarchies.[49]

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[1)] Any reasonable number of the corresponding references that would be admissible within the framework of the present article, which is not a special review, would be inadequate.

[2)] Referring to Ref. 11, we have in mind, strictly speaking, only the KP hierarchy; however, one of the principal observations of the present paper consists precisely in the universality of the construction for $\mathfrak{T}$ (and hence of the statements that follow below in the text) for a whole series of integrable hierarchies.

[3)] We refrain here from considering the subtleties associated with the fact that a natural interpretation of the flows of integrable systems is achieved in terms of the *co*-adjoint action of one of the subalgebras associated with the given r-matrix; see Ref. 13. As in most other applications, we assume the presence of an isomorphism between the Lie algebra and its dual space.

[4)] In relation to the citing of papers devoted to W-algebra in quantum field theory, and also to applications of the method of Drinfel'd and Sokolov to the study of these algebras, as well as to the field-theoretical interpretation and quantization of the Drinfel'd–Sokolov approach itself, we again find ourselves in the situation described in footnote 1.

[5)] In the classic papers (see Refs. 9 and 30 and the further bibliography in these papers) on algebro-geometric solutions, the τ-function and wave functions are true functions. Variants in which the wave function is a cross section of a bundle of J-differentials [or $(1-J)$-differentials, for ψ^*], and also the forms of the corresponding τ-functions, have appeared comparatively recently in a number of papers; see, e.g., the derivation of these objects from the operator formalism in Ref. 31.

6) For information on other approaches to the continuum limit of discrete hierarchies, see Ref. 20.

7) A possible formal reason for this is the consistency[3] of the $sl(N)$-Toda hierarchy and the mKdV hierarchy: The mKdV flows along the "third" set of times t_n preserve the $sl(N)$-Toda equations, and so it is natural to assume the existence of Virasoro transformations consistent with all xyt-equations.

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