Localization and persistent current in a one-dimensional disordered loop

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The mean value of persistent current $|j_n|$ in a disordered loop enclosing a magnetic flux has been calculated. Due to localization, this mean value decreases exponentially with increasing parameter L/l, where L is the length of the loop and l is the mean free path.

Since the beginning of the sixties the problem of persistent electric current in a ring of normal metal, enclosing the Aharonov–Bohm magnetic flux, has been discussed.[1–4] The paper by Büttiker, Imry, and Landauer,[5] in which the persistent current in a one-dimensional ($1D$) loop has been considered, has boosted the interest of theorists to this problem.[6–8] In 1990 the effect was measured at last in an aggregate of 10^7 mesoscopic copper rings.[9]

The persistent current is due to the following. The magnetic flux in the ring destroys time-reversal symmetry. Therefore the degeneracy of the states carrying current clockwise and counterclockwise is lifted. As a result, for a given occupation there arises uncompensated current flowing in this or in that direction, depending on the position of the Fermi level. We can choose the gauge in such a way that the presence of the magnetic flux ϕ will manifest itself only in the boundary condition

$$\psi_n(x+L)=\exp(2i\pi\Phi/\Phi_0)\psi_n(x). \tag{1}$$

Here L is the ring circumference and $\Phi_0 = hc/e$ is the flux quantum. An electron, having completed one revolution around the axis, finds itself in the same potential. This periodicity is related to the Bloch wave vector k_n, which is defined by the condition (1):

$$k_n=\frac{2\pi}{L}\left(n+\frac{\Phi}{\Phi_0}\right). \tag{2}$$

Thus, the spectrum ε_n depends periodically on Φ with a period Φ_0. The velocity

$$v_n=\frac{1}{\hbar}\frac{\partial\varepsilon_n}{\partial k_n}=\frac{Lc}{e}\frac{\partial\varepsilon_n}{\partial\Phi} \tag{3}$$

corresponds to the state with the index n. Therefore this state carries the current[1]

$$j_n=-\frac{ev_n}{L}=-c\frac{\partial\varepsilon_n}{\partial\Phi}. \tag{4}$$

At zero temperature the total current j is obtained by summing (4) over N lowest states. As shown in Ref. 7, in a $1D$ loop the correlations between the energy spectrum $\varepsilon_n(\Phi)$ and the currents j_n are so strong that each next higher (in energy) current j_N is sufficient to cancel the sum of all the previous currents. In other words, the total current j has the same sign, but a smaller magnitude than the current j_N of the last occupied level.

Since the magnetic flux Φ enters the problem only via the boundary condition (1), the considered effect depends, to a considerable degree, on the conservation of the phase coherence over the length L. But in a $1D$ disordered loop there is localization of the wave functions $\psi_n(x)$ on a scale l_c of the order of the mean free path l. Therefore, if $L \gg l_c$, the wave functions $\psi_n(x)$ have an exponentially weak dependence on the boundary condition (1). As a result, we should expect exponential attenuation of the persistent current: $j \sim j_n \propto \exp(-L/l_c)$. It turns out that a rigorous derivation of this intuitively clear formula is not simple. The thing is that the boundary condition (1) does not allow the formulation of a Markov process for the electronic wave function (it is not known beforehand with which values of ψ and $d\psi/dx$ to begin). However, it is possible to formulate the Markov process for the transfer-matrix.[10]

The aim of the present paper is to calculate the mean value of $|j_n|$ over a random potential when the eigenvalue ε_n coincides with the Fermi energy ε_F:

$$J=\left\langle\frac{1}{L}\sum_n\delta(\varepsilon_F-\varepsilon_n)\,|j_n|\right\rangle. \tag{5}$$

We use the quasiclassical condition

$$K_F=(2m\varepsilon_F)^{1/2}\gg 1/l. \tag{6}$$

We have succeeded in carrying out the calculations to the end only in the case of a weak magnetic field to a first nonvanishing order in the parameter

$$\Phi/\Phi_0 \ll 1. \tag{7}$$

Let us begin with construction of the Markov process for the transfer-matrix. Cutting the loop at any point, we unfold it into a disordered segment. The transfer matrix T connects the amplitudes of the waves traveling to the left and to the right before (A,B) and after (A',B') the disordered segment:

$$\begin{bmatrix}A'\\B'\end{bmatrix}=T\begin{bmatrix}A\\B\end{bmatrix}=\exp(2\pi i\Phi/\Phi_0)\begin{bmatrix}1/t^* & r/t\\ r^*/t^* & 1/t\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix}. \tag{8}$$

Since the magnetic flux is included here in the definitions of A' and B', the boundary conditions will be periodic:

$$A'=A,\quad B'=B. \tag{9}$$

The amplitudes of transmission, t, and reflection, r, are related by $|t|^2+|r|^2=1$. Therefore the T-matrix is parametrized in the following way:

$$T=\exp(2\pi i\Phi/\Phi_0)\begin{bmatrix}\operatorname{ch}\Gamma e^{i\alpha} & \operatorname{sh}\Gamma e^{i\beta}\\ \operatorname{sh}\Gamma e^{-i\beta} & \operatorname{ch}\Gamma e^{-i\alpha}\end{bmatrix}. \tag{10}$$

For $L=0$ we have

$$\Gamma=0,\quad \alpha=0,\quad \beta=0. \tag{11}$$

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If the length L of the disordered segment is increased by a small amount a, the transfer-matrix changes:

$$T(L+a)=\begin{bmatrix} 1+iK_Fa-i\delta & -i\gamma \\ i\gamma & 1-iK_Fa+i\delta \end{bmatrix} T(L). \tag{12}$$

Assume that the amplitudes of forward, δ, and backward, γ, scattering are random quantities of the white-noise type with a zero mean value

$$\langle\delta\rangle=\langle\gamma\rangle=0. \tag{13}$$

Their mean-square values are connected with the mean free paths for backward, l, and forward, l_f, scattering:

$$\frac{1}{l}=\frac{1}{a}\langle\gamma^2\rangle,\quad \frac{1}{l_f}=\frac{1}{a}\langle\delta^2\rangle. \tag{14}$$

From (12) we find the following increments of the transfer-matrix parameters:

$$\Delta\Gamma=-\gamma\sin(\alpha+\beta), \tag{15}$$

$$\Delta\alpha=K_Fa-\delta-\gamma\,\mathrm{th}\,\Gamma\cos(\alpha+\beta), \tag{16}$$

$$\Delta\beta=K_Fa-\delta-\gamma\,\mathrm{cth}\,\Gamma\cos(\alpha+\beta). \tag{17}$$

The periodic boundary conditions (9) reduce to the energy quantization rule[10]

$$f(\varepsilon_n)=\mathrm{ch}\,\Gamma\cos\alpha=\cos(2\pi\Phi/\Phi_0). \tag{18}$$

Using this rule, we can rewrite the expression (4) for the current carried by the nth level in the form[6,7]

$$j_n=\frac{e}{\hbar}\frac{\sin(2\pi\Phi/\Phi_0)}{f'(\varepsilon_n)}=\frac{e}{\hbar}\frac{\mathrm{tg}(2\pi\Phi/\Phi_0)}{\Gamma_\varepsilon'\,\mathrm{th}\,\Gamma-\alpha_\varepsilon'\,\mathrm{tg}\,\alpha}. \tag{19}$$

As usual, in the $1D$ case an important role is played by the phase. Its vanishing means that the energy quantization rule holds. To define such a phase in a natural manner, it is necessary to write first the expression for the s-matrix. The latter connects the waves (A,B') incident on the disordered segment with the scattered waves (A',B). Using Eq. (10), we find

$$\begin{bmatrix} A' \\ B \end{bmatrix}=S\begin{bmatrix} A \\ B' \end{bmatrix}$$

$$=e^{i\alpha}\begin{bmatrix} \cos\theta\exp(2\pi i\Phi/\Phi_0) & \sin\theta\, e^{i\beta} \\ -\sin\theta\, e^{-i\beta} & \cos\theta\exp(-2\pi i\Phi/\Phi_0) \end{bmatrix}\begin{bmatrix} A \\ B' \end{bmatrix}, \tag{20}$$

where $\cos\theta=1/\cosh\Gamma$. The eigenvalues of the S-matrix, $\lambda_{1,2}=\exp(i\varphi_{1,2})$ unambiguously define two scattering phases

$$\varphi_{1,2}=\alpha\pm\arccos[\cos\theta\cos(2\pi\Phi/\Phi_0)], \tag{21}$$

reduced to the interval $(-\pi,\pi)$. Now the quantization rule acquires the form

$$\det(S-1)=(e^{i\varphi_1}-1)(e^{i\varphi_2}-1)=0. \tag{22}$$

Therefore, for $\varepsilon=\varepsilon_n$ one of the phases $\varphi_{1,2}$ should vanish:

$$\frac{1}{L}\sum_n\delta(\varepsilon-\varepsilon_n)=\frac{1}{L}[\delta(\varphi_1)\varphi_{1\varepsilon}'+\delta(\varphi_2)\varphi_{2\varepsilon}']. \tag{23}$$

Thus, in the doubly connected geometry two phases arise, possessing the necessary qualities.

Let us use now the condition (7), meaning that the magnetic field is weak. The only place where the small parameter Φ/Φ_0 should be retained is the expression (19) for the current. Then the expression for the phases (21) becomes simpler:

$$\varphi_{1,2}=\alpha\pm\theta. \tag{24}$$

The unknown mean value (5) can be rewritten in the form

$$J=\frac{1}{L}\Big\langle[\delta(\alpha+\theta)(\alpha_\varepsilon'+\theta_\varepsilon')+\delta(\alpha-\theta)(\alpha_\varepsilon'-\theta_\varepsilon')]$$
$$\times\frac{e}{\hbar}\frac{2\pi\Phi/\Phi_0}{|\theta_\varepsilon'\,\mathrm{tg}\,\theta-\alpha_\varepsilon'\,\mathrm{tg}\,\alpha|}\Big\rangle$$
$$=\frac{e}{\hbar L}\left(\frac{2\pi\Phi}{\Phi_0}\right)\langle[\delta(\alpha+\theta)+\delta(\alpha-\theta)]\,|\mathrm{ctg}\,\theta|\rangle. \tag{25}$$

The cancellation of derivatives with respect to energy, which takes place under the condition (7), is a considerable simplification allowing to carry out the calculations to the end. In the opposite case, i.e., for $\Phi\sim\Phi_0$, we should have constructed the Markov process for six connected random variables θ, α, β, θ_ε', α_ε', and β_ε'.

If we introduce a new variable F using the relations

$$\cos\theta=\frac{1}{\mathrm{ch}\,\Gamma}=\left(\frac{2}{F+1}\right)^{1/2},\quad \sin\theta=\mathrm{th}\,\Gamma=\left(\frac{F-1}{F+1}\right)^{1/2}, \tag{26}$$

then, for the length of the disordered segment increased by a small value a, the increments ΔF, $\Delta\alpha$, and $\Delta\beta$ will amount to

$$\Delta F=-2\gamma\sin(\alpha+\beta)(F^2-1)^{1/2}, \tag{27}$$

$$\Delta\alpha=K_Fa-\delta-\gamma\cos(\alpha+\beta)\left(\frac{F-1}{F+1}\right)^{1/2}, \tag{28}$$

$$\Delta\beta=K_Fa-\delta-\gamma\cos(\alpha+\beta)\left(\frac{F+1}{F-1}\right)^{1/2}. \tag{29}$$

According to the quasiclassical condition (6), the variable $\alpha+\beta$ is rapid, i.e., the coefficients of the Fokker–Planck equation for the distribution function $W(L;F,\alpha,\beta)$ must be averaged over this variable. The second-order increments

$$\Delta\Delta F=4\gamma^2F,\quad \Delta\Delta\alpha=0,\quad \Delta\Delta\beta=0 \tag{30}$$

are found from the relations (27)–(29). Thus, the equation to be found has the form

$$l\frac{\partial W}{\partial L}=\Big\{-K_Fl\left(\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\right)+\frac{\partial}{\partial F}(F^2-1)\frac{\partial}{\partial F}$$
$$+\frac{\partial^2}{\partial\alpha\,\partial\beta}\left(\frac{l}{l_f}+\frac{1}{2}\right)+\frac{1}{2}\frac{\partial^2}{\partial\alpha^2}\left(\frac{l}{l_f}+\frac{1}{2}\frac{F-1}{F+1}\right)$$
$$+\frac{1}{2}\frac{\partial^2}{\partial\beta^2}\left(\frac{l}{l_f}+\frac{1}{2}\frac{F+1}{F-1}\right)\Big\}W(L;F,\alpha,\beta). \tag{31}$$

Its initial condition is

$$W(0;F,\alpha,\beta)=\delta(F-1)\delta(\alpha)\delta(\beta). \tag{32}$$

It follows from Eq. (31) that for $L>1/K_F$ stochastization over the rapid variables α and β occurs, and the solution no longer depends on them:

$$W(L;F,\alpha,\beta)|_{L\gg 1/K_F}=\frac{1}{(2\pi)^2}W(L;F). \tag{33}$$

The right-hand side of (33) satisfies the equation well-known in the theory of $1D$ localization[11-13]

$$l\frac{\partial W}{\partial L}=\frac{\partial}{\partial F}(F^2-1)\frac{\partial}{\partial F}W(L;F). \tag{34}$$

Its solution is found with the help of the Möller–Fock transformation ($F=\cosh u$),

$$W(L;F)=W(L;\mathrm{ch}\,\tilde{u})$$
$$=\frac{\exp(-L/4l)}{\pi^{1/2}}\left(\frac{l}{2L}\right)^{3/2}\int\limits_u^\infty dx\frac{x\exp(-x^2l/4L)}{(\mathrm{ch}\,x-\mathrm{ch}\,u)^{1/2}}. \tag{35}$$

Taking into account (33), we find the following expression for the mean value (25):

$$J=\frac{e}{\hbar L}\left|\frac{2\pi\Phi}{\Phi_0}\right|\frac{1}{\pi}\left\langle\left(\frac{2}{F-1}\right)^{1/2}\right\rangle. \tag{36}$$

Calculating the integrals, we finally get

$$J=\left(\frac{\pi}{2}\right)^{1/2}\frac{e}{\hbar l}\left|\frac{\Phi}{\Phi_0}\right|\left(\frac{2l}{L}\right)^{1/2}\exp\left(-\frac{L}{4l}\right). \tag{37}$$

This expression is obtained for $L\gg 1/K_F$. In particular, it is valid, when $L\sim l$.

Thus, we have calculated the mean value of the current modulus at the Fermi level (5). According to Ref. 7, the total current in the $1D$ loop should be of the same order of magnitude. In the regime of strong localization, $L\gg l$, the energy spectrum ε_n has an exponentially weak dependence on the magnetic flux in the loop. Note that as $l/L\to\infty$ the mean value (37) grows obeying a power law. The saturation occurs when L becomes of the order of the wavelength $1/K_F$.

At last, note once more that we have been able to carry out the calculations to the very end only due to the cancellation of the derivatives α'_ε and θ'_ε with respect to energy in Eq. (25). In the case of a strong magnetic field $\Phi\sim\Phi_0$, or while calculating higher moments of the current, such a complete cancellation does not occur, and the equations for the distribution function of six random variables, $W(L;F,\alpha,\beta,F'_\varepsilon,\alpha'_\varepsilon,\beta'_\varepsilon)$, should be solved.

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