Temperature dependence of the upper critical field of type II superconductors with fluctuation effects

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Fluctuations of the order parameter are taken into consideration in an analysis of the temperature dependence of the upper critical field of a type II superconductor with a three-dimensional superconductivity. This temperature dependence is of universal applicability, to all type II superconductors, if the magnetic fields and temperatures are expressed in appropriate units. This dependence is derived explicitly for the regions of strong and weak magnetic fields. The results are applied to high T_c superconductors, for which fluctuation effects are important. For these superconductors, the $H_{c2}(T)$ dependence is quite different from the linear dependence characteristic of the mean-field theory, over a broad range of magnetic fields.

INTRODUCTION

According to the mean-field theory (the Ginzburg–Landau theory), the upper critical field H_{c2} of a type II superconductor is a linear function of the temperature near T_c. One would expect that the corrections to this dependence for fluctuations would be very small in the case of conventional superconductors, because of their extremely small Ginzburg numbers ($\mathrm{Gi} \sim 10^{-11}$–$10^{-14}$). In the high T_c superconductors, in contrast, fluctuation effects are far larger ($\mathrm{Gi} \propto \xi_0^{-4}$, where ξ_0 is the correlation length at $T = 0$). For YBaCuO, for example, the Ginzburg number is[1] $\mathrm{Gi} \sim 10^{-3}$.[1] Accordingly, the $H_{c2}(T)$ dependence for the new superconductors may be very nonlinear even near T_c. This circumstance may be responsible for the curvature observed on plots of $H_{c2}(T)$ for the high T_c superconductors in many studies (e.g., Ref. 2).

Golubov and Dorin[3] included fluctuations in the order parameter in a study of the effect on $H_{c2}(T)$ of an electron-electron interaction at the microscopic level (in the BCS model). That study was restricted to dirty superconductors in first-order perturbation theory in this interaction. Shapiro[4] and Bulaevskiĭ *et al.*[5] have studied $H_{c2}(T)$ theoretically in the fluctuation region [$|\tau| < \mathrm{Gi}$, where $\tau \equiv (T - T_c)/T_c$] by a microscopic approach, without specifying the superconductivity mechanism. Through the use of the Ψ theory,[6] they found the result $H_{c2} \propto |\tau|^{4/3}$. There is a point to be noted about that result. In the functional of the Ψ theory which determines the free energy, only the simplest of the terms containing powers of the gradient, specifically $|(-i\hbar\nabla - 2e\mathbf{A}/c)\psi|^2$, was retained in Refs. 4 and 5; here $\mathbf{A}$ is the vector potential, and ψ the order parameter. This approach of ignoring terms with higher powers of the gradient is justified in the Ψ theory only for describing situations in which the variations in the order parameter occur over length scales much larger than the correlation length $\xi(\tau)$ at the given temperature.[6] [The functional of the Ψ theory is essentially a block Hamiltonian, in which a smoothing is carried out to scales on the order of $\xi(\tau)$.] This requirement is not satisfied, however, near the line $H_{c2}(\tau)$. The nucleating regions of the superconducting phase which appear have a length scale on the order of $\xi(\tau)$, as in the Ginzburg-Landau theory (Ref. 4). The result derived for $H_{c2}(\tau)$ in Refs. 4 and 5 thus cannot be regarded as having a solid foundation. Nevertheless, we would expect it to correctly convey the functional dependence of H_{c2} on τ in weak magnetic fields.

Below we show that the $H_{c2}(\tau)$ dependence is noticeably nonlinear not only inside the critical region ($|\tau| < \mathrm{Gi}$) but also outside it. Accordingly, in the present study[1)] we analyze this dependence for the case in which there are fluctuation effects over the entire temperature range with $|\tau| \ll 1$. We use the results and methods of the phase-transition theory of Ref. 8. In Sec. 1 we state the problem. In Sec. 2 we analyze the overall $H_{c2}(T)$ dependence and show that it is universal for all type II superconductors, if the magnetic fields and temperatures are expressed in appropriate units. In the next two sections of the paper we derive explicit expressions for the functional dependence of H_{c2} on T in the limiting cases of strong and weak magnetic fields. The results of this study are stated in the Conclusion.

1. STATEMENT OF THE PROBLEM

The field $H_{c2}(T)$ is generally understood as the magnetic field below which the superconducting phase appears at the given temperature. This definition requires some refinement when fluctuations of the order parameter are taken into consideration. We define the line $H_{c2}(T)$ in the (H, T) plane as the line of second-order phase transitions from the normal phase to the mixed state. We start from the assumption that such a line exists, without proving it. Let us examine the shape of this line.

In a study of the superconducting phase transition one must in general consider fluctuations of the vector potential along with fluctuations of the order parameter. We know, however, that the contribution of the fluctuations of the vector potential to the thermodynamic properties of a type II superconductor is comparatively small, provided that the Ginzburg–Landau parameter is sufficient large (Ref. 9, for example). The size of the critical region with respect to fluctuations of the electromagnetic field was estimated in Ref. 10. It was found that these fluctuations become important only under the condition $|\tau| \lesssim \mathrm{Gi}_A \sim \mathrm{Gi}\varkappa^{-3}$, where $\varkappa$ is the value of the Ginzburg–Landau parameter far from the transition point ($|\tau| > \mathrm{Gi}$). Since Gi_A is small even for the high T_c superconductors ($\varkappa \sim 10^2$, $\mathrm{Gi}_A \sim 10^{-9}$), fluctuations of

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the electromagnetic field can be ignored in a study of the $H_{c2}(T)$ dependence. Below we accordingly assume that the magnetic field is spatially uniform and equal to the given value H.

To describe the fluctuating field of the order parameter ψ we use the methods of Ref. 8. To determine H_{c2}, it is sufficient to consider the fluctuations in the order parameter only in the normal phase, i.e., above the line $H_{c2}(T)$. This approach is to be understood everywhere below. The part of the partition function in which we are interested and the binary correlation function are given by path integrals:

$$Z = \int \exp\left(-\frac{\mathscr{H}}{T}\right) D\psi D\psi^*, \quad (1)$$

$$\langle\psi^*(\mathbf{r})\psi(\mathbf{r}')\rangle = Z^{-1}\int \psi^*(\mathbf{r})\psi(\mathbf{r}')\exp\left(-\frac{\mathscr{H}}{T}\right) D\psi D\psi^* \quad (2)$$

where $\mathscr{H}$ is the Ginzburg–Landau Hamiltonian. We will be using dimensionless variables everywhere below. Lengths are expressed in units of $\xi_0 = (c/\alpha)^{1/2}$, and magnetic fields in units of $h = H/H_{c2}^{\mathrm{GL}}(0)$, where $H_{c2}^{\mathrm{GL}}(0) = \Phi_0/2\pi\xi_0^2$, and Φ_0 is the flux quantum. The dimensionless order parameter ψ is related to the dimensional order parameter used everywhere in Ref. 8 by $\psi = \psi_{\mathrm{dim}}[\alpha\xi_0^3/(2T)]^{1/2}$. The quantities c and α are the coefficients of the Ginzburg–Landau Hamiltonian, in the standard notation.[8] In dimensionless variables, this Hamiltonian is

$$\frac{\mathscr{H}}{T} = \int d\mathbf{r}\{|(-i\nabla+\mathbf{A})\psi|^2 + \tau|\psi|^2 + 2^{3/2}\pi\,\mathrm{Gi}^{1/2}|\psi|^4\}. \quad (3)$$

The reason for the numerical factor in front of $\mathrm{Gi}^{1/2}$ in the last term in (3) is that the Ginzburg number is defined here as in Ref. 1. For the dimensionless vector potential A we choose the gauge $A = (-hy, 0, 0)$.

Below we use a representation for ψ in addition to the coordinate representation. We expand the order parameter in the eigenfunctions ψ_λ of the operator $\hat{\mathscr{H}}_0 = (-i\nabla + A)^2 + \tau$: $\psi = \Sigma_\lambda c_\lambda \psi_\lambda$, $\hat{\mathscr{H}}_0\psi_\lambda = E_\lambda\psi_\lambda$. The index λ represents the set of wave-vector projections k_x, k_z and the nonnegative integer n: $\lambda = \{k_x, k_z, n\}$. The eigenvalues are $E_\lambda = \tau + k_z^2 + h(2n+1)$; the ψ_λ are normalized by $\delta_{nn'}\delta(k_x - k_x')\delta(k_z - k_z')$ and are the same as the eigenfunctions of a charged particle in a magnetic field. In this representation, Hamiltonian (3) is

$$\frac{\mathscr{H}}{T} = \sum_n \int d\mathbf{q} E_\lambda |c_\lambda|^2 + \frac{(\mathrm{Gi}\,h)^{1/2}}{2\pi^{1/2}} \sum_{n_1n_2n_3n_4} \int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3 d\mathbf{q}_4$$
$$\times\, \delta(\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}_3-\mathbf{q}_4) I(\lambda_1,\lambda_2,\lambda_3,\lambda_4) c_{\lambda_1}^* c_{\lambda_2}^* c_{\lambda_3} c_{\lambda_4}, \quad (4)$$

where $\mathbf{q} \equiv (k_x, k_z)$ is a two-dimensional vector, and the coefficient $I(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is given by

$$I(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \frac{\exp\{-{}^1/_2[(\varkappa_1-\varkappa_2)^2+(\varkappa_3-\varkappa_4)^2]\}}{(n_1!\,n_2!\,n_3!\,n_4!)^{1/2}}$$
$$\times \sum_{r=0}^{\min(n_1+n_2,\,n_3+n_4)} \frac{r!}{2^r} P_{n_1+n_2-r}^{(r-n_1,\,r-n_2)}(0) P_{n_3+n_4-r}^{(r-n_3,\,r-n_4)}(0)$$
$$\times H_{n_1+n_2-r}(\varkappa_1-\varkappa_2) H_{n_3+n_4-r}(\varkappa_3-\varkappa_4). \quad (5)$$

Here $\varkappa_i \equiv k_{xi}/(2h)^{1/2}$, $H_n(x)$ are the Hermite polynomials, and $P_n^{(\alpha,\beta)}(0)$ are the values of the Jacobi polynomials at the origin of coordinates. Here and below, all the orthogonal polynomials are defined as in Ref. 11. It can be shown that in this representation the binary correlation function defined by

$$\langle c_{\lambda_1}^* c_{\lambda_2}\rangle = Z^{-1}\int c_{\lambda_1}^* c_{\lambda_2} \exp\left(-\frac{\mathscr{H}}{T}\right) \prod_\lambda dc_\lambda\, dc_\lambda^*,$$

is diagonal with respect all its indices, is independent of k_x, satisfies

$$\langle c_\lambda^* c_{\lambda'}\rangle = \delta_{n,n'}\delta(k_x-k_x')\delta(k_z-k_z')G_n(k_z), \quad (6)$$

and is related to the correlation function in the coordinate representation by

$$G_n(k_z) = \int d\mathbf{r} \exp(ik_z z)\exp\left[-\frac{h}{4}(x^2+y^2)\right] L_n\left(\frac{h}{2}(x^2+y^2)\right)$$
$$\times \exp\left(ih\frac{xy}{2}\right)\langle\psi^*(\mathbf{r})\psi(0)\rangle, \quad (7)$$

where $L_n(x)$ are Laguerre polynomials. The quantity $\exp(ihxy/2)\times\langle\psi^*(\mathbf{r})\psi(0)\rangle$ in (7) is simply $\langle\psi^*(\mathbf{r})\psi(0)\rangle$ for the case in which the vector potential is chosen in the cylindrical gauge: $\mathbf{A} = [\mathbf{hr}]/2$. It is clear from symmetry considerations that this quantity depends on only $x^2 + y^2$ and $|z|$. This circumstance was utilized in the derivation of (6) and (7).

In a given magnetic field, we find the temperature of the phase transition from a normal phase to the mixed state in the standard way,[8] as the point at which a condensate appears:

$$[G_0(k_z=0, h, \tau_c)]^{-1} = 0. \quad (8)$$

We have a few words of explanation regarding Eq. (8). First, by virtue of its meaning, Eq. (8) should contain that correlation function which leads to the maximum value of τ_c. In the normal phase, it determines the occupation numbers of the least stable mode of the order parameter; it is this mode which leads to the appearance of a condensate at τ_c. It can be shown that, for a binary correlation function in any representation found by expanding ψ in any system of functions other than ψ_λ, the corresponding τ_c is no greater than that found with the help of G_n. This assertion follows from the diagonal nature of $\langle c_\lambda^* c_\lambda'\rangle$ with respect to all its indices [see (6)]. This diagonal nature is in turn a consequence of simply the symmetry properties of the system. Second, in the Gaussian approximation ($\mathrm{Gi}\to 0$), the quantities $G_n(k_z)$ can be calculated easily:

$$G_n^{(0)}(k_z) = \frac{1}{E_\lambda} = \frac{1}{\tau + k_z^2 + h(2n+1)}. \quad (9)$$

In this case, condition (8) leads to a result which has also been found in the mean-field theory: $\tau_c = -h$. The quantity $(G_0^{(0)})^{-1}$ vanishes earlier—i.e., at a higher temperature—than the other quantities $(G_n^{(0)})^{-1}$. It is natural to assume that this assertion remains in force in the general case $\mathrm{Gi}\neq 0$. It is for this reason that we selected $n = 0$ in (8). We will have some further comments regarding this assumption at the end of the fourth section of this paper.

Condition (8) means that as the phase-transition point is approached the correlation length along the magnetic field, ξ_z, goes off to infinity. This conclusion follows from the equation which is the inverse of (7):

$$\exp\left(ih\frac{xy}{2}\right)\langle\psi^*(\mathbf{r})\psi(0)\rangle=\frac{h}{(2\pi)^2}\exp\left[-\frac{h}{4}(x^2+y^2)\right]$$
$$\times\sum_{n=0}^{\infty}L_n\left(\frac{h}{2}(x^2+y^2)\right)\int dk_z\exp(-ik_z z)G_n(k_z).$$

It follows from this equation that the correlation lengths perpendicular to the magnetic field remain finite even at $\tau=\tau_c$. They are on the order of the magnetic length $1/h^{1/2}$. There is thus no reason to believe that the hypothesis of scale invariance would be valid for describing a phase transition in the case $h\neq 0$. These aspects of the behavior of the correlation lengths in the Gaussian approximation were analyzed in Ref. 12.

2. UNIVERSAL NATURE OF THE $H_{c2}(T)$ DEPENDENCE

Hamiltonian (3) and thus all the correlation functions depend on the three parameters τ, h, and Gi. We set $\tau<0$, since the line $h_{c2}(\tau)$ lies in this region in the (h,τ) plane. We introduce the new coordinates r' and the new order parameter ψ' by means of

$$\mathbf{r}=|\tau|^{-1/2}\mathbf{r}',\quad \psi=|\tau|^{1/2}\psi'.$$

It is thus simple to show that the parameters appear only in the combinations $h/|\tau|$ and $(\mathrm{Gi}/|\tau|)^{1/2}$ in transformed Hamiltonian (3). Going over to the new variables in Eqs. (1), (2), and (7), we find (for example) that the quantity $G_0(k_z)$ depends on the parameters in the following way:

$$G_0^{-1}(k_z,\tau,h,\mathrm{Gi})=|\tau|F_0\left(\frac{k_z}{|\tau|^{1/2}},\frac{h}{|\tau|},\frac{\mathrm{Gi}}{|\tau|}\right),$$

where F_0 is some function of its arguments. Consequently, condition (8) leads to the equation

$$F_0(0,h/|\tau|,\ \mathrm{Gi}/|\tau|)=0,$$

which determines the line of phase transitions in the (h,τ) plane. The three parameters appear in two combinations in this equation. Solving this equation, we find $h_{c2}/|\tau|=f(|\tau|/\mathrm{Gi})$, where f is a function of one argument. The temperature dependence of the upper critical field can thus be written in the form

$$\frac{h_{c2}}{\mathrm{Gi}}=\frac{|\tau|}{\mathrm{Gi}}f\left(\frac{|\tau|}{\mathrm{Gi}}\right), \tag{10}$$

It follows that this dependence is of universal applicability in the sense that it holds for all type II superconductors if h_{c2}/Gi and $|\tau|/\mathrm{Gi}$ are used as variables. The entire dependence on the constants characterizing the superconducting material (i.e., the Ginzburg number) is incorporated in the units in which the magnetic fields and temperatures are expressed. Explicit expressions for the function f for strong ($h\gg\mathrm{Gi}$) and weak ($h\ll\mathrm{Gi}$) magnetic fields are derived in the following sections of this paper.

3. STRONG MAGNETIC FIELDS

As we know,[8] a Feynman-diagram technique can be used to write the correlation functions for the order parameter as perturbation-theory series. Here we treat the last term in (3) as the perturbation. To obtain information on the function f in (10), we analyze the corresponding expansion of the mass operator Σ_0 ($\Sigma_0\equiv\tau+h+k_z^2-G_0^{-1}$) of the Green's function G_0 given in (6). Figure 1 shows the first few diagrams corresponding to this expansion. We use the representation in which the Hamiltonian is given by (4) and (5). The Green's functions in the Gaussian approximation, (9), then correspond to the solid lines in the diagrams, while factors proportional to $(\mathrm{Gi}h)^{1/2}I(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ correspond to the vertices. An integration is to be carried out over all the internal two-dimensional wave vectors q_i under the condition that the conservation laws for these vectors hold at each vertex. A summation is to be carried out over the internal indices n_i.

In analyzing the diagrams, we must bear the following point in mind: A Hamiltonian of the type in (3) corresponds to a renormalizable field theory. In this theory, only the first two diagrams in Fig. 1 are primitively divergent.[13,14] Consequently, for a theory of this sort, all the divergences in the diagrams at large momenta (or their dependence on the cutoff parameter, which in this case is equal to $1/\xi_0$ in order of magnitude) can be eliminated through a renormalization of the "mass": the superconducting transition temperature T_c. In other words, this can be done by introducing in Hamiltonian (3) a counterterm $(\delta\tau_1+\delta\tau_2)|\psi|^2$, in which the constants $\delta\tau_1$ and $\delta\tau_2$ (which are independent of τ and h) are chosen in order to cancel the divergences (the dependence on the cutoff parameter) in respectively the first and second diagrams in Fig. 1. Below we assume that we have gone through this renormalization procedure, so the contribution of any diagram to Σ_0 is finite and independent of the cutoff parameter.

We consider an arbitrary diagram with l vertices. We wish to analyze the behavior of the corresponding contribution to the mass operator $\Sigma_0^{(l)}$ as a function of the parameters h, τ, and Gi. For such a diagram, there are $2l-1$ internal lines. Of the l conservation laws for the two-dimensional vectors $\mathbf{q}_i$, one leads to the equality of the external $\mathbf{q}$'s. Consequently, when we take these laws into account we have $(2l-1)-(l-1)=l$ integrations over the internal wave vectors k_{zi} and k_{xi}. To determine how these integrals depend on the parameters, we go over to the new integration variables $\varkappa_i=k_{xi}/H(2h^{1/2}$, $\tilde{k}_{zi}=k_{zi}/\varepsilon^{1/2}$, where $\varepsilon\equiv\tau+h$. From each Green's function in (9) we single out one factor of ε^{-1}. The expression inside the integral and summation signs will then depend on only the ratio h/ε, which appears

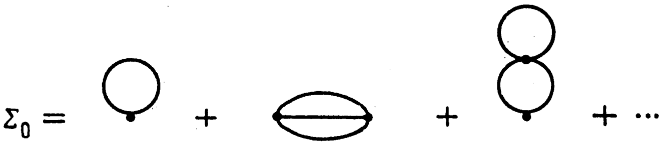

FIG. 1. The composition of the mass operator Σ_0 in powers of $\mathrm{Gi}^{1/2}$ of up to second order inclusively.

in the transformed $G_n^{(0)}$. We also note that all the integrals over $\varkappa_i$ can be calculated easily by taking account of the orthogonal nature of the Hermite polynomials which are incorporated in $I(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, according to (5). As a result, for an external $k_z = 0$ we find an expression of the following type for $\Sigma_0^{(l)}$ (we are omitting some inconsequential details associated with the renormalization):

$$\Sigma_0^{(l)}(k_z=0)=(\mathrm{Gi}\,h)^{l/2}\frac{(\varepsilon h)^{l/2}}{\varepsilon^{2l-1}}\sigma_l\left(\frac{h}{\varepsilon}\right). \tag{11}$$

Here $\sigma_l(h/\varepsilon)$ denotes the result of integrations over the internal $\tilde{k}_{zi}$ and $\varkappa_i$ and of all summations of products of transformed $G_n^{(0)}$ and factors $I(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Using (11), we easily find the universal functional dependence $h_{c2}(\tau)$ in (10) again.

According to (10), strong fields $h \gg \mathrm{Gi}$ are equivalent to small values of Gi. Consequently, if h is sufficiently large [Eq. (8)], then a root of equation h should lie near the straight line $\varepsilon = 0$ in the (h, τ) plane. This line corresponds to the line of phase transitions in the Gaussian approximation. In this region of parameter values, expression (11) simplifies further. According to (9), under the condition $\varepsilon \ll h$ the quantity $G_0^{(0)}$ is much larger than the other $G_n^{(0)}$'s, so we can restrict the summation over Landau levels to the single term with $n = 0$. In particular, all the sums in σ_l reduce to a single h/ε-independent term in which all the internal variables satisfy $n_i = 0$. In this single-level approximation, in which only the lower Landau level is taken into account from the outset, the quantity σ_l are constants, and $\Sigma_0(k_z = 0)$ is the product of ε and some function of a single argument:

$$\Sigma_0(k_z=0)=\varepsilon\sigma\left(\frac{\mathrm{Gi}^{1/2}h}{\varepsilon^{3/2}}\right). \tag{12}$$

We also note that in the case $\varepsilon \sim \mathrm{Gi}^{1/3}h^{2/3}$ all the terms in the series for the mass operator are on the same order of magnitude. Consequently, $\mathrm{Gi}^{1/3}h^{2/3}$ is the size of the fluctuation region at strong magnetic fields (at $h \gg \mathrm{Gi}^{1/3}h^{2/3}$). This result agrees with an estimate of the size of this region found from an analysis of the specific heat.[15]

Expression (12), which was derived through an analysis of a perturbation-theory series, is strictly valid within the convergence region of this series. Furthermore, in deriving (12) we used expression (9) for $G_n^{(0)}$, which is valid only for $\varepsilon > 0$. On the other hand, the root of Eq. (8) may lie at $\varepsilon < 0$, and it may apparently lie outside the convergence region of the perturbation-theory series. However, neither the boundary of this region nor the straight line $\varepsilon = 0$ is singled out in the (h, τ) plane by any physical consideration. The singular points of the exact Green's function lie exclusively on the true line of second-order phase transitions, $h = h_{c2}(\tau)$. The functional form of the mass operator should thus not change when we go outside the convergence region of the perturbation-theory series and when we go from positive to negative values of ε, provided only that we do not cross the phase-transition line. Consequently, in sufficiently strong magnetic fields, in the region in which the normal phase exists, and under the condition $|\varepsilon| \ll h$, the quantity $G_0^{-1}(k_z = 0) = \varepsilon - \Sigma_0(k_z = 0)$ can be written in the form

$$G_0^{-1}(k_z=0)=\mathrm{Gi}^{1/3}h^{2/3}R\left(\frac{\varepsilon}{\mathrm{Gi}^{1/3}h^{2/3}}\right), \tag{13}$$

which agrees with (12). Here $R(x)$ is some function of its argument. We note that the incorporation of other Landau levels (other than the zeroth) in the analysis of the mass operator will lead to a correction on the order of $(\mathrm{Gi}h)^{1/2}$ to the right side of (13).

Substituting (13) into (8), we find $\varepsilon_c = C_1 \mathrm{Gi}^{1/3}h^{2/3}$, where C_1 is the largest root of the equation $R(x) = 0$. Using the definition $\varepsilon = \tau + h$, we find the dependence of the transition temperature on the magnetic field:

$$\tau_c=-h+C_1\mathrm{Gi}^{1/3}h^{2/3}. \tag{14}$$

If we are to avoid a contradiction between (14) and the approximation $h \gg |\varepsilon_c|$, used in deriving this expression, we must require that the second term on the right side of (14) be much smaller than the first. This requirement refines the condition for a strong field. Solving Eq. (14) by an iterative method in h, we find, within quantities to $(\mathrm{Gi}/|\tau|)^{1/3}$ inclusive, the following temperature dependence for the upper critical field ($|\tau| \gg \mathrm{Gi}$):

$$h_{c2}(\tau)=-\tau+C_1\mathrm{Gi}^{1/3}(-\tau)^{2/3}. \tag{15}$$

For the function $f(x)$ which appears in (10) we find the following expression, for large values of the argument of this function ($x \gg 1$):

$$f(x)=1+C_1x^{-1/3}. \tag{16}$$

To determine the functional dependence in (15) more accurately, we would have to take account of the terms on the order of $(\mathrm{Gi}h)^{1/2}$ which we ignored in (13). In this case the corrections to (14) and (15) will depend on the behavior of the function $R(x)$ as $x \to C_1$. There is yet another circumstance to be noted. Since equality (13) is valid only in the single-level approximation, in deriving (14) we tacitly assumed somewhat more than the existence of a root of (8): We tacitly assumed that there also exists a root of the equation $R(x) = 0$. Otherwise, the functional form of the correction term in (14) would have been different. Finally, we note that a method similar to that used in the preceding section of this paper could be used to derive the result in (13) directly from a path integral representing G_0, if the single-level approximation were used from the outset in (4).

The line of phase transitions from the normal phase to the mixed state was studied in Ref. 16 in the single-level approximation, although it was not assumed there that this line corresponds to $H_{c2}(T)$ (and the concept of an upper critical field was not defined physically). The transition temperature was found in Ref. 16 as the point at which the coefficient of the quadratic term in a certain effective Hamiltonian changed sign. When there are four terms, that approach is generally incorrect. Nevertheless, the result found there leads to a functional dependence of h on τ which is the same as that in (15). This agreement appears to be a consequence of the circumstance that the correction to the result of the Ginzburg–Landau theory in (14) is proportional to the width of the fluctuation region, which is the only length scale which appears in any calculations in the single-level approximation.[17]

4. WEAK MAGNETIC FIELDS

In weak magnetic fields, $h \ll \mathrm{Gi}$, the fluctuation region becomes broader than the temperature interval $(\varepsilon < h)$ in which the single-level approximation could be valid. In this case, Landau levels in addition to the lowest should be taken into consideration in determining the very size of the critical region (this size turns out to be on the order of Gi). Throughout this region, a large number of these levels must be taken into account simultaneously in perturbation-theory calculations, so the single-level approximation is not valid for weak magnetic fields. In this situation we would not expect to obtain information on the function f in (10) by using the series of a perturbation theory in $\mathrm{Gi}^{1/2}$ for the analysis, as was done above. We instead examine the quantity G_0 at $\tau > 0$ by means of a perturbation theory in the magnetic field, in which the interaction $|\psi|^4$ is taken into account exactly. Then, staying inside the region of the normal phase, we continue the result to negative values of τ. We assume that everything which pertains to the case $h = 0$, $\tau > 0$ is known. Since weak fields correspond to $|\tau_c| \ll \mathrm{Gi}$, the values of the quantities at $h = 0$ need be known only inside the critical region. They have been studied well in this region,[8] since critical phenomena in superconductors in the absence of a magnetic field can be described by means of the hypothesis of gauge invariance.

Working in the **k** representation, we expand the order parameter in plane waves:

$$\psi(\mathbf{r}) = (2\pi)^{-3/2} \int d\mathbf{k}\, e^{i\mathbf{k}\mathbf{r}} b_{\mathbf{k}}.$$

In this representation, the Green's function $g(\mathbf{k})$, defined by

$$g(\mathbf{k}) \equiv \int d\mathbf{r} \langle \psi^*(\mathbf{r}) \psi(0) \rangle e^{i\mathbf{r}\mathbf{k}} = \int d\mathbf{k}' \langle b_{\mathbf{k}}^* b_{\mathbf{k}'} \rangle,$$

is related to $G_0(k_z)$ by an equation which follows from (7):

$$G_0(k_z) = \frac{1}{2^{1/2}\pi h} \int dk_x\, dk_y \times \exp\left(-\frac{k_x^2 + k_y^2}{2h} - \frac{i}{h} k_x k_y \right) g(k_x, k_y, k_z). \qquad (17)$$

The part of Hamiltonian (3) which depends on the magnetic field is

$$\frac{\mathcal{H}_{int}}{T} = \int d\mathbf{k}\, d\mathbf{k}'\, b_{\mathbf{k}}^* b_{\mathbf{k}'} \delta(k_x - k_x') \delta(k_z - k_z') \times [2ihk_x \delta'(k_y - k_y') - h^2 \delta''(k_y - k_y')]. \qquad (18)$$

Here $\delta'(x)$ and $\delta''(x)$ are the first and second derivatives of the δ-function. We will treat (18) as the perturbation of the Hamiltonian which we would obtain by setting $h = 0$ in (3). Figure 2 shows graphical representations of the corrections to the Green's function of up to second order in h, inclusively, at $\tau > 0$. There are vertices of two types (with one and two wavy lines), which describe the interaction of the order parameter ψ with the external magnetic field and which correspond to respectively the first and second terms in (18). The circles represent $\Gamma^{(n)}$, i.e., the exact vertex parts with n external ends for the $|\psi|^4$ interaction at $h = 0$, while the lines are associated with the exact Green's functions in the absence of the magnetic field: $g_0(\mathbf{k}) \equiv g(\mathbf{k}, h = 0)$. As an example, we write the expressions corresponding to the second and last diagrams in Fig. 2, within numerical factors:

$$ihg_0(\mathbf{k}) \int d\mathbf{p}\, p_x g_0(\mathbf{p}) \left[\frac{\partial}{\partial t_y} A(\mathbf{k}, \mathbf{p}, \mathbf{t}) \right]_{\mathbf{t}=\mathbf{p}}, \qquad (19)$$

$$h^2 g_0(\mathbf{k}) \int d\mathbf{p}\, g_0(\mathbf{p}) \left[\frac{\partial^2}{\partial t_y^2} A(\mathbf{k}, \mathbf{p}, \mathbf{t}) \right]_{\mathbf{t}=\mathbf{p}}. \qquad (20)$$

Here we have used

$$A(\mathbf{k}, \mathbf{p}, \mathbf{t}) \equiv g_0(\mathbf{t}) g_0(\mathbf{k}+\mathbf{t}-\mathbf{p}) \Gamma^{(4)}(\mathbf{k}, \mathbf{t}; \mathbf{p}, \mathbf{k}+\mathbf{t}-\mathbf{p}).$$

The differentiations with respect to t_y in (19) and (20) arose after the integrations by parts which eliminated the derivatives of the δ-functions in (18).

We now seek the dependence of the corrections to g_0 on the parameter τ. From Ref. 8 we have

$$g_0(\mathbf{k}) = \xi^{2-\eta} \tilde{g}_0(\mathbf{k}\xi),$$
$$\Gamma^{(2n)}(\mathbf{k}_1, \ldots, \mathbf{k}_{2n}) = \xi^{-\sigma_{2n}} \tilde{\Gamma}^{(2n)}(\mathbf{k}_1\xi, \ldots, \mathbf{k}_{2n}\xi), \qquad (21)$$
$$\sigma_{2n} = n(2-\eta) - 3(n-1),$$

where $\xi \propto \tau^{-\nu}$ is the correlation length in the fluctuation region at $\tau \ll \mathrm{Gi}$ and $h = 0$, and η and ν are critical exponents. Going over to the variables $\mathbf{k}\xi$, $\mathbf{p}\xi$, $\mathbf{t}\xi$ in (19) and (20), and using (21), we find that these expressions reduce to $ih\xi^{2(2-\eta)}$ and $h^2\xi^{2(2-\eta)+2}$, respectively, multiplied by

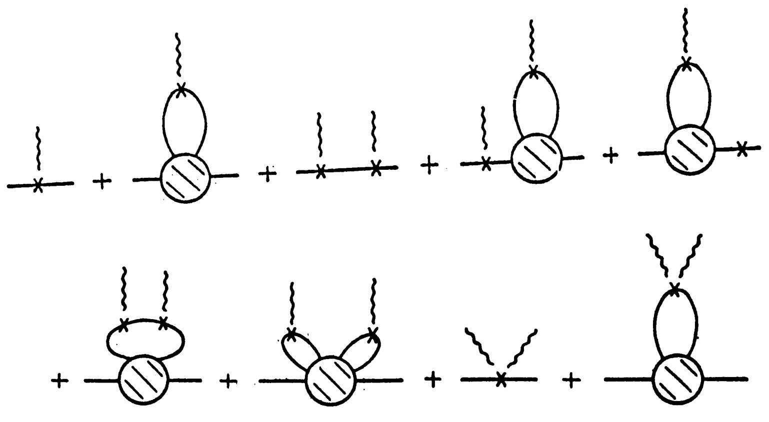

FIG. 2. Corrections to the Green's function g_0 of up to second order in the magnetic field inclusively.

certain functions $\mathbf{k}\xi$. Correspondingly, it is a straightforward matter to analyze the corrections to g_0 for all the other diagrams in Fig. 2. As a result we find that the structure of the function $g(\mathbf{k}, h, \tau)$ calculated within terms of order up to h^2, inclusively, is

$$g(\mathbf{k}, h, \tau)=\xi^{2-\eta}(\tilde{g}_0(\mathbf{k}\xi)+ih\xi^{2-\eta}a_{1,0}(\mathbf{k}\xi) +h^2\xi^{2(2-\eta)}a_{2,0}(\mathbf{k}\xi)+h^2\xi^{2-\eta+2}a_{0,1}(\mathbf{k}\xi)+\ldots),$$

where $a_{i,j}(x)$ are certain functions of one variable, and the term proportional to $h^2\xi^{3(2-\eta)}$ describes the contributions of the third through seventh diagrams.

Let us examine the form of the correction terms to g_0 in higher orders in h. An arbitrary diagram is a set of vertex parts $\Gamma^{(2n_i)}$, $i = 1,...,l$, all ends of which except the two outermost are connected in pairs by internal lines, with the result that a connected configuration is formed. Each such line (there is a total of $N=\Sigma_{i=1}^{l} n_i - 1$ such lines) contains at least one vertex of Hamiltonian (18) of one sort or another (see Fig. 3, which shows an example of such a diagram). We assume that there are s_1 vertices of the first type (with one wavy line) and s_2 of the second type (with two wavy lines), with $s_1+s_2 \geqslant N$. Let us examine the contribution of such a diagram to $g(\mathbf{k})$. Making use of the l conservation laws, one of which simply gives us the equality of the external wave vectors $\mathbf{k}$, we have $N-(l-1)$ three-dimensional integrations over internal wave vectors $\mathbf{p}_j$. Also appearing in this expression is the product of all the vertex parts and $N+2+s_1+s_2$ functions g_0. In addition, each vertex of the first type introduces a factor of the type ihp_{xj} and one differentiation with respect to the y component of the internal wave vector. A vertex of the second type contributes h^2 and two such differentiations. Going over to the new variables as we did earlier, using (21), and carrying out a simple calculation of the powers of ξ, we find that this contribution has the structure

$$\xi^{2-\eta}(ih\xi^{2-\eta})^{s_1}(h\xi^2h\xi^{2-\eta})^{s_2}a_{s_1,s_2}(\mathbf{k}\xi).$$

Consequently, $g(\mathbf{k}, h, \tau)$ depends on the magnetic field through the combinations $h\xi^{2-n}$ and $h\xi^2$:

$$g(\mathbf{k}, h, \tau)=\xi^{2-\eta}\tilde{g}(\mathbf{k}\xi, h\xi^2, h\xi^{2-\eta}).$$

Finally, a substitution of this expression into (17) gives us

$$G_0(k_z=0)=\xi^{2-\eta}\tilde{G}_0(h\xi^2, h\xi^{2-\eta}). \tag{22}$$

As in the preceding section of this paper, we continue G_0 out of the region in which the result in (22), derived by perturbation theory, is strictly valid, into the region $\tau<0$, to the line of the phase transitions. In addition, making use of the small value of the critical exponent η, we use the approximation $\eta=0$ below.[8] [We are justified in using this approximation to determine $h_{c2}(\tau)$ at least under the condition $\ln(\mathrm{Gi}/|\tau|) \ll 1/\nu\eta$, i.e., essentially throughout the fluctuation region, except in a small neighborhood of $\tau=0$, since ξ^{η} essentially reduces to a constant under this condition.] As a result we find

$$G_0(k_z=0,\tau,h)=\frac{1}{h}Q\left(\frac{\tau}{h^{1/2\nu}}\right), \tag{23}$$

where $Q(x)$ is some function of one argument. Here again we have made use of $\xi \propto \tau^{-\nu}$. From (8) and (23) we finally find

$$\tau_c(h) \propto h^{1/2\nu}, \quad h_{c2}(\tau) \propto |\tau|^{2\nu}. \tag{24}$$

Using $\nu \approx 2/3$ (Ref. 8), we find the final expression for the function $f(x)$ in (10) for small values of its argument ($x \ll 1$):

$$f(x)=C_2x^{4/3}, \tag{25}$$

where C_2 is a numerical coefficient. Note that (24) agrees qualitatively with the corresponding expressions in Refs. 4 and 5.

In formulating the problem we assumed that for each value of h it is G_0 among the various G_n which leads to the maximum value of τ_c. Actually, this assumption is not important for deriving the results of the present study, since in practice we have derived them without making use anywhere of the circumstance that it is G_0 rather than some other correlation function G_n which appears in (8). Only the values of the coefficients C_1 and C_2 in (16) and (25) depend on the particular correlation function which is in (8), but we have not determined those coefficients in the present study. In addition, the equivalence of strong fields to small values of Gi allows us to assert that this assumption is at any rate valid under the condition $h \gg \mathrm{Gi}$. Furthermore, since the curves corresponding to the functions (16) with different values of C_1 do not intersect, and a corresponding assertion holds for (25), then G_0 may be replaced by some G_n with $n \neq 0$ in (8) in the course of the decrease in h only at fields $h/\mathrm{Gi} \sim 1$. In this case there would be a change in slope of the $h_{c2}(\tau)$ line; such a change seems unlikely. However, even if it does occur, it would have no effect on the results found in this paper.

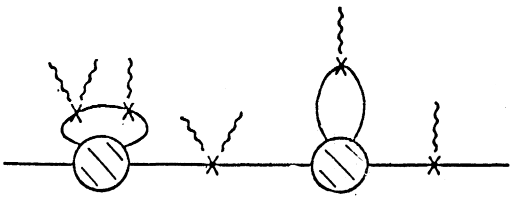

FIG. 3. Correction to the Green's function g_0 of seventh order in the magnetic field.

CONCLUSION

We have analyzed the temperature dependence of the upper critical field of a type II superconductor with fluctuations in the order parameter at temperatures $|\tau| \ll 1$. In the (H, T) plane, the line $H_{c2}(T)$ is defined as the line of second-order phase transitions from the normal phase to the mixed state. We have shown that the $H_{c2}(T)$ dependence, written in terms of the variables H_{c2}/H_{fl} and $(T_c-T)/T_{fl}$, is the same for all type II superconductors:

$$\frac{H_{c2}}{H_{fl}}=\frac{T_c-T}{T_{fl}}f\left(\frac{T_c-T}{T_{fl}}\right), \tag{26}$$

Here $f(x)$ is a universal function which does not depend on the constants characterizing the superconducting material; $T_{fl} \equiv T_c\mathrm{Gi}$; $H_{fl} \equiv H_{c2}^{\mathrm{GL}}(0)\mathrm{Gi}$; and $H_{c2}^{\mathrm{GL}}(0)$ is the value of the upper critical field at $T=0$ according to the Ginzburg-Landau theory. In strong fields $H \gg H_{fl}$ and weak fields $H \ll H_{fl}$

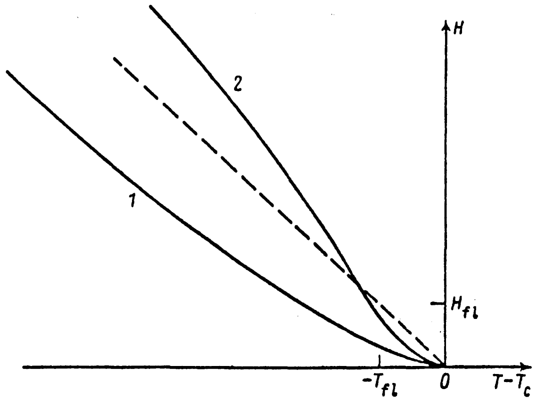

FIG. 4. Approximate overall behavior $H_{c2}(T)$. 1—$C_1<0$; 2—$C_1>0$. The dashed line show the results of the Ginzburg–Landau theory.

(or, equivalently, at temperatures $T_c - T \gg T_{fl}$ and $T_c - T \ll T_{fl}$), the function $f(x)$ is given by (16) and (25), respectively, in which C_1 and C_2 are numerical coefficients which have not been determined. Corresponding to these two limiting cases are different sizes of the fluctuation region along the temperature scale: T_{fl} for weak magnetic fields and $T_{fl}(H/H_{fl})^{2/3}$ for strong fields. In sufficiently strong magnetic fields, the experimental temperature dependence of the specific heat and that of the electrical resistance of a superconductor clearly exhibit an $H^{2/3}$ scaling (Refs. 18–20). Figure 4 shows two possible versions of the overall dependence $H_{c2}(T)$ corresponding to the different signs of C_1. We are apparently dealing with the case $C_1<0$.

For conventional superconductors, T_{fl} and H_{fl} are extremely small, and only the case of strong magnetic fields can be implemented experimentally. The ratio $(H/H_{fl})^{1/3}$ is quite high in this case. According to (16), the approximation $f(x)\approx 1$ is then quite accurate, so the result of the Ginzburg–Landau theory is always a good approximation for $H_{c2}(T)$ in these superconductors. In YBaCuO we have[2] $H_{c2}^{\mathrm{GL}}(0)\sim 10^3$ kOe. Making use of the value of the Ginzburg number for this superconductor,[1] we find $H_{fl}\sim 1$ kOe. Consequently, the deviations from a linear dependence $H_{c2}(T)$ should be seen experimentally over a broad range of magnetic fields. The field region $H\lesssim 10^2$ kOe has been studied in most detail experimentally. In this region, we could expect the Ginzburg–Landau theory and the result derived above to be applicable. The correction term in (16) is significant in this region, amounting to at least 20% of the leading term with $C_1\sim 1$. In the new superconductors, there is apparently no region in which the result $H_{c2}\propto|\tau|$ holds well.

As the phase-transition line is approached in a magnetic field, only the correlation length which is longitudinal with respect to H diverges; the transverse lengths remain finite (they are proportional to $H^{-1/2}$). We would accordingly expect that the nature of the singularities in the physical quantities upon a phase transition of this sort would be quite different than at $H=0$ (the singularities would apparently be weaker and would be seen only in the derivatives of physical properties with respect to the temperature or the magnetic field). A study of this question would clearly be of interest, and it would permit a correct experimental determination of the $H_{c2}(T)$ line.

The dependence $H_{c2}(T)$ was studied above under the assumption that the superconductivity is three-dimensional. In many of the new superconductors, however, there is apparently a transition from a three-dimensional behavior of the order parameter near T_c to a quasi-two-dimensional behavior far from T_c. This transition occurs at a certain temperature T_{cr}. The results found here apply to the region of the three-dimensional behavior and are apparently valid only under the condition $|\tau|\ll|\tau_{\mathrm{cr}}|$. Superconductors with a quasi-two-dimensional superconductivity, with low values of $|\tau_{\mathrm{cr}}|$, require a special study.[21]

[1)] The basic results of this study were published in Ref. 7.

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