Radiation by a rapidly decaying ring current

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If the current relaxation time in a ring is shorter than the time it would take light to propagate across the ring, the stored magnetic energy is radiated efficiently along the axis. A method is proposed for carrying out calculations on the current relaxation in a thin ring with allowance for the radiation and retardation. Some possible implementations of the effects found here are discussed.

INTRODUCTION

The radiation by any system of slowly varying currents is well understood (Ref. 1, for example). In this "magnetic dipole" case, the energy flux density P is given by the familiar formula

$$P=\frac{\ddot{m}^2\sin^2\theta}{4\pi c^3R_0^2}, \tag{1}$$

where $m = IS_R/c$ is the magnetic moment, I is the current, S_R is the area of the ring, θ is the angle between the observation point and the axis of the system, and R_0 is the distance to the observation point. The efficiency of the radiation in this case, by which we mean the ratio of the radiated energy to the stored energy, is given by

$$\eta\propto\frac{(\rho_0/c\tau_0)^3}{\ln(\rho_0/r)}. \tag{2}$$

It is assumed here that the radiation is being emitted by a ring current (Fig. 1) with a relaxation time τ_0 determined from, for example, the condition $I \sim I/\tau_0$. The proportionality factor (which is on the order of unity) depends on the shape of the current I. If the dipole approximation is valid, the efficiency is extremely small, as we easily see. We might add that the radiation does not have a pronounced spatial anisotropy.

In the other case in which a ring current is radiating as it decays rapidly, i.e., under the condition $\tau_0 \ll \rho_0/c$, the situation has been unclear, although it has been obvious that energy would be radiated very efficiently in the limit $\tau_0 \to 0$. For this range of relaxation times, there has been no quantitative study of the radiation efficiency, of the directional properties of the radiation, or of methods for realizing such relaxation times, to the best of our knowledge.

In this paper we are reporting a study of the radiation by a rapidly decaying ring current for relaxation times $r \ll c\tau_0$. We also examine the conditions under which this case can be realized.

In Sec. 1 we look at the general problem of the radiation by a monotonically decaying ring current. In Sec. 2 we examine the current relaxation process in a thin conducting ring under the assumption that the current has no effect on the state of the material. Here we take account of the inverse effect of the radiation and also retardation. We find the conditions under which a rapid current decay is possible. In Sec. 3 we take the Joule heating of the conductor into account in an examination of rapid current relaxation. We conclude with a discussion of the results and directions for further study.

1. RADIATION BY A RAPIDLY DECAYING RING CURRENT

Let us consider the field radiated by a monotonically decaying current in the geometry in Fig. 1. Since the problem is axisymmetric, it is a simple matter to write an expression for the only nonvanishing component of the vector potential, i.e., the azimuthal component A_φ (the scalar potential can be taken to be identically zero):

$$A_\varphi=\frac{\rho_0}{c}\int_0^{2\pi}d\varphi\cos\varphi\frac{I\left(t-(\rho_0^2+\rho^2+z^2-2\rho\rho_0\cos\varphi)^{1/2}/c\right)}{(\rho_0^2+\rho^2+z^2-2\rho\rho_0\cos\varphi)^{1/2}}. \tag{3}$$

Since we are interested in large distances ($R_0 \gg \rho_0$ and $R_0 \gg \rho_0^2/c\tau_0$), we can expand the expression in the radical in (3) in powers of $1/R_0$. As a result we find the expression

$$A_\varphi=\frac{\rho_0}{cR_0}\int_0^{2\pi}d\varphi\cos\varphi I(t-R_0/c+\rho\rho_0\cos\varphi/cR_0) \tag{4}$$

for the vector potential. For the electric field $E_\varphi = -A_\varphi/c$ we find

$$E_\varphi=-\frac{\rho_0}{c^2R_0}\int_0^{2\pi}d\varphi\cos\varphi\,\partial_tI(t-R_0/c+\rho\rho_0\cos\varphi/cR_0). \tag{5}$$

Up to this point we have not specified the time dependence of the current. To find some concrete results, we take the dependence $I(t)$ to be

$$I(t)=I_0(1-2/\pi\operatorname{arctg}(t/\tau_0))/2, \tag{6}$$

where τ_0 is a characteristic relaxation time of the current. We can then express the integral in (5) in terms of elementary functions:

$$E_\varphi=\frac{2^{1/2}\rho_0I_0}{c^2R_0\tau_0}F(\tau,\varepsilon),$$

where

$$F(\tau,\varepsilon)=\frac{1}{\varepsilon((\tau^2-\varepsilon^2-1)^2+4\tau^2)^{1/2}}\times\Big[[((\tau^2-\varepsilon^2-1)^2+4\tau^2)^{1/2}-\varepsilon^2-1+\tau^2]^{1/2}-\tau[((\tau^2-\varepsilon^2-1)^2+4\tau^2)^{1/2}+\varepsilon^2+1-\tau^2]^{1/2}\Big]. \tag{7}$$

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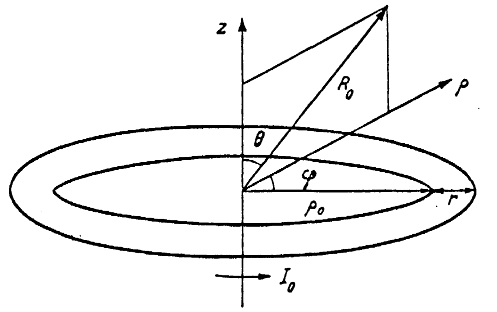

FIG. 1. Geometry of the problem.

[In (7) and below we are using $\tau = (t - R_0/c)/\tau_0$ and $\varepsilon = \rho_0 \sin\theta / c\tau_0$, and we are considering the case $\tau > 0$; for $\tau < 0$ we have $F(-\tau,\varepsilon) = -F(\tau,\varepsilon)$.]

Figure 2 shows the field as a function of the time and the observation angle. We see that in the case of a rapid current decay there is a direction, near the axis, along which the amplitude is a maximum. It also follows from Fig. 2 that the time between the maximum amplitudes of the pulse (positive and negative) increases with increasing observation angle. The reason is that an observer at direction near the axis of the ring receives the fields from all points of the ring nearly simultaneously, while an observer in the plane of the ring receives the pulse from the nearest half of the ring arrives first, and then, after a time $\Delta t = \rho_0/c$, the pulse from the other half.

For convenience we also write an expression for the spectrum of the radiation:

$$E_\varphi(\omega) = \int dt\, e^{i\omega t} E_\varphi(t) = \frac{4\pi\rho_0 I_0 i}{c^2 R_0} J_1\left(\frac{\omega\rho_0 \sin\theta}{c}\right)\exp(-|\omega\tau_0|).$$

For the energy flux density $P = c[EH]/4\pi$ we then find

$$P(\theta,\tau) = \frac{\rho_0^2 I_0^2}{2\pi c^3 R_0^2 \tau_0^2 \varepsilon^2((\tau^2-\varepsilon^2-1)^2+4\tau^2)} \times ([(\tau^2-\varepsilon^2-1)^2+4\tau^2]^{1/2}(1+\tau^2)-\varepsilon^2-1-2\tau^2-\tau^4+\tau^2\varepsilon^2). \quad (8)$$

If the current decays slowly ($\rho_0 \ll c\tau_0$), expression (8) naturally reduces to the magnetic-dipole approximation (1).

A quantity of importance in practice is the total energy which crosses a unit area in a given direction:

$$W(\theta) = \int dt P(\theta, t).$$

This integral can be expressed in terms of elliptic integrals of the first and second kinds:

$$W(\theta) = \frac{\rho_0^2 I_0^2}{\pi c^3 R_0^2 \tau_0^2 \varepsilon^2}\left(\frac{2+\varepsilon^2}{(1+\varepsilon^2)^{1/2}} K\left(\frac{\varepsilon}{(1+\varepsilon^2)^{1/2}}\right) - 2(1+\varepsilon^2)^{1/2} E\left(\frac{\varepsilon}{(1+\varepsilon^2)^{1/2}}\right)\right). \quad (9)$$

Figure 3 shows the behavior of the concentration coefficient $D(\theta)$ $[D(\theta) = 2W(\theta)/\int W(\theta)\sin(\theta)d\theta]$ as a function of the "rapidity parameter" of the process ($\rho_0/c\tau_0$). We see that with decreasing relaxation time a maximum arises in the energy flux density near the axis of the ring. The height of this maximum increases without bound. This spatial distribution of the radiation is qualitatively different from that in the dipole case, $\tau_0 \to \infty$.

Integrating expression (9) over the angle, we find the total amount of energy which is radiated:

$$W_{tot} = \frac{2\pi\rho_0 I_0^2}{c^2}\left(\ln(\nu+(1+\nu^2)^{1/2}) - \frac{2}{\nu}((1+\nu^2)^{1/2}-1)\right). \quad (10)$$

In (10) and below, the quantity $\nu = \rho_0/c\tau_0$ is the rapidity parameter of the process. If we now assume that all the initial energy W_{st} is stored in the magnetic field and that it has the value

$$W_{st} = \frac{2\pi\rho_0 I_0^2}{c^2}\ln(\rho_0/r), \quad (11)$$

then we can easily find the radiation efficiency $\eta = W_{tot}/W_{st}$:

$$\eta = \frac{\ln(\nu+(1+\nu^2)^{1/2}) - 2((1+\nu^2)^{1/2}-1)/\nu}{\ln(\rho_0/r)}. \quad (12)$$

For slow processes, expression (12) becomes the magnetic-dipole expression [see (2)], with a proportionality factor of 1/12. Figure 4 shows a plot of the radiation efficiency versus

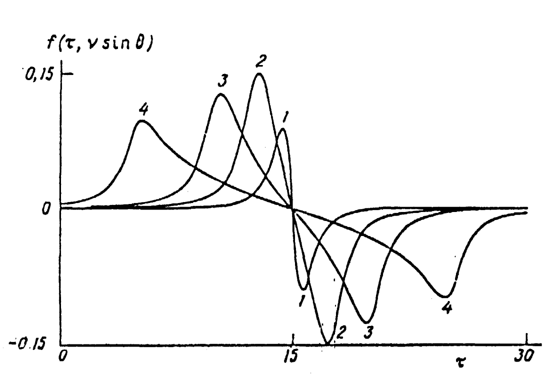

FIG. 2. Field as a function of the time and the angle in the case $\rho_0/c\tau_0 = 30$. 1—$\theta = 1°$; 2—5°; 3—10°; 4—20°.

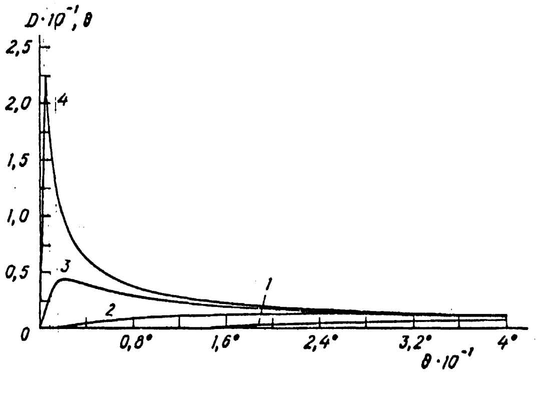

FIG. 3. Directional pattern of the radiation. 1—$\rho_0/c\tau_0 = 1$; 2—10; 3—100; 4—1000.

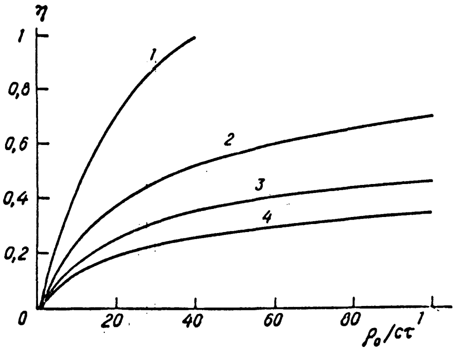

FIG. 4. Efficiency of the radiation versus $\rho_0/c\tau_0$. 1—$\rho_0/r = 10$; 2—10^2; 3—10^3; 4—10^4.

the speed parameter ν and the "thinness" $\zeta = \rho_0/r$. We see that, as we suggested, if the relaxation is fast enough energy will be radiated efficiently. If $c\tau_0 \sim r$, the efficiency becomes greater than unity, implying that our approximation is not valid in this region. This is the result which we would expect, since at such short times the distribution of the current over the cross section of the conductor would become important, while we have completely ignored that distribution in our original equation, (3).

2. CURRENT RELAXATION IN A RING IN THE CASE OF LOW INITIAL CURRENTS

To determine whether fast relaxation processes are possible in principle, we consider the case in which the current does not alter the state of the conductor. In other words, we consider the case of low currents. It is easy to see that a current can be regarded as low if two conditions hold:

a) The Joule heating or (what is essentially the same thing) the stored energy is smaller than the internal energy of the conductor:

$$LI^2/2c^2 < Mw_p,$$

where L, M, and w_p are the inductance, the mass, and the heat of vaporization of the conductor. For ordinary conductors we can assume that the current is low if $I < r$, where the current is expressed in milliamperes, and the radius in centimeters.

b) The electric field does not alter the nature of the conductivity, i.e., does not cause breakdown. If we restrict the discussion to ordinary conductors and to critical fields on the order of 50 kV/cm, we can write another condition under which a current can be judged low: $j < 150$ mA/mm^2, where j is the current density.

Under these conditions the relaxation time can be estimated roughly from $\tau_0 \sim L/Rc^2$ [$R = 2\pi\rho_0/\sigma S$, σ, S, and $L = 4\pi\rho_0 \ln(\rho_0/r)$ are the resistance, conductivity, cross-sectional area, and inductance of the conductor]. The rapidity parameter can be written as

$$\nu = 0{,}5\rho_0 c/\sigma S/\ln(\rho_0/r).$$

We see from this expression that if the radius is sufficiently large the current in the ring will decay rapidly, and the radiation will be efficient. Just how accurate this estimate is is not clear at the outset, since the concept of an inductance which we are using here is valid only in the quasisteady case. The case of a rapidly decaying current is by definition not quasisteady. To determine the actual relaxation time in the case $\rho_0 \gg \sigma S/c$, we need to consider the radiation reaction.

We can do this in a first approximation by incorporating the dipole radiation in the energy balance equation:

$$\frac{d}{dt}\left(\frac{LI^2}{2c^2}\right) = -I^2R - \frac{2}{3c^3}\ddot{m}^2$$

or

$$LI\dot{I}/c^2 = -I^2R - \frac{2\pi^2\rho_0^4}{3c^5}\ddot{I}^2. \tag{13}$$

If we take that path, however, we still do not know the range of applicability. In the case of intense radiation, Eq. (13) is clearly incorrect, and we would have to start from the general Maxwell equations. They lead in the usual way to the wave equation

$$(\nabla^2 - \partial_t^2/c^2)\mathbf{E} = \frac{4\pi}{c^2}\partial_t\mathbf{j}, \quad \mathbf{j} = \sigma\mathbf{E}. \tag{14}$$

In order to solve Eq. (14), we need to specify initial conditions. There are two ways to do this.

First, we could assume that there is no electric field anywhere at $t = 0$ [$\mathbf{E}(r,0) = 0$]. On the other hand, the time derivative of the field is not zero [$\partial_t\mathbf{E}(r,0) \neq 0$]. This case is realized when there is a sufficiently fast phase transition from a superconducting state to a normal state.

Second, we can assume that the field is nonzero at $t = 0$ [$\mathbf{E}(r,0) \neq 0$], but its derivative is zero [$\partial_t\mathbf{E}(r,0) = 0$].

From the standpoint of the efficiency of the radiation, these cases are extremely similar. In the present paper we will discuss only the case with $E(0) \neq 0$. We will also assume that the initial electric field is concentrated in the interior of the conductor. This assumption makes it possible to pursue the analytic calculations quite far. In practice, this assumption means that the energy of the electric field is negligible in comparison with the energy of the magnetic field. Under these assumptions, Eq. (14) can be written as[2]

$$(\nabla^2 - \partial_t^2/c^2)\mathbf{E} = \frac{4\pi}{c^2}[\partial_t\mathbf{j} - (\sigma\delta(t) + \partial_t\delta(t)/4\pi)\mathbf{E}(r,0)] = \mathbf{J}(r,t), \tag{15}$$

where we can assume $-\infty < t < \infty$, and we can assume that all quantities are zero at $t < 0$.

Equation (15) can be solved easily with the help of the retarded Green's function

$$G_0(r,t) = -\delta(t - r/c)/4\pi r$$

(Ref. 2, for example):

$$\mathbf{E}(\mathbf{r},t) = \int d^3r'\,dt'\,G_0(|\mathbf{r}-\mathbf{r}'|, t-t')\mathbf{J}(\mathbf{r}',t'). \tag{16}$$

Equation (16) is exact. In the case of a thin ring, i.e., under the condition $\zeta \gg 1$, we can replace $\mathbf{J}(\mathbf{r}',t)$ inside the integral by

$$J^*(t) = S^{-1}\int dS'\,J_\varphi(\mathbf{r}',t).$$

In other words, we can assume that the field and the current are both uniform over the cross section of the conductor. This assumption is completely reasonable, since at characteristic relaxation times $\tau_0 \sim L/c^2R$, the skin thickness is $\delta = c/(2\pi\sigma\omega)^{1/2}$ at $\omega \sim 1/\tau_0$. The skin thickness is thus greater than the thickness of the conductor, implying that the current distribution is approximately uniform. We should of course restrict the discussion to parameter values such that the condition $\tau_0 \gg r/c$ holds.

After averaging Eq. (16) over the cross section of the conductor, we find an integral equation for

$$E^{\bullet}(t) = S^{-1} \int E_\varphi(\mathbf{r}, t)\, dS:$$

$$E^{\bullet}(t) = -\int G(t-t')\,\partial_{t'}(\sigma E^{\bullet}(t'))\,dt' + (\partial_t G(t)/4\pi + \sigma G(t))E_0^{\bullet}. \tag{17}$$

Now all the "geometry" of the problem is in the function $G(t)$, which is given by

$$G(t) = \frac{\rho_0\theta(t)}{c^2 t S}\int dS\, dS' \int d\varphi \cos\varphi\, \delta[ct - (\rho^2+\rho'^2+(z-z')^2 - 2\rho\rho'\cos\varphi)^{1/2}]. \tag{18}$$

Here we have used the polar coordinates (ρ, φ, z); $dS = d\rho dz$; the integration is carried out over the cross-sectional area of the conductor forming the ring; and $\theta(\tau)$ is the unit step function. Integrating over φ, we can rewrite (18) as

$$G(t) = \frac{2\theta(t)}{c\rho_0 S}\int dS\, dS' \frac{\rho^2+\rho'^2+(z-z')^2-c^2t^2}{((2\rho\rho')^2-(\rho^2+\rho'^2+(z-z')^2-c^2t^2)^2)^{1/2}}. \tag{19}$$

The integration in (19) is carried out over the points at which the expression in the radical is nonnegative.

Using the relation $j = \sigma E$ for the total current flowing through the ring, we find from (18)

$$I(t) = -\sigma\int_0^t G(t-t')\,dI(t') + \frac{1}{4\pi}\frac{dG(t)}{dt} I_0, \tag{20}$$

where $I_0 = I(t=0)$ and $t > 0$. Note that expression (20) is valid only if the conductivity is not too low ($\pi r\sigma/c > 1$), since otherwise the distribution of the current over the cross section of the conductor becomes an important consideration. Actually, this condition is not restrictive. Interestingly, because of the discontinuity of the initial value $E(r,0)$ at the surface of the conductor we have $\partial_t I(0) \neq 0$.

To solve (20) we need to specify the function $G(t)$. This function is difficult to find analytically. However, for thin circular rings (and we are restricting the discussion to such rings), we can rewrite (19) as

$$G(t) = \frac{8\pi r}{c}\frac{\zeta^2 - 2\tilde{t}^2}{\zeta} 8 f(\tilde{t}) g(t, \zeta) \quad \text{for} \quad 0 < t < \zeta,$$

$$G(t) = 0 \quad \text{for} \quad \tilde{t} < 0 \text{ and } \tilde{t} > \zeta \tag{21}$$

where $\zeta = \rho_0/r$, $\tilde{t} = ct/2r$,

$$f(x) = x - \frac{4}{3\pi}[(1+x^2)E(x) - (1-x^2)K(x)], \quad 0 < x < 1,$$

$$f(x) = x - \frac{4x}{3\pi}[(1+x^2)E(1/x) + (1-x^2)K(1/x)], \quad x \geq 1,$$

$$g(x,z) = \frac{4}{3\pi^2(x+z)^{1/2}}\int_{\max(-1,x+1-z)}^{1} \frac{dk}{(k-x+z-1)^{1/2}} \times [(1+k^2)E'(k) - 2k^2K'(k)]$$

(K, K', E, and E' are elliptic integrals). The function $g(x,z)$ describes the effect of the remote regions of the ring on the current. For sufficiently thin rings ($\zeta \to \infty$), this function can be replaced by $1/(8\zeta)$. The function $f(x)g(x,z)$ is plotted in Fig. 5.

At $t > 2\rho_0/c$ the inhomogeneous term in (20) vanishes, and the exact solution of this equation becomes

$$I(t) \propto \exp(-\beta t), \tag{22}$$

where the coefficient β satisfies the transcendental equation

$$\beta = \left(\sigma \int G(t)\exp(\beta t)\,dt\right)^{-1}. \tag{23}$$

Interestingly, by expanding the current in the integral in (20) in powers of $t - t'$ for the same region, we easily find an equivalent differential equation (admittedly, of infinite order) to describe the current relaxation process with radiation:

$$IR = -\frac{L}{c^2}\dot{I} + \frac{2\pi\rho_0^3}{3c^4}\dddot{I} - \frac{2\pi^2\rho_0^4}{3c^5}\ddddot{I} + \ldots$$

$$= -\frac{L}{c^2}\dot{I} - \frac{\pi\rho_0}{c^2}\sum_{n=2}^{\infty}\left[-\frac{4\rho_0}{c}\right]^n \frac{\Gamma^2(n/2)(1-n)}{\Gamma(n+2)\Gamma(n)} I^{(n+1)}(t). \tag{24}$$

The coefficients in this series are decreasing coefficients in the quasisteady case, so it is natural to regard (24) as a generalization of the quasisteady relaxation equation which is found when only the first term on the right side is retained. The second term in (24) describes the effect of the radiation

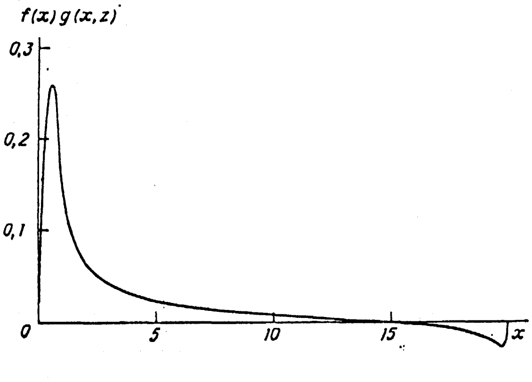

FIG. 5. Plot of the function $f(x)g(x, z)$ at $z = 20$.

due to the instant at which the current is turned on (due to the discontinuity in the first derivative). The third term describes the effect of the dipole radiation and therefore differs from the corresponding term in (13) only by a total derivative (and so forth). Expression (22) is of course a solution of (24).

More interesting from our standpoint are the non-quasisteady regime and times $0<t<2\rho_0/c$, for which (22) and (24) are not completely correct. The effect of the field at parts of the ring remote from the point under consideration is unimportant in this region, and we can actually treat the current decay in an infinite conducting cylinder ($\rho_0/r=\infty$). In this case $G(t)$ can be described by the following expression, in place of (21):

$$G(t)=\frac{8\pi r}{c}f(\tilde{t}) \quad \text{for} \quad 0<t<\infty.$$

Using this expression, we find that Eq. (20) has the exact solution

$$I(t)=\frac{I_0 i}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{\omega}e^{-i\omega t}\times\left[1-\frac{i\pi\omega J_1(\omega r/c)H_1^{(1)}(\omega r/c)}{\omega+4\pi i\sigma[1-i\pi J_1(\omega r/c)H_1^{(1)}(\omega r/c)]}\right].$$

An important point is that for $t\gg r/c$ this expression is the same as the exact solution of the problem of a decaying current in a cylinder:

$$I(t)=\frac{I_0 i}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{\omega}e^{-i\omega t}\times\left[1+\frac{2\omega J_1(kr)H_1^{(1)}(\omega r/c)/(kr)}{ckJ_1(kr)H_0^{(1)}(\omega r/c)-\omega J_0(kr)H_1^{(1)}(\omega r/c)}\right],$$

where $k=\omega c^{-1}(1+4\pi i\sigma/\omega)^{1/2}$. This agreement is further support for the validity of our approach.

Figure 6 shows a representative result of the calculations on the relaxation in an Al ring (the finite value of ρ_0/r has been taken into account here). Analysis of this figure reveals that for the parameter values chosen the solution with radiation is approximately the same as the solution of the quasisteady equation without radiation, $L\dot{I}/c^2=-IR$. The greatest distinction is that at small values of the time the exact solution decays more rapidly than does the solution of the quasisteady equation without radiation. Figure 7 shows the radiation efficiency for a ring as a function of the speed parameter $\nu=\rho_0/c\tau_0$. Comparison of this behavior with (12), in which the replacement $\tau_0\to\tau_0/\pi$ is made to reconcile the time scales, shows that the efficiency of the radiation for the quasiexponential current under consideration here is higher than that for the current shape described by expression (6).

Overall, the results of this section of the paper lead to the assertion that under the condition $\rho_0/r\gg\pi r\sigma/c$ the initial magnetic energy will be efficiently converted into the energy of coherent radiation.

3. CURRENT RELAXATION IN A RING WITH JOULE HEATING

If the conditions for a low current are not satisfied, the relaxation process becomes considerably more complicated, since the resistance of the conductor increases in the course of the heat evolution. This increase in resistance leads in turn to an increase in the loss, a decrease in the relaxation time, and thus an increase in the efficiency of the radiation process. Furthermore, if the initial current is sufficiently high the conductor material may expand, with the result that the resistance may increase by several orders of magnitude, and the efficiency of the radiation may become comparable to unity. If the current is too high, the relaxation times may be so short that the induced electric fields cause breakdown and thus a sharp decrease in the resistance, since the conductivity of the breakdown plasma is close in order of magnitude to the conductivity of a good metal. If the goal is to achieve an efficient radiation of stored energy, then both the geometry of the system and the initial currents should be chosen to maximize the heating of the conductor, but at the same time to prevent breakdown effects. The discussion below is restricted to the case in which the maximum electric field is less than 50 kV/cm.

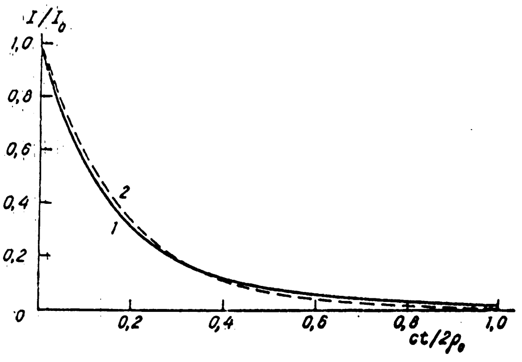

FIG. 6. Current relaxation in a ring with $\rho_0=300$ m and $r=0.03$ mm, without consideration of heating. 1—Exact solution; 2—quasisteady solution.

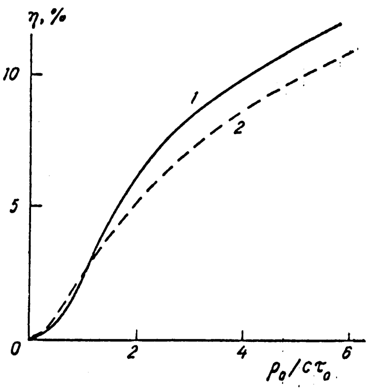

FIG. 7. Efficiency of the radiation by a rapidly decaying current versus $\rho_0/c\tau_0$ with $\rho_0/r=10^7$ and without heating. 1—Exact solution; 2—the current is approximated by the expression $I/I_0=0.5[1-2\arctan(\pi t/\tau_0)/\pi]$.

The heating of a conductor is an extremely complicated phenomenon, since numerous competing processes occur: thermal spreading, later hydrodynamic expansion, a redistribution of the current over the cross section of the conductor, heat conduction, and more. However, a qualitative description of this process is sufficient for our purposes. We will accordingly use a simple model in which the resistance of the conductor is determined exclusively by the thermal energy evolved during the heating of the conductor. The dependence of the resistance on the heat which has been evolved has been studied in many papers. A good approximation of this dependence was proposed in Ref. 3. Although that approximation is based on a study of "exploding" aluminum foils, we will use it to study the relaxation of high currents in a ring.

According to the approximation proposed in Ref. 3, the relative increase in the resistance R (in comparison with its initial value R_i) is described by the following function of the thermal energy w which is evolved:

1. Heating to the melting point,

$$R/R_i=1+6{,}25w,\quad 0\leqslant w\leqslant 0{,}64\ \text{kJ/g}.$$

2. Melting,

$$R/R_i=5+5{,}8(w-0{,}64),\quad 0{,}64\leqslant w\leqslant 1{,}02\ \text{kJ/g}.$$

3. Heating of the liquid,

$$R/R_i=7{,}02+5{,}06(w-1{,}02),\quad 1{,}02\leqslant w\leqslant 2{,}5\ \text{kJ/g}.$$

4. Explosion,

$$R/R_i=14{,}5\exp\left(0{,}42(w-2{,}5)\right),\quad w\geqslant 2{,}5\ \text{kJ/g}. \qquad (25)$$

Approximation (25) is of course not valid at arbitrary values of the thermal energy which is evolved. If there is a pronounced specific heat evolution, the resistance decreases to values on the order of the resistance of a normal metal. To estimate the maximum specific energy at which (25) is valid, we can draw on two facts. First, according to Ref. 3, expression (25) is definitely valid at $w\leqslant 10$ kJ/g. Second, an inspection of the data of Ref. 4 reveals that in the absence of hydrodynamic dispersal the resistance reaches its maximum at a temperature on the order of 40 eV, which corresponds to $w\approx 50$ kJ/g. On this basis we can apparently assume that the expression (25) is valid at $w\leqslant 20$ kJ/g. Taking this point into account, we will consider only situations in which the condition $w<20$ kJ/g holds. We should bear in mind that, if the range of applicability of (25) is broader, then the radiation may be even more efficient; if it is narrower, then in order to achieve the same results we should alter the geometry of the system, e.g., increase the radius of the ring. Another point which must be kept in mind is that for other materials there may be other approximations of the type in (25), and this circumstance could again lead to a change in the efficiency of the radiation from that found here, in one direction or the other.

Once we have found the resistance as a function of the energy supplied, $R(w)$, we can easily work from (20) to write self-consistent equations for the current relaxation:

$$R(w)I(t)=-\frac{2\pi\rho_0}{S}\int_0^t G(t-t')\,dI(t')+\frac{R_i}{4\pi}\frac{dG(t)}{dt}I_0, \qquad (26)$$

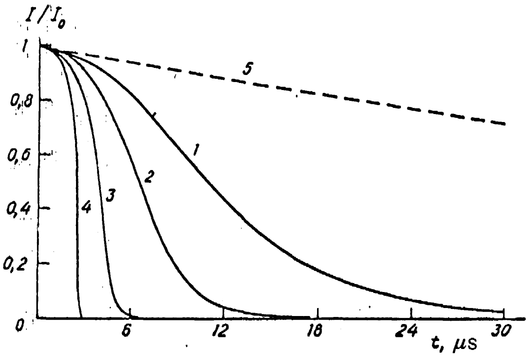

FIG. 8. Current relaxation in a ring with $\rho_0=3$ m and $r=0.7$ mm with heating. 1—$I_0=110$ kA; 2—170; 3—240; 4—300; 5—without heating.

$$M\frac{dw}{dt}=I^2R(w) \qquad (27)$$

(M is the total mass of the ring).

Figure 8 shows the nature of the relaxation for various values of the initial current. Note that at high values of the initial current the curve drops off progressively more sharply; this behavior means that the conductor explodes. Since there is a rapid change in the current from a value near its initial value to a vanishing value, we would expect that efficient radiation of energy might be possible here. Note that the heating of the conductor takes considerably more time that the explosion itself, so that the process can be initiated by (for example) arranging the transition of a current-carrying superconducting ring to the normal state over a time short in comparison with the heating time. Such a transition is completely feasible.

Analysis of (26) and (25) shows that, as soon as the specific stored energy density w_0 becomes greater than 2.5 kJ/g (we recall that we are discussing an Al ring as an example), a maximum is reached in dI/dt or the voltage $U=IR$. This maximum is interesting because, according to (5), it is the behavior of the current near this maximum which determines the radiation field. In this neighborhood the solution of (26) can be approximated by an expression

$$I(t)=I^*\left(1-2/\pi\,\mathrm{arctg}\,(t/\tau_0)\right)/2. \qquad (28)$$

If the voltage maximum is sufficiently sharp (or, equivalently, if $w_0\gg 2.5$ kJ/g), this will be a good approximation if we choose $I^*=2.16/w_0^{1/2}$ and

$$\tau_0=1{,}25\sigma_0\left(\frac{r^2}{c^2}\right)\ln(\rho_0/r)\exp(-0{,}42w_0). \qquad (29)$$

As the specific stored energy increases, the quantity I^* decreases in a power-law fashion, and τ_0 decreases in an exponential fashion. In other words, as w_0 is increased energy should be radiated more efficiently.

Knowing the parameters of the approximating current, (28), we can then easily calculate the rapidity parameter of the relaxation process, using (9) and (10):

$$\nu=\frac{0{,}8\rho_0 c}{\sigma_0 r^2\ln(\rho_0/r)}\exp(0{,}42w_0). \qquad (30)$$

We also find the radiation efficiency

TABLE I. Parameters of the relaxing current at an initial current density $j_0 = 400$ kA/mm^2 and at a maximum field $E_{max} < 50$ (100) kV/cm.

ρ_0, m	r, mm	I, MA	η, %	W_0, kJ/g	E_0, MJ	τ, ns	ν
10	0,28	0,1	1,8	15	0,6	7,4	4,5
	(0,29)	(0,11)	(2,8)	(16,5)	(0,7)	(4,2)	(7,9)
20	0,26	0,08	2,8	14	1	9,2	7,2
	(0,28)	(0,1)	(4,2)	(16,7)	(1,4)	(3,9)	(16,9)
30	0,26	0,08	3,9	15	1,6	7,7	12,9
	(0,28)	(0,1)	(4,9)	(16,7)	(2,0)	(3,9)	(25,2)
40	0,26	0,08	4,3	15	2	8,2	16,1
	(0,27)	(0,09)	(5,4)	(16,7)	(2,7)	(3,9)	(34)
50	0,26	0,08	4,9	15	2,6	7,5	22
	(0,27)	(0,09)	(5,8)	(16,6)	(3,2)	(4,1)	(41)

$$\eta=\frac{4{,}65}{w_0}\,\frac{\ln(\nu+(1+\nu^2)^{1/2})-2((1+\nu^2)^{1/2}-1)/\nu}{\ln(\rho_0/r)} \qquad (31)$$

and the maximum electric field in the ring,

$$E_{max}=\frac{8{,}5}{r(\ln(\rho_0/r))^{1/2}}\exp(0{,}42w_0), \qquad (32)$$

where E_{max} is in units of volts per centimeter, r is in centimeters, and w_0 is in kilojoules per gram. We have of course used the expression $W_{st} = LI_0^2/2c^2$, rather than $LI^*/2c^2$, for the stored energy. Tables I and II show some typical parameter values of the relaxing current which were calculated from these formulas.

Interestingly, an unbounded increase in the initial energy (even if the limitation imposed by breakdown is lifted) leads to an efficiency $\eta = 1.95/\ln(\rho_0/r)$, not $\eta = 1$, as might be expected at first glance.

The calculations above were carried out for a specific functional dependence of the resistance on the energy evolved, that in (25). If we are instead interested in estimating the efficiency in the dipole approximation, there already exists an exact formula which is valid in this case, regardless of the dependence of the resistance on the energy evolved:

$$\eta=\frac{1}{48\pi^2}\left(\frac{\rho_0 c}{\sigma_0 r^2}\right)^3\frac{K(w_0)}{\ln^4(\rho_0/r)},$$

where

$$K(w_0)=\int_0^1 dx\,y(x)\left(y(x)-2(1-x)\frac{dy}{dx}\right)^2,$$

and $y(x) = R(xw_0)/R_i$. If we use the approximation (25) for $R(w)$, we naturally find an expression which agrees with (31).

4. CONCLUSION

We have analyzed the radiation which occurs during free relaxation of a current in a ring. We have found that if the current decreases sufficiently rapidly stored energy will be radiated efficiently. The energy flux density radiated can reach high values in directions near the axis of the ring.

If the heating of the ring can be ignored, efficient radiation is possible only for rings of sufficiently large diameter. With $\rho = 300$ m, $r = 0.03$ mm, and a current on the order of 3 kA, for example, the efficiency is 0.1. This result means, by the way, that an energy on the order of 10–100 kJ is radiated on a time on the order of 10^{-6} s. An important point is that a radiator of this sort can be used repeatedly.

If parameter values at these scales are unacceptable, we should apparently make use of the increase in the resistance during the flow of a current through a conductor in order to achieve an efficient radiation of energy. It has been shown here that, for ring diameters of several tens of meters and for initial current densities on the order of hundreds of kiloamperes per square millimeter, it is again possible to achieve a

TABLE II. Parameters of the relaxing current for an initial current density $j_0 = 100$ kA/mm^2 and a maximum field $E_{max} < 50$ (100) kV/cm.

ρ_0, m	r, mm	I, MA	η, %	W_0, kJ/g	E_0, MJ	τ, ns	ν
20	1,27	0,51	0,5	18	31,4	34,8	1,9
	(1,33)	(0,56)	([illegible],1)	(20)	(37,6)	(18,8)	(3,5)
40	1,2	0,45	1	18	53,9	45	2,9
	(1,29)	(0,52)	(2,2)	(20)	(71,5)	(17)	(7,6)
60	1,2	0,45	1,8	18	84	35	5,7
	(1,26)	(0,5)	(2,7)	(20)	(102)	(18)	(11)
80	1,2	0,44	2,4	18	111	33	8,5
	(1,24)	(0,48)	(3,2)	(20)	(130)	(18)	(14,7)
100	1,17	0,43	2,6	18	132	35	9,4
	(1,23)	(0,48)	(3,6)	(20)	(162)	(18)	(18,3)

radiation efficiency on the order of 3–5%. Parameter values in these ranges are recommended for testing the validity of these results.

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