Mesoscopic fluctuations of the Josephson current through small junctions

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In this paper we calculate the critical current as a function of the magnetic field in a Josephson junction, allowing for fluctuations in the order parameters in one of the superconducting sides. We show that above the temperature T_c at which transition to the superconducting state occurs, a Josephson current may emerge. A sharp drop in the critical current occurs in the range of strong magnetic fields ($\Phi \gg \Phi_0$), and random "mesoscopic" oscillations are observed on the I_c vs H curve. Below T_c the order-parameter phase fluctuations are found to produce a drop in the critical current and distortions in the "Fraunhofer" dependence.

1. INTRODUCTION

In Ref. 1 it was shown that the critical current in disordered Josephson junctions (in such junctions the current density or the Josephson phase difference depends in a random manner on the coordinates in the junction plane) may vary considerably from specimen to specimen and that random "mesoscopic" oscillations are observed on the critical current vs magnetic field curve. Josephson junctions in which the fluctuations of the superconductivity order parameter are large also exhibit these properties. Studies of the current flow in such systems are of great interest because they make it possible to investigate the fluctuations in the absolute value and phase of the order parameter and find the microscopic parameters of a superconductor, among them the coherence length and mean free path.

Changes in the density of the Josephson current caused by fluctuations of the order parameter in one of the superconducting sides at $T > T_c$ have been studied theoretically in Ref. 2 and for $T < T_c$ in Refs. 3 and 4. The Josephson current density at a point in the junction was determined by the mean value of the order parameter, but the critical current through the Josephson junction, which is determined by the order-parameter correlator, was not calculated either. For this reason, in Secs. 2 and 3 we determine the real dependence of the critical current on the magnetic field in a Josephson junction with allowance for fluctuations in the order parameter at temperatures above and below T_c.

2. THE FLUCTUATION JOSEPHSON CURRENT FOR $T > T_c$

Let us consider a Josephson junction maintained at a temperature T close to the critical temperature T_c of one of the superconducting sides. Such a superconductor is characterized for $T > T_c$ by an inhomogeneous fluctuation order parameter $\Delta(r)$. The critical current in a small Josephson junction (the junction width L is less than the Josephson penetration depth λ_j) placed in a magnetic field $\mathbf{H}$ that is parallel to the junction plane is given by the following formula:[5]

$$I_c^2 = \frac{j_0^2}{\Delta_0^2} \iint d^2\rho_1\, d^2\rho_2 \Delta(\rho_1, 0)\Delta^*(\rho_2, 0) \exp\left[i\frac{2\pi\Phi}{\Phi_0 L}(x_1 - x_2)\right], \quad (1)$$

where $\rho = (x,y)$ stands for the coordinate in the junction plane, j_0 and Δ_0 the density of the critical current and the absolute value of the order parameter at $T = 0$, Φ the magnetic flux through the Josephson junction, and Φ_0 the quantum of magnetic flux.

To find the mean critical current we average Eq. (1) over the different values of the fluctuation order parameter. The mean current can be written as a functional integral in the following manner:[6]

$$\overline{I_c^2} = \frac{j_0^2}{\Delta_0^2} \iint d^2\rho_1\, d^2\rho_2 \exp\left[i\frac{2\pi\Phi}{\Phi_0 L}(x_1 - x_2)\right] \times \int D\Delta(\mathbf{r})\Delta(\rho_1, 0)\Delta^*(\rho_2, 0) \times \exp\left[-\frac{1}{T}\int d^3\mathbf{r}\left(C\left|\frac{\partial\Delta}{\partial\mathbf{r}}\right|^2 + a|\Delta|^2 + \frac{b}{2}|\Delta|^4\right)\right], \quad (2)$$

where a, b, and C are the Gor'kov coefficients in the Ginzburg-Landau functional.[7]

For $T \gg T_c$ outside the critical region [$\tau = (T - T_c/T_c \gg \tau_{cr}$;] see Ref. 6], we keep only the second-order terms in the Ginzburg-Landau functional. Then, performing the Fourier transformation $\Delta(\mathbf{r}) = \Sigma_\mathbf{q} \Delta_\mathbf{q} \exp(i\mathbf{q}\cdot\mathbf{r})$, we get

$$\overline{I_c^2} = \frac{j_0^2}{\Delta_0^2} \iint d^2\rho_1\, d^2\rho_2 \exp\left[i\frac{2\pi\Phi}{\Phi_0 L}(x_1 - x_0)\right] \times \prod_{\mathbf{q}=1}^{N} \iint d\,\mathrm{Re}\,\Delta_\mathbf{q}\, d\,\mathrm{Im}\,\Delta_\mathbf{q} \times \left[\frac{V(Cq^2+a)}{\pi T}\right]^N \left\{\sum_\mathbf{q} |\Delta_\mathbf{q}|^2 \exp[i\mathbf{q}_\perp(\rho_1 - \rho_2)]\right\} \times \exp\left[\frac{V}{T}(Cq^2+a)|\Delta_\mathbf{q}|^2\right]$$
$$= \frac{j_0^2}{\Delta_0^2} \iint d^2\rho_1\, d^2\rho_2 \exp\left[i\frac{2\pi\Phi}{\Phi_0 L}(x_1 - x_2)\right] \frac{T}{V} \times \sum_\mathbf{q} \frac{\exp[i\mathbf{q}_\perp(\rho_1 - \rho_2)]}{(Cq^2+a)}, \quad (3)$$

where V is the volume of the superconductor, $\mathbf{q}_\perp$ the wave vector in the junction plane, and N the total number of wave vectors, which in the final result must be set to infinity.

The fluctuations in the order parameter are large in small superconductors. Hence, let us find the critical current in a Josephson junction whose superconducting side is a thin film with a thickness d much smaller than $\xi(T) = \xi(0)\tau^{-1/2}$, where $\xi(0)$ is the coherence length in the superconductor at $T = 0$. Carrying out the usual replacement $V^{-1}\Sigma_\mathbf{q} = d^{-1}\int d^2\mathbf{q}(2\pi)^{-2}$, we can write Eq. (3) in the

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form

$$\overline{I_c^2}=\frac{j_0^2T}{\Delta_0^2\,dC}\iint d^2\rho_1\,d^2\rho_2\exp\left[i\frac{2\pi\Phi}{\Phi_0L}(x_1-x_2)\right]\times\int\frac{d^2\mathbf{q}}{(2\pi)^2}\frac{\exp[i\mathbf{q}(\rho_1-\rho_2)]}{q^2+(1/\xi)^2}\qquad(4)$$

and calculating the integral with respect to $\mathbf{q}$ we get

$$\overline{I_c^2}=\frac{j_0^2T}{\Delta_0^2 2\pi\,dC}\iint d^2\rho_1\,d^2\rho_2\exp\left[i\frac{2\pi\Phi}{\Phi_0L}(x_1-x_2)\right]\times K_0\left(\frac{|\rho_1-\rho_2|}{\xi}\right).\qquad(5)$$

If the width L of the Josephson junction is much greater than ξ, the main contribution to the current is provided by such vectors ρ_1 and ρ_2 for which $|\rho_1-\rho_2|\sim\xi\ll L$, and the dependence of the mean critical current on the magnetic field strength has the form

$$\overline{I_c^2}=I_1^2=\frac{j_0^2TS}{\Delta_0^2 2\pi\,dC}\left[\left(\frac{2\pi\Phi}{\Phi_0L}\right)^2+\frac{1}{\xi^2}\right]^{-1}.\qquad(6)$$

As will be shown below, however, the fluctuations in the critical current are of the same size ($\sim I_1$) as the mean critical current, and therefore the real dependence of the critical current on the magnetic field is not determined solely by the mean current [i.e., by Eq. (6)]. To find this dependence we must first calculate the correlator $K(\Delta\Phi)=\langle I_c^2(\Phi_1)I_c^2(\Phi_2)\rangle-\langle I_c^2(\Phi_1)\rangle\langle I_c^2(\Phi_2)\rangle$, where $\Delta\Phi=\Phi_2-\Phi_1$. The correlator is determined by the average of the product of four order parameters:

$$\langle I_c^2(\Phi_1)I_c^2(\Phi_2)\rangle=\frac{j_0^4}{\Delta_0^4}\iiiint d^2\rho_1\,d^2\rho_2\,d^2\rho_3\,d^2\rho_4\times\exp\left[i\frac{2\pi\Phi_1}{\Phi_0L}(x_1-x_2)+i\frac{2\pi\Phi_2}{\Phi_0L}(x_3-x_4)\right]\times\langle\Delta(\rho_1)\Delta^*(\rho_2)\Delta(\rho_3)\Delta(\rho_4)\rangle.\qquad(7)$$

If the junction surface area S is much greater than ξ^2, the average of four order parameters splits into a sum of products of two order parameters. Using Eqs. (2)–(4), we find

$$\langle I_c^2(\Phi_1)I_c^2(\Phi_2)\rangle=\frac{j_0^4}{\Delta_0^4}\iiiint d^2\rho_1\,d^2\rho_2\,d^2\rho_3\,d^2\rho_4\times\exp\left[i\frac{2\pi\Phi_1}{\Phi_0L}(x_1-x_2)+i\frac{2\pi\Phi_2}{\Phi_0L}(x_3-x_4)\right]\left[K_0\left(\frac{|\rho_1-\rho_2|}{\xi}\right)\times K_0\left(\frac{|\rho_3-\rho_4|}{\xi}\right)+K_0\left(\frac{|\rho_1-\rho_4|}{\xi}\right)K_0\left(\frac{|\rho_2-\rho_3|}{\xi}\right)\right],\qquad(8)$$

and the correlator can be written in the form

$$K(\Delta\Phi)=\frac{j_0^4}{\Delta_0^4}\iint d^2\rho_1\,d^2\rho_4K_0\left(\frac{|\rho_1-\rho_4|}{\xi}\right)\times\exp\left[i\frac{2\pi\Phi_1}{\Phi_0L}(x_1-x_4)\right]\iint d^2\rho_2\,d^2\rho_3\times\exp\left[i\frac{2\pi\Phi_2}{\Phi_0L}(x_2-x_3)\right]K_0\left(\frac{|\rho_2-\rho_3|}{\xi}\right)\exp\left[i\frac{2\pi\Delta\Phi}{\Phi_0L}(x_4-x_2)\right].\qquad(9)$$

Since the main contribution to (9) is provided by vectors ρ_1, ρ_2, ρ_3, and ρ_4 such that $|\rho_1-\rho_4|$ and $|\rho_2-\rho_3|$ are on the order of $\xi\ll L$, we get

$$K(\Delta\Phi)=I_1^2(\Phi_1)I_1^2(\Phi_2)\frac{\sin^2(\pi\Delta\Phi/\Phi_0)}{(\pi\Delta\Phi/\Phi_0)^2}.\qquad(10)$$

Thus, in the case of the real I_c vs Φ dependence, random oscillations with a period on the order of Φ_0 are observed superposed on the smooth decay predicted by Eq. (6). Such behavior of the fluctuation critical current resembles the mesoscopic dependence of the conductivity of small metallic specimens on the magnetic field strength[8] and is characteristic of disordered Josephson junctions.[1]

3. THE FLUCTUATION JOSEPHSON CURRENT AT $T<T_c$

For $T<T_c$ we write Eq. (2) in the form

$$\overline{I_c^2}=\frac{j_0^2}{\Delta_0^2}\iint d^2\rho_1\,d^2\rho_2\exp\left[i\frac{2\pi\Phi}{\Phi_0L}(x_1-x_2)\right]\int D|\Delta(\mathbf{r})|\times D\varphi(\mathbf{r})\Delta(\rho_1,0)\Delta^*(\rho_2,0)\exp\left[-\frac{1}{T}\int d^3\mathbf{r}\left[C\left(\frac{\partial\Delta}{\partial\mathbf{r}}\right)^2+C|\Delta|^2\left(\frac{\partial\varphi}{\partial\mathbf{r}}\right)^2+a|\Delta|^2+\frac{b}{2}|\Delta|^4\right]\right],\qquad(11)$$

where $|\Delta|$ and φ are the fluctuating absolute value and phase of an order parameter. Far from the critical temperature range the main contribution to the mean current is provided by the fluctuations of phase, while the absolute value of the order parameter is assumed constant and can be derived from the mean-field theory.[7] Then

$$\overline{I_c^2}=j^2(T)\iint d^2\rho_1\,d^2\rho_2\exp\left[i\frac{2\pi\Phi}{\Phi_0L}(x_1-x_2)\right]\int D\varphi(\mathbf{r})\times\exp\left\{i[\varphi(\rho_1)-\varphi(\rho_2)]-\frac{1}{T}\int d^3\mathbf{r}C|\Delta|^2\left(\frac{\partial\varphi}{\partial\mathbf{r}}\right)^2\right\}$$

$$=j^2(T)\iint d^2\rho_1\,d^2\rho_2\exp\left[i\frac{2\pi\Phi}{\Phi_0L}(x_1-x_2)\right]\prod_{q=1}^{N}\int d\,\mathrm{Re}\,\varphi_{\mathbf{q}}\,\mathrm{Im}\,\varphi_{\mathbf{q}}\times\left[\frac{2VC|\Delta|^2q^2}{\pi T}\right]^{N/2}\exp\left[i\varphi_{\mathbf{q}}(\exp i\mathbf{q}\rho_1-\exp i\mathbf{q}\rho_2)-\frac{V}{T}C|\Delta|^2\mathbf{q}^2|\varphi_{\mathbf{q}}|^2\right],\qquad(12)$$

where $j(T)$ is the density of the critical current through the Josephson junction without allowance for fluctuations.

In Eq. (12) we have taken into account only the smooth long-wave fluctuations of the phase and ignored vortex fluctuations, which are responsible for the Thouless–Kosterlitz transition and become important at temperatures exceeding $T_{\mathrm{T-K}}$. The temperature interval $T_c-T_{\mathrm{T-K}}$ proves, however, to be exceptionally narrow, of order $t_c\tau_{\mathrm{cr}}$, so that our approach becomes invalid. At temperatures below $T_{\mathrm{T-K}}$ the vortex fluctuations contribute practically nothing to the order-parameter correlator.[9,10]

Calculating the integrals with respect to $\varphi_{\mathbf{q}}$ and going over from summation over $\mathbf{q}$ to integration, we obtain

$$\overline{I_c^2}=j^2(T)\iint d^2\rho_1\,d^2\rho_2\exp\left[i\frac{2\pi\Phi}{\Phi_0L}(x_1-x_2)\right]\times\exp\left\{\frac{T}{2Cd|\Delta|^2}\int\frac{d\mathbf{q}}{2\pi\mathbf{q}}[1-J_0(\mathbf{q}|\rho_1-\rho_2|)]\right\}.\qquad(13)$$

Setting the upper limit in the integral with respect to $\mathbf{q}$ equal to ξ^{-1}, we get

$$\overline{I_c^2}=j^2(T)\int d^2\rho_1\int d^2\rho_2 f(|\rho_1-\rho_2|)\exp\left[i\frac{2\pi\Phi}{\Phi_0L}(x_1-x_2)\right],\qquad(14)$$

where

$$f(z)=\begin{cases} (\xi/z)^\gamma, & z\gg\xi, \\ 1, & z\ll\xi, \end{cases}$$

$$\gamma=\frac{T}{4\pi Cd|\Delta|^2}.$$

The I_c vs H dependence specified by Eq. (14) is in all respects the same as the dependence of the critical current on the magnetic field strength for a Josephson junction with Abrikosov vortexes.[11] Hence, using the results of Ref. 11, we find that for low magnetic fields ($\Phi/\Phi_0 \ll 1$) and a specimen in the form of a disk of radius L,

$$\overline{I_c^2}=j^2(T)\frac{(8\pi)^{1/2}}{2-\gamma}L^4\left(\frac{\xi}{2L}\right)^\gamma\frac{\Gamma((3-\gamma)/2)}{\Gamma(3-\gamma)}. \tag{15}$$

In the opposite case of high magnetic fields ($\Phi/\Phi_0 \gg 1$) we can employ the asymptotic behavior of the Bessel function for large values of the independent variable to obtain

$$\overline{I_c^2}=j^2(T)8\pi L^4\left(\frac{\xi}{2L}\right)^\gamma\left\{\frac{\gamma\pi}{4}\frac{\Gamma(1-\gamma/2)}{\Gamma(1+\gamma/2)\alpha^{2-\gamma}}+\frac{1}{\alpha^3}\left[\frac{\Gamma((3-\gamma)/2)}{\Gamma((1+\gamma)/2)\alpha^{-\gamma}}-\frac{\sin 2\alpha}{2}\right]\right\}, \tag{16}$$

with $\alpha = 2\pi\Phi/\Phi_0$. The first term in Eq. (16) describes the deviation from the Fraunhofer dependence due to the fluctuations of the Josephson phase in the junction.

In the same manner as in Sec. 2 we can calculate the correlator $K(\Delta\Phi)$ by averaging the four order parameters. Using Eqs. (12) and (13), we arrive at the following expression for the correlator $\langle I_c^2(\Phi_1)I_c^2(\Phi_2)\rangle$:

$$\langle I_c^2(\Phi_1)I_c^2(\Phi_2)\rangle = j^4(T)\iiiint d^2\rho_1 d^2\rho_2 d^2\rho_3 d^2\rho_4 \exp\left[i\frac{2\pi\Phi_1}{\Phi_0 L}(x_1-x_2)+i\frac{2\pi\Phi_2}{\Phi_0 L}(x_3-x_4)\right]\left\{\frac{\xi^2|\rho_1-\rho_3|\,|\rho_2-\rho_4|}{|\rho_1-\rho_2|\,|\rho_1-\rho_4|\,|\rho_2-\rho_3|\,|\rho_3-\rho_4|}\right\}^\gamma. \tag{17}$$

In high fields ($\alpha \gg 1/\gamma$), where the first term in (16) determines the size of the critical current, the main contribution to (17) is provided by vectors $\rho_1, ..., \rho_4$ such that either $|\rho_1-\rho_2|$ and $|\rho_3-\rho_4|$ are much smaller than L or $|\rho_2-\rho_3|$ and $|\rho_1-\rho_4|$ are small. In each case we still have formula (10) for $K(\Delta\Phi)$. Thus, at temperatures below T_c as well the I_c vs H curve exhibits random mesoscopic oscillations with a period of order Φ_0.

4. DISCUSSION

The above calculations suggest that even at temperatures above T_c there can be a fluctuation Josephson current. Equation (6) implies that $I_c(\Phi=0)$ can be written as

$$I_c(0)=I_0\frac{\xi}{L}\frac{1}{(p_F d)^{1/2}}\begin{cases}(p_F l)^{-1/2}, & l\ll\xi_0,\\ (p_F\xi_0)^{-1/2} & l\gg\xi_0,\end{cases} \tag{18}$$

where I_0 is the critical current in the Josephson junction at absolute zero.

This formula has a simple physical meaning. For $T > T_c$ the fluctuation contribution to the Josephson current is provided by regions whose size is of order $\xi(T)$ and where the order parameter is correlated. The number n of such regions is of order $(L/\xi(T))^2$ and in each the direction of the current is random. Hence, the net current must be proportional to $n^{1/2}\xi^2(T) \sim I_0\xi/L$.

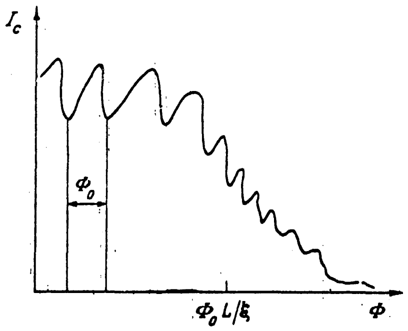

FIG. 1. The fluctuation critical current in a Josephson junction as a function of the magnetic field strength at $T > T_c$.

Note that Eq. (18) can be written in the following form:

$$I_c(0)=I_0\tau^{-1/2}\left(\frac{\xi_0}{p_F^2L^2d}\right)^{1/2}, \tag{19}$$

which, obviously, does not depend on the mean free path l. To observe the fluctuation current in a small Josephson junction in experiments, one must select a superconducting side with a high value of ξ_0 and a small thickness d.

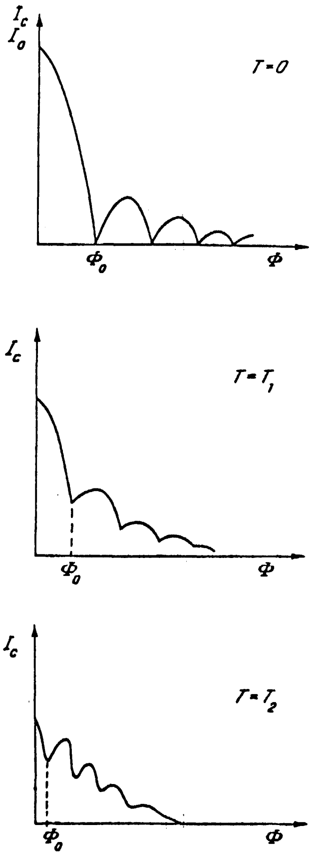

FIG. 2. The critical current in a Josephson junction as a function of the magnetic field strength at $T < T_c$; the temperatures T_1 and T_2 are such that $0 < T_1 < T_2 < T_c$.

An approximate $I_c(\Phi)$ curve corresponding to Eqs. (6) and (10) is depicted in Fig. 1. These equations show that critical-current decay begins at high magnetic fields, $\Phi\sim\Phi_0L/2\pi\xi\gg\Phi_0$. Random mesoscopic oscillations of the critical current with a period of the order of Φ_0, however, should be observed for $\Phi\sim\Phi_0$.

Let us now discuss the $I_c(T,\Phi)$ curve in the temperature range below T_c. Formula (15) for $I_c(T,0)$ can be written as

$$I_c(T,0)=I_0(T)\exp\left[\frac{\ln(L/\xi)}{dp_F^2l|\tau|}\right], \tag{20}$$

where $I_0(T)$ is the critical current in the Josephson junction without allowance for fluctuations. This shows that current fluctuations become significant in a temperature range where $|\tau|\sim\tau_{cr}\ln(L/\xi(0))\gg\tau_{cr}$ holds.

As we approach T_c, the $I_c(\Phi)$ curve deviates from the usual Fraunhofer pattern [the first term in Eq. (16)] and decay of the critical current as a function of the magnetic field strength becomes weaker (Fig. 2). In the region of high magnetic fields ($\alpha\gg\gamma^{-1}=|\tau|/\tau_{cr}$), random mesoscopic fluctuations with a characteristic period of the order of Φ_0 appear on the I_c vs Φ curve.

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