Phenomenological description of spiral waves arising under radiant heating of metals

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A phenomenological equation describing spiral-like thermal structures arising when a liquid film is formed on a metallic surface heated by a beam of laser radiation is suggested and studied.

1. INTRODUCTION

When a metal is heated by laser radiation, a liquid phase of either the metal itself or its compound (e.g., its oxide) is formed on its surface. The liquid layer can strongly influence the heating dynamics and lead to the formation of various surface structures.[1,2] Furthermore, one often has to deal with liquid films in such applied problems as laser doping and welding of metals, lasting information recording, etc. Therefore it is interesting to study various aspects of the dynamics of thin liquid layers on metallic surfaces in the field of laser radiation.

In Ref. 3 experiments are reported on vanadium heating by a confined beam of continuous radiation of a CO_2- or a YAG-laser in the air, the laser power being equal to 150 W and the beam radius to about 1 cm. A liquid film of V_2O_5 was formed on the metallic surface. It has turned out that, in spite of the radiation power being constant and the shape of intensity distribution being smooth, various spatially inhomogeneous thermal structures are observed: one-dimensional (in the form of a series of moving points) and two-dimensional (in the form of rotating spirals with a different numbers of arms) (see Figs. 1–3).

Spiral waves (reverberators) have been observed in various media and many times simulated numerically and analytically (see Ref. 4 and the references therein). However, a feature of the systems studied is that reverberators exist also in homogeneous media, and the inhomogeneity either influences their evolution and interaction or is necessary only at the initial stage of their formation. As for the system studied in Ref. 3, the existence itself of spiral waves is caused by external inhomogeneity.

The aim of the present study is the construction of a phenomenological model qualitatively describing the experiments of Ref. 3.

2. PROBLEM FORMULATION. INSTABILITY OF INHOMOGENEOUSLY HEATED LIQUID FILM

Consider the following problem. An axially symmetric beam of continuous laser radiation is incident on a hairline-thin infinite planar layer of incompressible liquid absorbing this beam. The inhomogeneity of intensity distribution in the beam gives rise to inhomogeneous distribution of temperature T over the plane of the layer and, as a consequence, due to the temperature dependence of surface tension $\sigma(T)$, to inhomogeneous distribution of the latter over the liquid surface. The spatial inhomogeneity of the σ distribution leads to formation of hydrodynamical currents flowing in the direction of $\nabla\sigma$ at the upper boundary of the liquid. As the radiation intensity grows, the velocity of hydrodynamical streams increases, and, at a certain critical value, the flow loses its stability. This effect is accompanied by breaking of the axial symmetry of the velocity and temperature distributions in the liquid. The arising pattern of secondary currents is the observed spiral-like structure.

To describe the dynamics and shape of secondary currents, we assume that the beam size is much larger than the characteristic wavelengths of arising perturbations. This allows to consider the main flow as locally one-dimensional, having only the radial component of the velocity. The convective instability of the liquid layer caused by the temperature dependence of the surface tension σ under inhomogeneous heating along one coordinate has been studied in Ref. 5. Below we list those results of this study which are most important for us.

Let an infinite thin layer of incompressible liquid be heated in such a way that the temperature at its upper boundary grows linearly along the x axis. This is effected by heat supply (in our case, by means of radiation) across the upper boundary. Thermal expansion of the liquid is absent, and $\sigma(T) = \sigma_0 - \gamma\sigma(T - T_0)$. The steady-state flow is found by solving simultaneously the equations of hydrody-

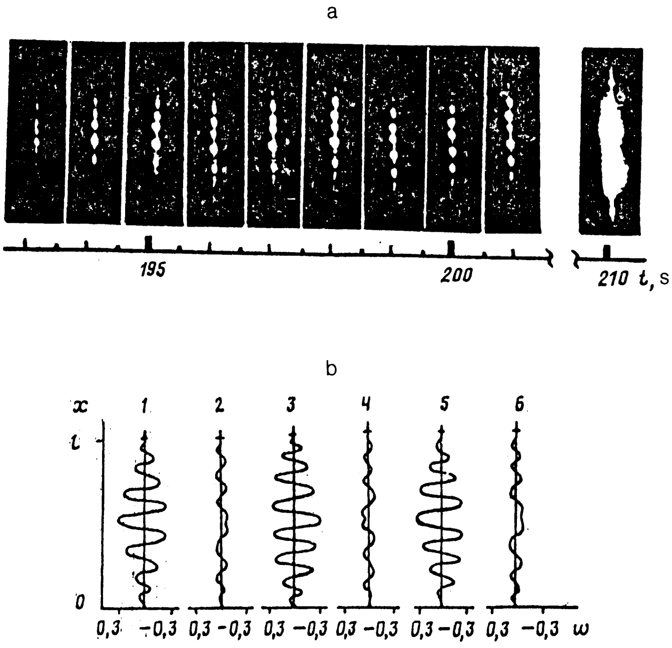

FIG. 1. (a) The photograph of the surface of a vanadium target heated by a CO_2-laser.[3] (b) The solutions of Eq. (3) at different times $t = 11.7$ (*1*), 48.6 (*2*), 63.3 (*3*), 105.0 (*4*), 132.0 (*5*), and 160.8 (*6*).

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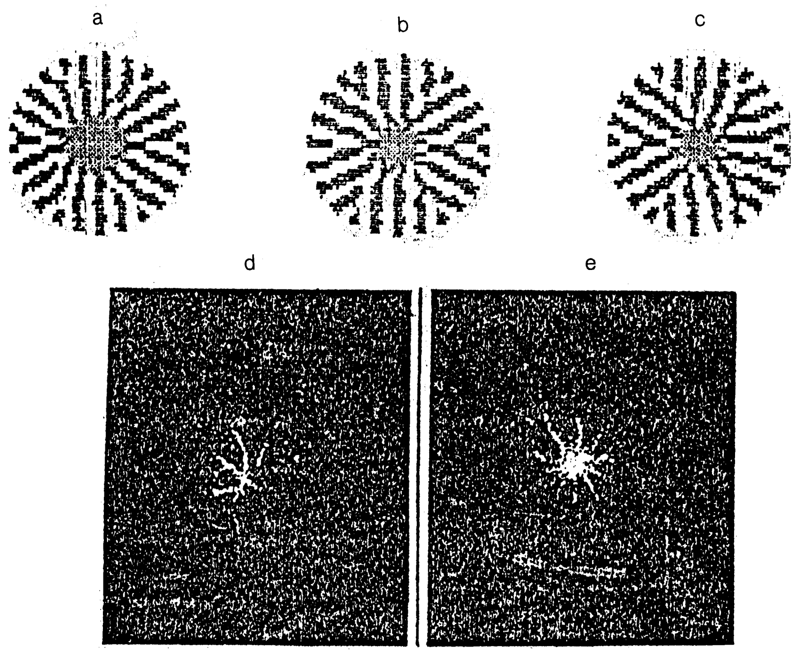

FIG. 2. The results of numerical study of Eq. (2) at different times $t = 230$ (a), 560 (b), and 3210 (c), and the experimental photographs from Ref. 3 [(d)—24 s, (e)—30 s]. The parameters have the following values: $\beta = 0.1$, $v_0 = -0.01$, $a = 0$, $A = 1$, $B = 1$, $C = 0$, $\psi = 80°$ and $R = 26.2$.

namics and heat conduction. Then its stability is analyzed by solving numerically the relevant spectral problem.

Since the coefficients of the equations obtained do not explicitly depend on the time t and coordinates (x,y) (the z axis is vertical, and the x axis is along the temperature gradient, $y \perp x$, $y \perp z$), the perturbations can be considered proportional to $\exp(pt + ik_x x + ik_y y)$. The result is a spectral function $p(k_x;k_y;b)$ having the temperature gradient b as a parameter. For a certain value of $b = b_c$ an instability can develop, i.e., the components k_x and k_y, for which $\text{Re}p > 0$, appear. The analysis performed in Ref. 5 has shown the following.

1) The instability arises, first of all, for $k_x \neq 0$, $k_y \neq 0$, i.e., the most stable modes make a certain nonzero angle ψ with the x-axis.

2) At the moment of stability loss ($\text{Re}p = 0$) it turns out that $\text{Im}(p) \neq 0$. This means that convective billows due to instability move with a finite phase velocity.

3. DERIVATION OF A PHENOMENOLOGICAL EQUATION OF LIQUID-FILM DYNAMICS

Problem formulation: derive a phenomenological equation for weakly supercritical structures on the basis of the above data concerning the perturbation spectrum. First, we

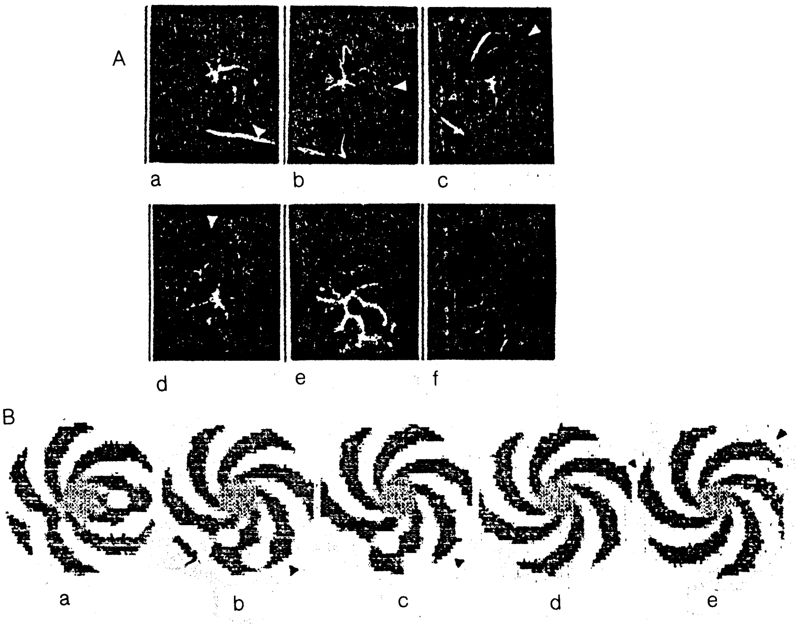

FIG. 3. A—the photographs of spiral-like structures arising on the vanadium surface under the heating by a beam of CO_2-laser radiation. The beam diameter d is 2.4 cm, its power is 150 W. The time $t = 174.5$ s (a), 175.0 (b), 175.5 (c), 176.0 (d), 183.3 (e), and 183.5 (f) is measured from the moment, when the laser is switched on.[3] B—the results of numerical study of Eq. (2) at different times $t = 2400$ (a), 4431 (b), 4443 (c), 4527 (d), and 4566 (e). The parameters have the following values: $\beta = 0.1$, $v_0 = -0.1$, $a = 0$, $A = 1$, $B = 1$, $C = 0$, $\psi = 45°$ and $R = 20.67$. The time period is $T = 84$.

find the form of the linear part of the equation in the presence of translational symmetry along the y axis. In the case of weak supercriticality the behavior of the system is determined, in view of the subordination principle,[6] by unstable modes. The plot of any function of general position depending on two variables is close to a paraboloid near its maximum. Therefore we can write for the increment p near the instability threshold (in the space of wave vectors)

$$\mathrm{Re}\,p=\beta-[A(k_x^2-k_{0x}^2)^2+2C(k_x^2-k_{0x}^2)(k_y^2-k_{0y}^2)+B(k_y^2-k_{0y}^2)^2],$$

where β is the supercriticality parameter, A, B, and C are constants satisfying the condition of positive definiteness of the corresponding quadratic form, $AB-C^2>0$, and the wave vectors $(\pm k_{0x}, \pm k_{0y})$ define the directions of the most unstable modes, for which $\mathrm{Re}(p)$ is a maximum. The imaginary part of the increment of perturbations that move with a nonzero phase velocity $\mathbf{v}(\mathbf{k})$, is

$$\mathrm{Im}\,p=-i\mathbf{k}\mathbf{v}(\mathbf{k}).$$

For small values of supercriticality we are interested in the behavior of $\mathrm{Im}(p)$ only in the vicinity of unstable modes. The simplest approximation of the function $\mathbf{v}(\mathbf{k})$ describing drifting billows whose fronts make an angle ψ with the x axis, is $v_x=v_0/\cos\psi$, $v_y=0$. This means that all structures drift along the x axis, and v_0 is the drifting velocity of the billows, inclined at the angle ψ to the x axis, in the direction of $\mathbf{k}_0$. Other approximations of $\mathbf{v}=\mathbf{v}(\mathbf{k})$ is also possible.

To pass on from the perturbation spectrum to the equation for a real order parameter w, which may be interpreted as a deviation of the temperature from the equilibrium value, we make the change of variables $\mathbf{k}\rightarrow -i\nabla$ and $p\rightarrow\partial/\partial t$. Furthermore, it is necessary to add nonlinear terms, which would eliminate the instability. By analogy with the Swift-Hohenberg equation,[7] we write the nonlinearity in the form aw^2-w^3. Thus, we arrive at the equation

$$\frac{\partial w}{\partial t}=\beta w-A\left(k_{0x}^2+\frac{\partial^2}{\partial x^2}\right)^2 w-2C\left(k_{0x}^2+\frac{\partial^2}{\partial x^2}\right)\left(k_{0y}^2+\frac{\partial^2}{\partial y^2}\right)w-B\left(k_{0y}^2+\frac{\partial^2}{\partial y^2}\right)^2 w-\frac{v_0}{\cos\psi}\frac{\partial w}{\partial x}+aw^2-w^3. \quad (1)$$

Here $|\mathbf{k}_0|$ is of the order of $1/d$, where d is the width of the liquid layer, and the sign of v_0 coincides with the sign of $d\sigma/dT$ (which is arbitrary for solutions).

Equation (1) has a certain generality, since it describes the loss of stability by modes whose wave vectors make a certain angle ψ with the x axis. In fact, the spectral function always has such a form near the maximum [with the form of $\mathrm{Re}p$ preserved under the substitutions $k_x\rightarrow -k_x$, $k_y\rightarrow -k_y$], and the behavior of the damped modes is not important due to the subordination principle. Note that for $A=B=C$ and $v_0=0$ this equation reduces to the Swift–Hohenberg equation with the maximum of the linear growth rate on the ring $|\mathbf{k}|=k_0$.

Up to now it was implied that the vector $\nabla\sigma$ points to the negative x axis. If the external inhomogeneity is given in the form of the field $\sigma(x,y)$ at the upper boundary of the liquid, then, for the σ inhomogeneity scales much larger than $1/k_0$, it is natural to make the change: $\partial/\partial x\rightarrow\mathbf{n}_1\nabla$, $\partial/\partial y\rightarrow\mathbf{n}_2\nabla$, where $\mathbf{n}_1$ is the unit vector along $\nabla\sigma$, and $\mathbf{n}_2$ is the unit vector in the perpendicular direction. We can assume that $\beta, A, B, C, k_{0x}, k_{0y}$, and v_0 are smooth functions of the space coordinates. The specific form of these functions should be chosen in such a way that they reproduce the form of the perturbation spectrum for the locally one-dimensional main flow at a given point (if the corresponding dependences are known).

Taking all this into account, we arrive, in the case of axial symmetry, at the equation

$$\frac{\partial w}{\partial t}=\beta w-A\left(k_r^2+\frac{\partial^2}{\partial r^2}\right)^2 w-2C\left(k_r^2+\frac{\partial^2}{\partial r^2}\right)\left(k_\varphi^2+\frac{\partial^2}{r^2\partial\varphi^2}\right)w-B\left(k_\varphi^2+\frac{\partial^2}{r^2\partial\varphi^2}\right)^2 w-\frac{v_0}{\cos\psi}\frac{\partial w}{\partial r}+aw^2-w^3. \quad (2)$$

Here v_0 is the phase velocity of the drift of the most unstable structure, r is the polar radius, φ is the polar angle and $k_r/k_\varphi=\cot\psi$. As for the dependence of the parameters in Eqs. (1) and (2) on the gradient b, the data of Ref. 5 do not allow to determine the corresponding relations. Therefore we will consider them, for the sake of simplicity, to be constant. In any case, in the framework of the suggestions made above, the variations of these quantities at distances of order $1/k_0$ should be small. Numerical estimates of the parameters made using the results of Ref. 5 give values of b_c of the order of tens of K/cm.

4. ANALYSIS OF THE PHENOMENOLOGICAL EQUATION IN ONE DIMENSION

In the series of experiments described in Ref. 3 the heating conditions were closest to those of the one-dimensional problem. A laser beam was focused with the help of a cylindrical mirror onto the plane of a target. The spot had the form of a strongly elongated ellipse. In the process of heating hot points formed near the center of the target and propagating to its edges were observed (Fig. 1a). The dynamics of these structures can be described phenomenologically in the framework of the one-dimensional version of the model (2). In one dimension Eq. (2) has for $k_0=1$ the form

$$\frac{\partial w}{\partial t}=\beta(x)w-\left(1+\frac{\partial^2}{\partial x^2}\right)^2 w-v(x)\frac{\partial w}{\partial x}+aw^2-w^3. \quad (3)$$

The supercriticality $\beta(x)$, according to its physical meaning, is negative in the region far from the center of the one-dimensional laser beam, where the temperature gradients are not large. Furthermore, it can be negative in the middle of the beam, where the radial component of the main flow velocity is small. It is also evident from the meaning of the quantity $v(x)$ that it should have different signs on different sides of the temperature distribution maximum. Equation (3) should be supplemented by boundary conditions

$$\left.\frac{\partial w}{\partial x}\right|_{x=0,l}=\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=0,l}=0, \quad (4)$$

where l is the size of the region. Note that the problem (3) in a finite region for $v=0$ contains a Lyapunov functional,[7] and, therefore, its solutions tend to stationary states for long times. Furthermore, separation of the variables in Eq. (3) linearized near $w=0$ gives rise, in this case, to a self-conjugate eigenvalue problem, and the increments of corresponding modes turn out to be real. The term $v\partial w/\partial x$ destroys the self-conjugacy of the problem and the Lyapunov functional. Therefore stable nonstationary regimes become possible. There is no fundamental difference between one and two dimensions in this case. Oscillations arise due to interaction between modes. Let us see how this happens in the case of two interacting modes.

For $\beta(x) = \text{const}$ and $v(x) = 0$, the solutions of the boundary problem (3), (4) linearized near the point $w = 0$ are cosines. Consider two neighboring interacting modes. Let the solution have the form $w = s_1 \cos(k_1 x) + s_2 \cos(k_2 x)$.

Substituting this solution into (3) and averaging over fast oscillating phases, we find for $a = 0$

$$\dot{s}_1 = (p_1 + V_{11}) s_1 + V_{12} s_2 - {}^3/_4 s_1 (s_1^2 + 2 s_2^2), \tag{5a}$$

$$\dot{s}_2 = V_{21} s_1 + (p_2 + V_{22}) s_2 - {}^3/_4 s_2 (s_2^2 + 2 s_1^2), \tag{5b}$$

where V_{ij} are corresponding matrix elements of the perturbation $v(x)$.

In the absence of perturbation ($V_{ij} = 0$) for $p_1, p_2 > 0$ there is an unstable node at the point $s_1 = 0$, $s_2 = 0$. When the perturbation is switched on, the roots of the characteristic equation of the system (5), linearized near $s_1 = 0$ and $s_2 = 0$, are given by the expression

$$\lambda_{1,2} = \frac{p_1 + p_2}{2} + \frac{V_{11} + V_{22}}{2} \pm \left[\frac{(p_1 - p_2 + V_{11} - V_{22})^2}{4} + V_{12} V_{21} \right]^{1/2}. \tag{6}$$

If, as it usually takes place, $V_{12} V_{21} < 0$, then, for $|V_{12} V_{21}| > \frac{1}{4}(p_1 - p_2 + V_{11} - V_{22})^2$, the instability of the zero equilibrium position has an oscillating character. As perturbation increases, asymmetrization of the phase portrait of the system (5) occurs. For a certain critical value of the perturbation all its nonzero equilibrium positions vanish. If the zero equilibrium position is unstable at the moment of confluence of the last saddle-node pair, then a stable limit cycle with zero frequency is created from the separatrix loop. If the interacting modes are assumed to be close to each other, the cycle is created when $|V_{12}| > |p + V_{11}|/2 \cdot 6^{1/2}$. If the perturbation increases further, the zero equilibrium position can become stable for $V_{11} + V_{22} < 0$ [see (6)]. Then the oscillating regime vanishes softly via an Andronov-Hopf bifurcation. In the two-mode approximation this occurs at $p_1 + p_2 + V_{11} + V_{22} = 0$.

If the region dimensions are large enough, not only two, but many modes of the relevant linear boundary problem can have positive increments. This leads both to a shift of the oscillation thresholds and to the onset of more complicated regimes.

Numerical analysis of Eq. (3) has confirmed the conclusion that oscillating regimes are created with zero frequency as the drift term increases. The boundary-value problem (3), (4) has been solved by an implicit scheme, using a five-point run with nonlinearity iteration.[8] Figure 1b shows the $w(x)$ dependences for different times in the case of the oscillating regime. Here $l = 18.3\pi$, $\beta(x) = -0.3(16/l^2)(x - l/4)(x - 3l/4)$, and $v(x) = -0.7(2x/l - 1)$. The initial conditions are $w(x) = 0.3\cos(18\pi x/l)$. The period T is equal to 112. The observed picture agrees qualitatively with the sequence of the photographs in Fig. 1a.

5. ANALYSIS OF THE PHENOMENOLOGICAL EQUATION IN TWO DIMENSIONS

Consider now Eq. (2) describing the time evolution of the function $w(x,y)$. Let us show that the introduced equations can describe the spiral-like structures. In fact, the perturbations have locally the form of quasiharmonic billows making everywhere the same angle ψ with the radius-vector. Therefore it is evident that the line of constant phase should be close to a logarithmic spiral.[9] Since the period of the structure remains roughly the same, as new arms of the spiral are created as the distance from the center increases.

Since Eq. (2) has a term describing drift along the radial coordinate, the many-arm spirals will move either away from the origin or towards it. If for an arbitrary polar angle φ the spiral twists in the same direction, its motion can be interpreted as rotation. Both twist directions are equiprobable, but they cannot coexist. In the case of a purely cubic nonlinearity one of the structures necessarily suppresses the other through nonlinear interaction. This can be shown on the basis of the reduced equations for the mode amplitudes in the case of translational symmetry.

Thus, spirals twisted in opposite directions cannot coexist in the same spatial region. Nevertheless, their coexistence is possible in different regions. The results of numerical calculation show that while moving along the angular or radial coordinate the direction of the spiral twist often changes. This can apparently be accounted for by the fact that in such a way it is easier to satisfy the condition of constancy of the structure period when moving away from the center.

We give now the results of the numerical analysis of Eq. (2) for $C = 0$. The problem has been solved by an implicit scheme using separation of the angular and radial coordinates. At every half-step the solution has been found by the method of a five-point run[8] with boundary conditions

$$\left.\frac{\partial w}{\partial r}\right|_{r=R} = \left.\frac{\partial^3 w}{\partial r^3}\right|_{r=R} = 0.$$

Nonlinear terms have been introduced into the coefficients of the run.

Figures 2a–2c show an example of numerical analysis of Eq. (2) with boundary conditions

$$w|_{t=0} = 0{,}3\cos(927{,}47\varphi + 282{,}9r).$$

Figures 2d and 2e show the photographs of the structure recorded in Ref. 3 for a medium focusing of the radiation beam (diameter d equals 1.8 cm). It is seen that the number of arms increases with the distance from the center.

The numerical results close to the structures shown in Fig. 3A have been obtained by introducing the relations

$$k_r = (R/3r)^{1/2}\cos\psi, \qquad k_\varphi = (R/3r)^{1/2}\sin\psi$$

for initial conditions

$$w|_{t=0} = 0{,}3\cos(15\varphi + 2{,}23r + 0{,}17),$$

(see Fig. 3B).

Figure 4 shows an example of a rotating spiral-like structure with the number of arms increasing with the distance from the center. The initial conditions are

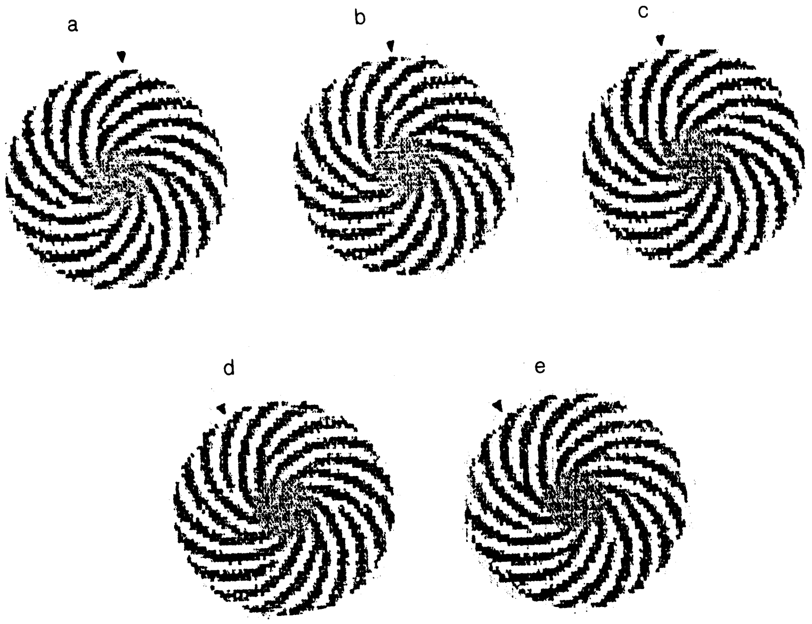

FIG. 4. The results of numerical study of Eq. (2) at different times $t = 2205$ (a), 2241 (b), 2277 (c), 2313 (d), and 2349 (e); $k_r = \cos\psi$, $k_\varphi = \sin\psi$. The parameters have the following values: $\beta = 0.1$, $v_0 = -0.1$, $a = 0$, $A = 1$, $B = 1$, $C = 0$, $\psi = 45°$, and $R = 40.338$. The period T of the quasiharmonic process, at the points $r = R$ is 108.

$$w|_{t=0} = 0{,}145 \exp\left[-\frac{(r-0{,}66R)^2}{(0{,}16R)^2}\right]\cos(18\varphi+0{,}25\pi r+0{,}17)$$
$$+0{,}1\exp\left[-\frac{(r-0{,}33R)^2}{(0{,}09R)^2}\right]\cos(6\varphi+0{,}25\pi r+0{,}17).$$

Owing to the nonuniform rotation velocity at different distances from the center, the arms reclose.

It is seen from Figs. 2–4 that by varying the coefficients in Eq. (2) we can simulate various experimental situations.

Numerical studies have shown that, as the Swift–Hohenberg equation, Eq. (2) has many different attractor solutions (including nonstationary ones in our case). The question of their regions of attraction has not been studied in detail.

6. CONCLUSIONS

The main conclusions of the present study are the following. The structures in the form of hot points and rotating spirals with many arms, observed in the experiments on vanadium heated by laser radiation, can be described as a result of interaction of weakly supercritical secondary flows due to loss of stability by thermocapillary convection. Such a convection is caused by spatially inhomogeneous distribution of σ over the surface of the liquid vanadium oxide. The structures mentioned above can be qualitatively described by a phenomenological equation for an order parameter obtained with the help of data on the perturbation spectrum near the threshold of stability loss.

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