# Strongly nonlinear waves in a plasma

V. S. Krivitskiĭ and S. V. Vladimirov

*General Physics Institute, Russian Academy of Sciences, Moscow*
(Submitted 6 November 1991)
Zh. Eksp. Teor. Fiz. **101**, 1510–1521 (May 1992)

Using kinetic theory we obtain general expressions which describe nonlinear longitudinal waves in a plasma. In the particular case of a cold plasma with immobile ions the solutions we find are the same as the results well known from hydrodynamical considerations. We analyze the qualitative differences which appear in a plasma where the particles have a finite temperature.

## 1. INTRODUCTION

The schemes for plasma accelerators of charged particles which make it possible to achieve high acceleration rates and energy imply excitation, in some way or other, of a strong longitudinal plasma wave in whose field an efficient acceleration takes place (see, e.g., the schemes for acceleration by beat waves,[1] by a wake wave excited by a bunch of charged particles,[2] or by short pulses of electromagnetic radiation[3,4]). The large amplitudes of the excited Langmuir waves necessarily require that one take into account nonlinear effects. Most often one restricts oneself in that case to the use of perturbation theory to consider effects up to and including the third order in the amplitude of the field wave (which leads, in particular, to a nonlinear frequency shift proportional to the square of the field amplitude). Already in this first nonvanishing order there appears a qualitative difference from the results of the linear theory: for instance, under well defined conditions a modulational instability develops which (in the one-dimensional case) leads to the formation of envelope solitons (see, e.g., Ref. 5). The possibility to use solitons of a longitudinal wave for acceleration is attractive because of their stability and unchanging structure. One then must bear in mind that the use of a finite number of terms in the perturbation theory series is valid only for sufficiently small wave amplitudes. On the other hand, both from the point of view of theoretical requirements for acceleration and in the appropriate experiments, the amplitudes of the excited longitudinal waves are rather large.

A theory exactly taking into account the nonlinearity of a one-dimensional plasma wave was already produced in the fifties.[6–9] However, this theory was based upon the equations of cold collisionless hydrodynamics (nonrelativistic[6] and relativistic;[7–9] see also more recent papers, e.g., Refs. 10 and 11) so that the range of its applicability is limited to a plasma with a temperature $T = 0$. Moreover, the problem itself of the validity of the hydrodynamic approximation for nonlinear motions in a collisionless cold plasma is nontrivial (we note that usually insufficient attention is paid to this problem). On the other hand, for arbitrary distributions with nonvanishing characteristic particle velocities (thermal, "two-temperature," etc. distributions) the hydrodynamic description is invalid (even in the linear limit when to obtain a valid dispersion one must introduce "manually" an adiabatic index into the equations).

The considerations of Ref. 12, based on kinetic theory, have shown that taking the particle velocity distribution into account introduces qualitatively new features into the description of a (one-dimensional) nonlinear plasma wave. In particular, they analyzed the possibility of the existence of particles trapped by the wave and if their distribution is the right one any form of nonlinear waves can be obtained.[12]

In what follows we develop in detail and analyze methods which enable us to obtain exact nonlinear wave solutions for arbitrary particle distributions in the plasma. It is important that in principle some of these solutions cannot be obtained in the framework of the hydrodynamic approximation. In particular, we indicate the possibility of the existence of solitary (solitonlike, but not envelope solitons as in the Zakharov system) nonlinear waves even in the case when the role played by the trapped particles is negligibly small.

## 2. GENERAL RELATIONS

We consider a plasma with an electron component which is described by the collisionless kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + e\mathbf{E}\frac{\partial f}{\partial \mathbf{p}} = 0, \qquad (2.1)$$

where $f$ is the electron distribution function and $e = -|e|$ is the electron charge. We are interested in longitudinal waves; we assume that there are no external fields. We restrict ourselves in what follows to the one-dimensional situation when the excitation propagates along the $x$-axis.

We supplement Eq. (2.1) with the Poisson equation

$$\operatorname{div}\mathbf{E} = \frac{\partial E}{\partial x} = 4\pi e\delta n = 4\pi e\int dp\,(f - f_0), \qquad (2.2)$$

where $f_0$ is the (stationary and uniform) equilibrium distribution function. In equilibrium conditions the electron and ion charges are balanced.

We shall look for a solution of the set (2.1) and (2.2) in the form of a wave propagating along the $x$-axis with a velocity $u$: $E = E(\tau)$ where $\tau = t - x/u$. We introduce the potential $\varphi$ of the electric field of the wave:

$$E = -\partial\varphi/\partial x.$$

As is usually done, we write the perturbation of the distribution function $\delta f = f - f_0$ in the form of a series

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$$\delta f=\sum_{j=1}^{\infty}\delta f^{(j)} \qquad (2.3)$$

in powers of the field $E$: $\delta f^{(j)}\sim E^{j}$.

By solving the set (2.1) to (2.3) we can obtain the following equation which describes oscillations of the field of the wave (we omit the intermediate calculations):

$$\frac{d^2}{dT^2}Z(T)=-\frac{\partial W(Z)}{\partial Z}, \qquad (2.4)$$

i.e., "an equation for oscillations of a particle of mass 1 in the potential $W(Z)$" (we choose $W(0)=0$):

$$W(Z)=-\sum_{j=1}^{\infty}\frac{Q_j}{(j+1)!}Z^{j+1},$$

$$Q_j=\frac{(mu)^j}{n_0}\int dp\left(\frac{1}{1-v/u}\frac{\partial}{\partial p}\right)^j f_0, \qquad (2.5)$$

where $n_0=\int dpf_0$ is the unperturbed density; for convenience we introduced the following dimensionless variables:

$$Z=-e\varphi/(mu^2),\quad T=\omega_{pe}\tau;$$

and the plasma frequency

$$\omega_{pe}^2=4\pi n_0e^2/m.$$

Integrating (2.4) once we have

$$^1/_2(dZ/dT)^2+W(Z)=C=\text{const}. \qquad (2.6)$$

When there are several kinds of particles (e.g., electrons and ions) present we must substitute for the potential $W(Z)$ the sum of the contributions $W=W_{\text{electr}}+W_{\text{ion}}$ since the Poisson equation (2.2) is linear in the charge density.

## 3. COLD PLASMA

In the case of a distribution function $f_0=n_0\delta(p)$, $p=mv$ describing a cold nonrelativistic plasma with $T=0$, the potential (2.5) is equal to

$$W(Z)=1+Z-(1+2Z)^{1/2}$$

(see Fig. 1); the convergence radius of the series (2.5) is in that case equal to $\frac{1}{2}$. We cannot continue the potential $W(Z)$ to the interval $Z<-\frac{1}{2}$ as it would there take on complex values.

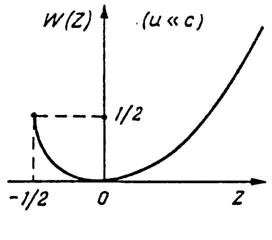


FIG. 1.

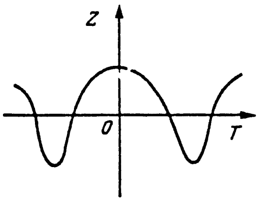


FIG. 2.

The equation of motion (2.4) in the potential

$$W(Z)=1+Z-(1+2Z)^{1/2}$$

can be easily integrated. The solution $Z(T)$ describes a periodic nonharmonic wave (see Fig. 2); the period $P$ is equal to $2\pi(=2\pi/\omega_{\text{pe}})$ and is independent of the amplitude. These results are the same as the results of a hydrodynamic approach.[6]

Taking the motion of the ions of mass $m_i$ into account gives

$$W(Z)=1-(1+2Z)^{1/2}+\frac{m_i}{m}\left(1-\left(1-\frac{m}{m_i}2Z\right)^{1/2}\right)$$

(see Fig. 3). The solutions $Z(T)$ are in general expressed in terms of elliptic integrals. The wave period $P$ depends on the amplitude when the motion of the ions is taken into account and is equal to

$$P=2\pi\left(\frac{m_i}{m+m_i}\right)^{1/2}\left(1-\frac{15}{8}\frac{m}{m+m_i}C\right).$$

In a sufficiently strong wave the velocity of the oscillations of the particles approaches the light velocity $c$. Using (2.5) one can also find the general relativistic expression for $W(Z)$ in a cold plasma:

$$W(Z)=[1+Z-(1+2Z+Z^2u^2/c^2)^{1/2}]/(1-u^2/c^2)$$

(see Fig. 4 for $u<c$, Fig. 5 for $u=c$, and Fig. 6 for $u>c$). This result is also the same as the result of the hydrodynamic approach.[7-9] The solution can again be expressed in terms of elliptic functions; the wave period for small amplitudes is equal to

$$P=2\pi\left(1+\frac{3}{8}C\frac{u^2}{c^2}\right)$$

(cf. Ref. 5).

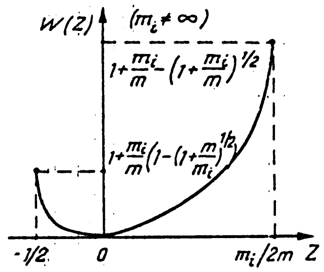


FIG. 3.

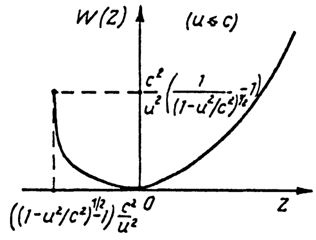


FIG. 4.

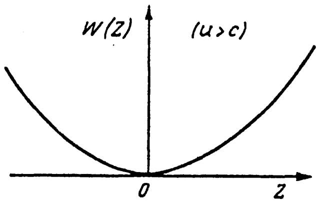


FIG. 6.

## 4. THERMAL CORRECTIONS

The electron distribution function for a nonrelativistic plasma with a nonvanishing temperature $T$ is

$$f_0 = \frac{n_0}{(2\pi)^{1/2} m v_T} \exp(-v^2/2v_T^2),$$

where $v_T^2 = T/m$. For small $v_T/u \ll 1$ the potential (2.5) is approximately equal to

$$W(Z) = 1 + Z - (1+2Z)^{1/2}$$

$$+ (v_T^2/u^2)\{2 + (1+2Z)^{-1/2} - 3(1+2Z)^{-1/2}\}$$

(these are the first two terms of an asymptotic expansion in $v_T^2/u^2$ which is not applicable near the point $Z = -\frac{1}{2}$). For small amplitudes $C \ll 1$ the wave period is

$$P = 2\pi/(1 + 3v_T^2/u^2)^{1/2} - 15\pi C v_T^2/u^2 + o(C v_T^2/u^2)$$

(cf. Ref. 13).

The required wave solution vanishes when the minimum of the function $W(Z)$ vanishes at the point $Z = 0$; this occurs if the velocity ratio $u/v_T$ becomes less than some critical value $a_{cr} \sim 1$ (as in the linear theory).

## 5. CHARGE DENSITY. WAVE ENERGY

According to (2.2) one can express the charge density oscillations

$$\delta n = \int dp\,(f - f_0)$$

in terms of the wave field $Z$:

$$\delta n/n_0 = d^2Z/dT^2 = -dW/dZ.$$

It is clear that the inequality $n = n_0 + \delta n > 0$ must be satisfied (by putting from the beginning in Sec. 3 $f_0 = n_0\delta(p)$ we have lost the possibility to require that the stronger condition $f = f_0 + \delta f > 0$ be satisfied). There can be "wave-breaking" in two cases: when $\delta n = -n_0$ (corresponding to $dW/dZ = 1$) or when $\delta n = +\infty$ ($dW/dZ = -\infty$). It is just this second case which is realized at the point $Z = -\frac{1}{2}$ for the function $W(Z)$ which describes a nonrelativistic cold plasma—see Fig. 1. (The wave breaks also in the relativistic case when $u < c$ for a similar reason—see Fig. 4).

The energy density $U$ of the oscillations is

$$U = (4\pi)^{-1} \int_{-\infty}^{t} dt\,\mathbf{E}\,\partial\mathbf{D}/\partial t, \tag{5.1}$$

where $\mathbf{D}$ is the electric induction connected with the polarization charges $\delta q = e\delta n$, which occur in the medium, by the equation

$$\operatorname{div}\mathbf{D} = \operatorname{div}\mathbf{E} - 4\pi\delta q.$$

Expressing $U$ in terms of the variables $Z$ and $T$ and using (2.6) we find that

$$U = n_0 m u^2 C.$$

Generally speaking, for a rigorous derivation we must first consider a nonvanishing external charge density $q_{\rm ext}$ and only afterwards let the latter tend to zero—this is necessary to remove a certain vagueness in (5.1), since for $q_{\rm ext} = 0$ we integrate a quantity which is equal to zero over an infinite interval, or else we have for the wave field $\mathbf{D} = 0$.

We see that apart from a dimensional factor $n_0 m u^2$ the energy density $U$ of the wave is the same as the integration constant $C$ in (2.6), i.e., as the "energy of the oscillation of a ficticious particle of mass 1 in the potential $W(Z)$." For wave velocities $u$ which are less than $c$ the wave energy density is thus bounded from above by some quantity (and hence $Z$ can take on values only in a range which is bounded both from below and from above by well defined finite limits). For $u = c$ the energy $C$ can take on arbitrarily large positive values and $Z$ can oscillate between the limits $-1$ and $+\infty$. For $u > c$ the energy $C$ can have any (positive) magnitude, and $Z$ can take on any value.

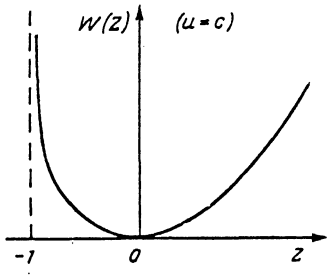


FIG. 5.

## 6. NONPERTURBATIVE KINETIC APPROACH

One can obtain the result (2.5) directly from the kinetic equation (2.1) without using an expansion in powers of the electric field (cf. Ref. 12). Indeed, if we look for a solution of the kinetic equation in the form $f = f(\tau, p)$, $\tau = t - x/u$ with the function

$$E = E(E) = -\operatorname{grad}\varphi(\tau) = \frac{1}{u}\frac{d\varphi(\tau)}{d\tau},$$

(2.1) reduces to the equation (for the sake of simplicity we consider the nonrelativistic case):

$$\left(1-\frac{v}{u}\right)\frac{\partial f}{\partial \tau}+\frac{e}{mu}\frac{d\varphi(\tau)}{d\tau}\frac{\partial f}{\partial v}=0. \qquad (6.1)$$

The general solution of this equation is

$$f=F\left(\frac{m(v-u)^2}{2}+e\varphi(\tau)\right), \qquad (6.2)$$

where $F$ is an arbitrary (differentiable) function (see Ref. 12).

In contrast to Ref. 12 it will be convenient for us to change in (6.2) at once from the argument

$$\mathscr{E}=m(v-u)^2/2+e\varphi$$

(the energy) to the argument $p = mv$ (the momentum). Indeed, the formal way of writing (6.2) assumes (if the function $F$ is single-valued) that the values of $f$ for two different arguments $v$ corresponding to the same $\mathscr{E}$ are the same. However, this requirement is completely unnecessary [this fact is especially obvious in the $\varphi = 0$ case: the solution of the free kinetic equation with $E = 0$ is an arbitrary function $f(v)$ of the velocity which does not at all need to satisfy the equation $f(-v) = (v)$]. Taking into account what we have said we can instead of (6.2) write the solution of (6.1) in the form

$$f=f_0(mu+m[(v-u)^2+2e\varphi(\tau)/m]^{1/2}). \qquad (6.3)$$

We note that if we use (6.2) instead of (6.3) there will not be a "good" limit for $\varphi\to 0$ (i.e., the unperturbed state of the plasma in the "correct" linear theory—see Ref. 12) in the theory developed here. In (6.3) $f_0$ is an arbitrary (differentiable) function. The reason why we use for (6.3) the same symbol $f_0$ as earlier in Secs. 2 and 3 will become clear in what follows.

It is important that one must take the function $[\cdots]^{1/2}$ in (6.3) not as an "algebraic" one, but as a branch of an analytical function (chosen after we have made in the complex $v$ plane the appropriate cut connecting the points $u\pm(-2e\varphi/m)^{1/2}$) for which the value of the square root in (6.3) tends to $v$ as $v\to+\infty$; therefore as $v\to-\infty$ the value of the root in (6.3) tends also to $v$ (and not to $|v|$).

The electron density $n$ is

$$n=\int dp f_0, \qquad (6.4)$$

where $p = mv$. We shall assume that there is no "external" electromagnetic field (i.e., a field which is not a wave field) and that the total electron and ion charges balance. The ion density $n_0$ is then equal to $n_0 = n|_{\varphi=0}$ (we assume that $m_i = \infty$).

The following is obvious: one must substitute the solution (6.3) into the Poisson equation:

$$-\frac{1}{u^2}\frac{d^2\varphi}{d\tau^2}=\operatorname{div}E=4\pi e\left[\int dp f_0-n_0\right]. \qquad (6.5)$$

Changing to the dimensionless variables which we used before

$$Z=-e\varphi/mu^2,\quad T=\omega_{pe}\tau,\quad \omega_{pe}=(4\pi n_0e^2/m)^{1/2},$$

we obtain an equation which describes the oscillations of the field in the wave:

$$d^2Z/dT^2=G(Z)-G(0), \qquad (6.6)$$

where

$$G(Z)=\frac{mu}{n_0}\int_{-\infty}^{+\infty}d\xi f_0(mu(1+[(\xi-1)^2-2Z]^{1/2})), \qquad (6.7)$$

while $G(0) = 1$ by virtue of the normalization condition (6.4) which we use.

We compare (6.6) and (6.7) with the results of Sec. 2. According to (2.4) the right-hand side of (6.6) is

$$-dW(Z)/dZ=\sum_{j=1}^{\infty}Q_jZ^j/j!,$$

where the $Q_j$ are defined in (2.5). Expanding (6.7) in a power series in $Z$ we find (replacing the integration variable $\xi$ by $\zeta=\xi-1$):

$$G(Z)=\frac{mu}{n_0}\int_{-\infty}^{+\infty}d\zeta f_0(mu(1+(\zeta^2-2Z)^{1/2}))$$

$$=G(0)+\frac{mu}{n_0}\int_{-\infty}^{+\infty}d\zeta\sum_{j=1}^{\infty}\frac{Z^j}{j!}\frac{d^j}{dZ^j}f_0(mu(1+(\zeta^2-2Z)^{1/2}))|_{Z=0}$$

$$=1+\frac{mu}{n_0}\int_{-\infty}^{+\infty}d\zeta\sum_{j=1}^{\infty}\frac{Z^j}{j!}\left(\frac{dp}{dZ}\frac{d}{dp}\right)^j f_0(p)|_{Z=0}$$

$$=1+\frac{(mu)^j}{n_0}\int_{-\infty}^{+\infty}dp\sum_{j=1}^{\infty}\frac{Z^j}{j!}\left(\frac{1}{1-v/u}\frac{d}{dp}\right)^j f_0(p)$$

$$=1+\sum_{j=1}^{\infty}\frac{Q_j}{j!}Z^j, \qquad (6.8)$$

which is exactly the same as the results of Sec. 2 (in (6.8) the variable is

$$p=mu(1+(\zeta^2-2Z)^{1/2}).$$

It is now clear why we used in (6.3) the same symbol $f_0$.

We must here make a few remarks. Firstly, writing down (6.7) presupposes that the argument of the function $f_0$ for $Z>0$ and $\xi$ close to unity becomes complex. This will not give any trouble if we assume that $f_0$ is analytic.

The function $f_0$ is arbitrary and the oscillations of the electrical field $Z$ in the wave can therefore also have an arbitrary form. However, in that sense even the linear oscillations can correspond to a completely arbitrary dispersion relation $\omega=\omega(k)$ (for an appropriate choice of the "unperturbed" distribution function $f_0$). The situation is different if for some reason or other we assume that $f_0$ corresponds to a well defined function (for instance, Maxwellian); in that case both the linear dispersion law and the nonlinear regime of the electric field in the wave become at once uniquely defined.

The leeway in finding the solution of the kinetic equation (2.1) is also present in the procedure of finding (2.3) as a sum of a perturbation-theory series. Indeed, (2.5) is based on using a "particular solution of the inhomogeneous equation" for $\delta f^{(l+1)}$:

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)\delta f^{(l+1)}=-eE\frac{\partial}{\partial p}\delta f^{(l)}, \qquad (6.9)$$

$l=0,1,2,...$; this particular solution is (in Fourier components)

$$\delta f_{\omega k}^{(l+1)}=-i(\omega-kv)^{-1}e\int d\omega'\, dk' E_{\omega' k'}\partial\delta f^{(l)}_{\omega-\omega',k-k'}/\partial p.$$

However, formally one can add to this solution an arbitrary general solution of the homogeneous equation, proportional to $\delta(\omega-kv)$ (in terms of Ref. 12 this corresponds to particles "trapped" by the field of the wave).

The procedure of expanding in a power series in $Z$ in (6.8) is correct if the function $f_0$ is analytic. The assumption that $f_0$ is analytic means that there cannot be a singular $\delta$-function term, proportional to $\delta(v-u)$ in $f_0$.

## 7. PLASMA WITH $T\neq 0$

The nonperturbative kinetic theory considered in the preceding section makes it possible to take correctly into account thermal effects without having recourse to an expansion in $v_T^2/u^2$. We can therefore now give an answer to the problem of the behavior of the function $W(Z)$ in the neighborhood of the point $Z=-\frac{1}{2}$ (in the nonrelativistic case) and also to the left of that point.

We consider the integral (6.7), substituting for $f_0(p)$ the Maxwell distribution from Sec. 4. We shall assume to begin with that the quantity $Z$ is negative (bearing in mind our study of the behavior of $W(Z)$ in the neighborhood of the point $Z=-\frac{1}{2}$). The argument of the function $f_0$ is then real for any real $\xi$ [see (6.7)].

The path of integration over the variable $\zeta\equiv\xi-1$ is shown in Fig. 7. One must in that case understand the integral (6.7) as a principal-value integral because of the necessity to connect the branch points $\pm i(2|Z|)^{1/2}$ of the root in (6.7) by a cut. The necessity of introducing a cut is thus, in turn, dictated just by the fact which we have already discussed before (Sec. 6): as $Z\to 0$ the branch of the root in (6.7) which we choose must go over into $\zeta$ and not into $|\zeta|$.

Integrating over $\zeta$ along the sides of the cut is in the present case (for $Z<0$) not necessary as the momentum $p$ and with it the variable $\zeta$ are real according to (6.5) to (6.7). Integration along the cut would lead to $\operatorname{Im} G\neq 0$ (and hence to $\operatorname{Im} W\neq 0$) which is physically meaningless.

For $Z>0$, on the other hand, the cut lies on the path of integration in (6.7) along the real axis—see Fig. 8. In this connection the problem arises of how to integrate over $\zeta$ for $-(2Z)^{1/2}<\zeta<(2Z)^{1/2}$ and also of how to interpret in that case the corresponding results (nonvanishing imaginary parts of the distribution function $f_0$, of the "potential" $W$, and so on). We shall return to this problem in what follows.

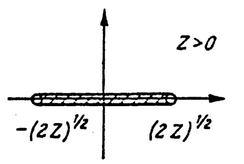

FIG. 8.

According to (6.7) we have for the potential $W(Z)$:

$$W(Z)=Z-\frac{mu}{(2\pi mT)^{1/2}}\int_0^Z dz\int_{-\infty}^{+\infty} d\zeta \times\exp\left[-\frac{u^2}{2v_T^2}\left(1+\zeta^2-2z+2(\zeta^2-2z)^{1/2}\right)\right]. \qquad (7.1)$$

It is convenient to study this integral by introducing the notation $-\eta^2\equiv 2z$, $-H^2\equiv 2Z$ and performing the conformal transformation $\tau=(\zeta^2+\eta^2)^{1/2}$. The integration over $\tau$ must then be taken along the real axis from $-\infty$ to $-\eta$ and then from $+\eta$ to $+\infty$. Writing $y\equiv\tau+1$ and changing the order of integration in (7.1) we then obtain

$$W(Z)=Z+\frac{u}{v_T(2\pi)^{1/2}}\iint_S dy\,d\eta\,(y-1)\exp\left(-\frac{u^2y^2}{2v_T^2}\right)\eta\left((y-1)^2-\eta^2\right)^{-1/2}, \qquad (7.2)$$

where the region $S$ is shown hatched in Fig. 9. We emphasize that the main contribution to the integral in (7.2) as $T\to+0$ comes from the neighborhood of the section $0<\eta<H$ for $y\approx 0$.

Omitting straightforward algebraic calculations we bring (7.2) to the form

$$W(Z)=Z+\frac{u}{v_T(2\pi)^{1/2}}\{I_1+I_2+I_3\}, \qquad (7.3)$$

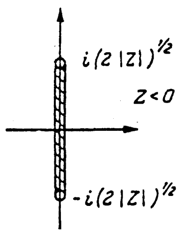

FIG. 7.

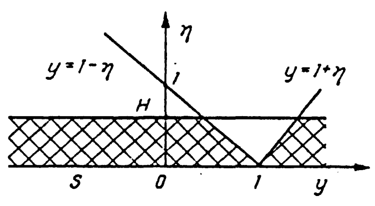

FIG. 9.

where

$$I_1=\int_{-\infty}^{1-H} dy(y-1)\exp\left(-\frac{u^2y^2}{2v_T^2}\right)[y-1+((y-1)^2+2Z)^{1/2}],$$

(7.4)

$$I_2=\int_{1-H}^{1+H} dy(y-1)^2\exp\left(-\frac{u^2y^2}{2v_T^2}\right), \quad (7.5)$$

$$I_3=\int_{1+H}^{+\infty} dy(y-1)\exp\left(-\frac{u^2y^2}{2v_T^2}\right)[y-1-((y-1)^2+2Z)^{1/2}].$$

(7.6)

We draw attention to the different signs in front of the root in $I_1$ and $I_3$; in (7.4) and (7.6) the roots must already be understood in the "algebraic" sense.

Analyzing (7.3) we find that the function $W(Z)$ has the shape shown in Fig. 10. Asymptotically as $Z\to-\infty$ we have, for instance, for $I_3$ [we multiply the numerator and denominator of the integrand by $(y-1+((y-1)^2+2Z)^{1/2})$]:

$$|I_3|<\left|\int_{1+H}^{+\infty} dy\frac{y-1}{y-1+((y-1)^2+2Z)^{1/2}}(-2Z)\right.$$

$$\left.\times\exp\left(-\frac{u^2y^2}{2v_T^2}\right)\right|<|2Z|\int_{1+H}^{+\infty} dy\exp\left(-\frac{u^2y^2}{2v_T^2}\right)$$

$$\times<|2Z|\int_{H}^{+\infty} dy\cdot\ldots\sim\frac{v_T^2}{u^2}\frac{1}{(2|Z|)^{1/2}}\exp\left(-\frac{u^2}{v_T^2}|Z|\right) \quad (7.7)$$

(we bear in mind that $H^2=-2Z$; i.e., the contribution $I_3$ is exponentially small as $Z\to-\infty$. A similar estimate is obtained also for $I_1$. For $I_2$ we have: $I_2\to(2\pi)^{1/2}\times(v_T/u)(1+v_T^2/u^2)$ as $Z\to-\infty$. As $Z\to-\infty$ we have thus $W(Z)=Z+1+v_T^2/u^2+o(1)$.

The function $W(Z)$ of (7.3) has a maximum for $Z=Z_{\max}\approx-\frac{1}{2}$ (for $T\to+0=0$ we have $Z_{\max}=-\frac{1}{2}$). The function $W(Z)$ is continuously differentiable for $T\neq0$; the discontinuity in the derivative $dW/dZ$ as $T\to+0$ is connected with the appearance of a singularity of the exponential in the integrand of (7.2):

$$\frac{u}{v_T(2\pi)^{1/2}}\exp\left(-\frac{u^2y^2}{2v_T^2}\right)\to\delta(y) \quad \text{for} \quad v_T\to+0.$$

Figure 9 explains the appearance of a kink in $W(Z)$ for $Z=-\frac{1}{2}$ as $T\to+0$: since the whole contribution to the integral is concentrated in that case on the section $0\leqslant\eta\leqslant H$, $y=0$, for $H=1$ (i.e., $Z=-\frac{1}{2}$) the second term in (7.2) (i.e., the integral) becomes equal to unity and ceases to increase when $H$ increases further. For $H>1$ (i.e., $Z<-\frac{1}{2}$) we thus have $W(Z)=Z+1$. The function $W(Z)$ has therefore as $T\to+0$ the shape shown in Fig. 11. The maximum of $W(Z)$ gets sharper (compare with Fig. 10). We have thus given an answer (which in principle could not be obtained in hydrodynamics) to the problem of the behavior of $W(Z)$ for $Z<-\frac{1}{2}$.

The "potential" $W(Z)$ shown in Fig. 10 admits the existence of solitary-wave type solutions of Eq. (6.6) [or (2.15)]: for $C=W(Z_{\max})$ the period $P$ of the oscillations tends to infinity. For any (even arbitrarily small!) temperature $T$ there thus occurs not wave breaking, as in hydrodynamics, but the formation of a solitonlike state $Z(t-x/u)$. (Of course, the problem of whether such a solution is a "true soliton" goes beyond the framework of the present approach.)

One must in this connection be reminded of Ref. 14 where solitary self-consistent *BGK* type solutions[12] were found which describe the combined motion of a packet of plasma waves and an electromagnetic bunch (in that case there is no braking of the wake wave which may be of interest from the point of view of collective acceleration methods).

Taking ions into account makes changes little in principal: the second maximum (as $T\to+0$—the second kink, see Fig. 3) at $Z\approx m_i/2m_e$ lies above the "admissible" region for the existence of oscillations. However, for $m_i=m_e$ (i.e., in an electron-positron plasma) both maxima correspond to the same energy: degeneracy sets in—see Fig. 12. In contrast to the $m_i=\infty$ case considered above the solitonlike solutions have in this case the shape not of a solitary hump but of a "kink," i.e., there are different ($\pm\frac{1}{2}$ as $v_T\to+0$) asymptotes in the limit as $(t-x/u)\to\mp\infty$.

For $T\neq0$ the potential $W(Z)$ has in the relativistic case when $u<c$ qualitatively the same form as in the nonrelativistic limit (see Figs. 10 and 11).

We now turn to the problem of how for $Z>0$ one should integrate "over the cut" from $\zeta=-(2Z)^{1/2}$ to $\zeta=(2Z)^{1/2}$ (see Fig. 8). The argument of the function $f_0$ and hence the distribution function itself as well as the "potential" $W$ then become complex quantities, which is physically meaning-

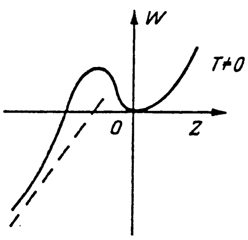


FIG. 10.

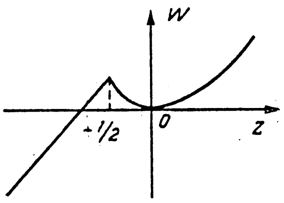


FIG. 11.

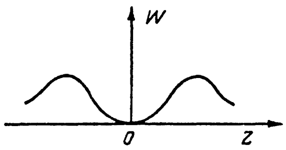


FIG. 12.

less. We thus are faced with the problem of what is the distribution function for $Z>0$ and $-(2Z)^{1/2}<\zeta<(2Z)^{1/2}$. Particles with such momenta are trapped by the wave; their distribution can be arbitrary, in general, being not at all connected with the distribution function of the other particles. Mathematically this means that the trapped particles are described by another function of the form (6.3), where we now must take for them cuts from $-\infty$ to $-(2Z)^{1/2}$ and from $(2Z)^{1/2}$ to $+\infty$. To obtain information about the number and the distribution of the trapped particles one must solve the initial problem, describing the formation of the nonlinear wave, exactly and this goes beyond the framework of the present discussion. We note incidentally that for similar reasons the functions of the form (6.3) referring to the regions $\zeta<-(2Z)^{1/2}$ and $\zeta>(2Z)^{1/2}$ (i.e., corresponding to particles moving "to theleft" and "to the right") can also in no way be connected with one another.

## 8. CONCLUSION

The expressions we have obtained for the "potential" $W(Z)$ assumed that there are no trapped particles (or that their role is negligible). As was already made clear in Ref. 12 one can, generally speaking, through an appropriate choice of the trapped-particle distribution, obtain any shape of $W(Z)$ and hence any shape of wave. Of course, some particles are always trapped by the wave; but in order to find out how many one must study the trapping process itself in detail.

It is rather obvious to assume that for $u\sim c$ (which is the most interesting case for the acceleration problem) the number of trapped particles with subluminal velocities cannot be large since in the corresponding experiments the bulk of the plasma particles have velocities of the order of thermal velocities which are much smaller than $c$.

Therefore: what can the function $f_0$ be in a strongly nonlinear wave? How can we connect the function $f_0$ describing a stationary wave solution with the function $f_0$ corresponding to the "unperturbed" state of the plasma prior to the excitation of the wave (by some well defined method)? Could one say that the "final" $f_0$ excited by the wave by some adiabatic means is the same as the "initial" $f_0$ (and because of this it makes sense to consider for $f_0$ a Maxwell distribution function)? By what excitation method can there appear the initial number of trapped particles? The answers to these questions go beyond the framework of the present discussion (which is based upon solutions which are functions of the argument $t-x/u$) and requires a study of the dynamics of the wave excitation process.

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Translated by D. ter Haar