Evolution of a Riemann wave in dispersive hydrodynamics

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We use the slow modulation method to study the breaking of a Riemann wave in dispersive hydrodynamics. The generalized hodograph method is used to find an exact analytical solution of the Whitham equations describing the dissipationless shock wave zone. We study Korteweg–de Vries hydrodynamics and the nonlinear Schrödinger equation.

1. INTRODUCTION

The breaking of a simple Riemann wave[1] leads in dispersive hydrodynamics to the appearance of a dissipationless shock wave (DSW)—a continuously expanding region filled with nonlinear small-scale waves. The use of the modulational Whitham equations[2] which recently have attracted the attention of a large number of researchers (see, e.g., the review in Ref. 3 and the literature cited there) has turned out to be very efficient for describing the structure of DSW. In a number of important cases [Korteweg–de Vries (KdV) hydrodynamics, the nonlinear Schrödinger (NLS) equation, the sine-Gordon equation, and so on] the modulational system can be written in a symmetric Riemann form.[4–6] If the physical conditions make it possible to specify some of the Riemann variables, the modulational equations have analytical solutions in the shape of simple and quasisimple waves.

Gurevich and Pitaevskiĭ were the first to apply the Whitham method in Ref. 7 to the problem of a shock wave in dispersive hydrodynamics (Gurevich–Pitaevskiĭ problem); they studied the structure of a simple DSW arising under the conditions of a sharp initial discontinuity, and they also gave a numerical analysis of the breaking of a simple hydrodynamic wave. Further developments of this problem were obtained in Refs. 8 and 9 where a more general class of solutions—quasisimple waves arising when a Riemann wave breaks at the boundary of a stationary gas—was studied.

In the present paper we construct a general analytical solution of the modulational KdV and NLS equations for the problem of the breaking of an arbitrary monotonic profile. In principle the possibility of constructing such solutions is connected with the use of the generalized (multidimensional) hodograph method which was proposed by Tsarev in Ref. 10 (see also Ref. 3). A noticeable feature of the generalized hodograph method consists in that it enables one to linearize the modulational system even though the number of Riemann invariants r_i exceeds the number of independent variables (x and t). Such a linearization turns out to be possible for the modulational KdV and NLS systems and for other systems which possess the semi-Hamiltonian property.[10,3] It is also very important that the boundary value problem for the matching of the solutions in the DSW region with the solutions of ordinary hydrodynamics can be formulated particularly simply in the generalized hodograph system. We show in Sec. 2 that the nonlinear Gurevich–Pitaevskiĭ problem with conditions on an unknown boundary can be reduced in r-space to the solution of a linear system with linear conditions on prescribed boundaries.

The direct solution of this system, however, is difficult since for KdV and NLSE it contains rather complicated combinations of complete elliptical integrals. Using the potential representation for the group velocities established by the present authors in Ref. 11 it is possible to reduce the (vector) system of Tsarev equations to a linear scalar second-order system not containing elliptical integrals. Its derivation is given in Sec. 3. In the KdV and NLS case this set of equations is of the Euler–Poisson type obtained recently in Refs. 12 and 13 by direct calculations. In the Appendix we also obtain a scalar equation of a different form which describes the slow modulations of the sine-Gordon equation.

An important property of the generalized hodograph method is that any two (i,j) of the Tsarev equations can be solved independently provided the r_k ($k \neq i,j$) invariants are fixed. This makes it possible during the construction of the three-dimensional solution in r-space to solve a two-dimensional Goursat problem at each stage, i.e., to use essentially the usual $(x;t)-(r_i,r_j)$ hodograph transformation. The required solution of the Gurevich–Pitaevskiĭ problem is constructed in Sec. 4. It contains two arbitrary functions describing the initial monotonic hydrodynamical profile and has a singularity with a continuous first derivative if these functions are different. Quasisimple waves are briefly considered in Sec. 5; the solutions for these are obtained by a simple reduction of the general formulae of Sec. 4.

The appropriate equations and some important solutions of the Gurevich–Pitaevskiĭ problem were obtained earlier by the present authors for KdV hydrodynamics in Ref. 11.

2. GUREVICH–PITAEVSKIĬ PROBLEM AND GENERALIZED HODOGRAPH METHOD

A simple Riemann wave is described by the equation

$$\partial_t r+V(r)\partial_x r=0, \tag{1}$$

which has the solution

$$x-V(r)t=W(r), \tag{2}$$

where $W(r)$ is the function which is the inverse of the initial profile $r=r_0(x)$.

a. KdV wave-breaking

We consider for definiteness the breaking of a simple wave in dispersive KdV hydrodynamics:

$$\partial_t u+u\partial_x u+\partial^3_{xxx}u=0. \tag{3}$$

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Let the wave-breaking occur initially at $t=0$ at the point $x=0$, $r=0$ (Fig. 1) where

$$U(x,0)\equiv r(x,0)=\begin{cases} r_0^+(x)\leqslant 0, & x\geqslant 0,\\ r_0^-(x)>0, & x<0,\end{cases} \tag{4}$$

with the inverse function (Fig. 1b)

$$W(r)=\begin{cases} W_+(r), & r\leqslant 0,\\ W_-(r), & r>0.\end{cases} \tag{5}$$

The DSW region which appears after the wave (2), (5) breaks is described by three functions

$$r_3(x,\ t)\geqslant r_2(x,\ t)\geqslant r_1(x,\ t)$$

(Fig. 2) which satisfy the modulational Whitham system obtained by averaging the KdV equation (3) over the family of solutions of the form $u(kx-\omega t)$, where k is the wavenumber and ω is the frequency. The modulational KdV system can be written in the Riemann form

$$\partial_t r_i + V_i(r)\partial_x r_i = 0, \tag{6}$$

where there is no summation over repeated indices ($i=1,2,3$), $r\equiv(r_1,r_2,r_3)$, and the group velocities have the "potential" representation[11]

$$V_i=\frac{\partial_i\omega}{\partial_i k}=U-\lambda\frac{\partial_i U}{\partial_i\lambda}\,. \tag{7}$$

Here

$$U(r)=\frac{\omega}{k}=\frac{1}{3}\sum_{i=1}^{3} r_i, \tag{8}$$

$$\lambda(r)=\frac{2\pi}{k}=6^{1/2}\cdot 2\int_{r_1}^{r_2} d\tau\left[\prod_{j=1}^{3}(\tau-r_j)\right]^{-1/2} \tag{9}$$

are the phase velocity and the wavelength. Equation (7) is of a general nature and follows from the conservation law for the number of waves

$$\partial_t k(r)+\partial_x\omega(r)=0, \tag{10}$$

which is always satisfied under conditions of single-phase averaging and which is, of course, a consequence of Eqs. (6).[2,14]

The required solution which describes the evolution of a DSW confined between the unknown boundaries $x=x^-(t)$ (trailing edge) and $x=x^+(t)$ (front) satisfies the conditions on the curves $x^\pm(t)$ for the matching of the corresponding branches $r_\pm(x,t)$ of the "exterior" solution of (2) with the solution (r_1,r_2,r_3) of the "interior" modulational equations (6) (Fig. 2):[7]

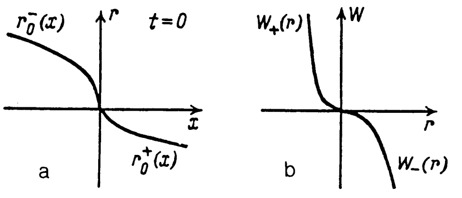

FIG. 1. Initial data for the wave-breaking problem (a) and the inverse function (b).

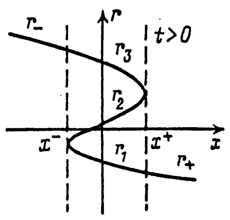

FIG. 2. Riemann invariants as functions of x in a DSW.

$$\begin{aligned} &r_3(x^-,\ t)=r_-(x^-,\ t), \quad r_2(x^-,\ t)=r_1(x^-,\ t)\\ &r_1(x^+,\ t)=r_+(x^+,\ t), \quad r_2(x^+,\ t)=r_3(x^+,\ t)\,. \end{aligned} \tag{11}$$

The hodograph transformation is found to be very efficient for solving this kind of problem; it was first applied by Gurevich, Krylov, and Mazur[9] to the problem of a quasisimple KdV wave in which two invariants (r_2 and r_3) were changing while the third one was fixed ($r_1\equiv 0$), with the wave moving through an unperturbed medium. It is natural to study the three- (and the more-) dimensional problem (6), (11) using the generalized hodograph method, a brief version of which is the following.

The vector generalization of the Riemann solution (2) for the system (6) has the form

$$x-V_i(r)t=W_i(r),\ i=1,\ldots,N, \tag{12}$$

where N is the order of the system. However, the functions $W_i(r)$ cannot be arbitrary but must satisfy the compatibility conditions

$$\frac{\partial_i W_j}{W_i-W_j}=\frac{\partial_i V_j}{V_i-V_j}. \tag{13}$$

The overdetermined system (13) is solvable in the KdV and NLS cases which have been studied.[3,10] The general solution of the compatibility equations (13) determines all symmetries, i.e., equations of the form

$$\partial_\tau r_i + W_i(r)\partial_x r_i = 0, \tag{14}$$

which commute with the original system (6) $(\partial^2_{t\tau} r_i = \partial^2_{\tau t} r_i)$.

In the KdV case the system (13) is defined in the region $r_1\leqslant r_2\leqslant r_3, r_1\leqslant 0, r_3\geqslant 0$ between the planes $r_2=r_1$ (trailing edge) and $r_2=r_3$ (leading edge) (Fig. 3). The Gurevich–Pitaevskiĭ conditions (11) in r-space take a simple form:

$$\begin{aligned} &W_1(r_1,\ 0,\ 0)=W_+(r_1),\\ &W_3(0,\ 0,\ r_3)=W_-(r_3). \end{aligned} \tag{15}$$

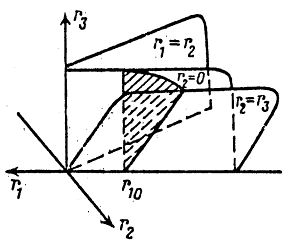

FIG. 3. Definition region in r space.

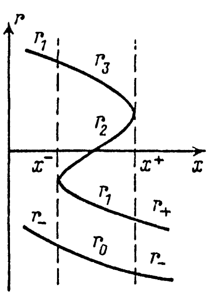

FIG. 4. Riemann invariants of the average NLS equation as functions of x for $t>0$.

Conditions (15) guarantee the matching of the solution of the system (13) with the appropriate branches of the function $W(r)$ [see (5)] on the fronts. Indeed, using explicit expressions for the velocities V_i one finds easily from Eqs. (13) that

$$(\partial_2+\partial_3)W_1|_{r_2=r_3}=0,\ (\partial_1+\partial_2)W_3|_{r_2=r_1}=0 \tag{16}$$

for bounded solutions W_1 and W_3, i.e., the values of W_1 are carried along without change from the r_1-axis along the line $r_2=r_3$ (and similarly for W_3 and the trailing front $r_2=r_1$).

We have thus instead of the nonlinear problem (6), (11) with conditions at unknown boundaries a linear system (13) in r-space which satisfies the linear conditions (15) on given boundaries.

The system (12) has an important property: it can be integrated independently in any $r_k=\text{const}$ plane. This "two-dimensional" structure makes it possible to split the solution of the wave-breaking problem into several stages, and we shall make effective use of this in what follows.

b. NLS wave-breaking

We can write the nonlinear Schrödinger equation with defocusing

$$2i\partial_t\Psi+\partial_{xx}{}^2\Psi-2|\Psi|^2\Psi=0$$

by means of the change of variables

$$\Psi=\rho^{1/2}\exp(i\varphi),\quad \varphi_x=v$$

in the form of a hydrodynamic system:[8]

$$\partial_t\rho+\partial_x(\rho v)=0, \tag{17}$$

$$\partial_t v+v\partial_x v+\partial_x\rho-\frac{1}{4}\partial_x\left(\frac{\partial_{xx}{}^2\rho}{\rho}-\frac{(\partial_x\rho)^2}{2\rho^2}\right)=0.$$

The system (17) describes one-dimensional flows without dissipation in the case of positive dispersion.

Averaging a stationary wave over a period leads[6,8] to a Riemann system of the form (6), which in contrast to the modulational KdV system consists of four equations for the invariants $r_3\geqslant r_2\geqslant r_1>r_0$ (Fig. 4). The characteristic velocities are, as before, described by the general formulae (7) in which

$$U(r)=\frac{1}{4}\sum_{j=0}^{3}r_j, \tag{18}$$

$$\lambda(r)=2^{1/2}\int_{r_2}^{r_3}\left[-\prod_{j=0}^{3}(\tau-r_j)\right]^{-1/2}d\tau. \tag{19}$$

The role of the "exterior" equations in the wave-breaking problem is now played by the equations of Eulerian hydrodynamics with $\gamma=2$. The problem of the breaking of a simple wave moving to the right corresponds to $r_0=\text{const}$. Introducing new variables

$$r_i'={}^3/_4(r_i-r_0)$$

and changing to a moving coordinate system,

$$x'=x-r_0t,$$

we arrive at a three-dimensional problem, the formulation of which was analyzed in the preceding subsection.

3. SCALAR POTENTIAL AND LINEAR EQUATIONS NOT CONTAINING ELLIPTIC INTEGRALS

The difficulties arising when we study directly the system (13) for the KdV and NLS equations are primarily connected with the presence of complete elliptic integrals on the right-hand sides. There is, however, a procedure which makes it possible to reduce the system (13) in its general form to a linear system of second order equations without elliptic integrals.

We note first of all an important consequence of Eqs. (13). Using the potential representation (7) we can easily prove that

$$\partial_j(\partial_i kW_i)=\partial_i(\partial_j kW_j), \tag{20}$$

i.e., the required functions W_i can also be written in a potential form:

$$W_i=\frac{\partial_i(kf)}{\partial_i k}=f-\lambda\frac{\partial_i f}{\partial_i\lambda}, \tag{21}$$

where f is some unknown function which has the meaning of a generalized phase velocity. Indeed, let us consider the conservation law

$$\partial_\tau k(r)+\partial_x(kf(r))=0 \tag{22}$$

which commutes with (10). Some of the equations (22) have the obvious meaning of conservation laws for the number of waves for the higher equations of the corresponding hierarchy. Using (14) to change explicitly in (22) to the r variables we find Eq. (21).

Equations (7) and (21) make it possible to convert the compatibility equations (13) to scalar equations: instead of N functions W_i we have a single unknown function f satisfying a combined overdetermined set of second-order equations:

$$\frac{\partial_i(\partial_j f/\partial_j U)}{\partial_i f/\partial_i U-\partial_j f/\partial_j U}=\frac{\partial_i(\partial_j\lambda/\partial_j U)}{\partial_i\lambda/\partial_i U-\partial_j\lambda/\partial_j U},\quad i\neq j. \tag{23}$$

We note the obvious solution $f=\lambda$ (which is not the same as the simplest solution $f=U$) which, however, bears no relation to the wave-breaking problem.

If we use Eqs. (8) and (18) for $U(r)$ we can reduce the set (23) for the KdV and NLS equations to the simpler set[11]

$$\frac{\partial_{ij}{}^2 f}{\partial_i f-\partial_j f}=\frac{\partial_{ij}{}^2\lambda}{\partial_i\lambda-\partial_j\lambda}. \tag{24}$$

It is interesting to note that the set (24) is a consequence of (23) not only in the $\partial_i U=$ const case (KdV,NLS) but also when the known function $U(r)$ satisfies the same system:

$$\frac{\partial_{ij}^2 U}{\partial_i U-\partial_j U}=\frac{\partial_{ij}^2\lambda}{\partial_i\lambda-\partial_j\lambda}.$$

Using the integral representations (11) and (19) for $\lambda(r)$ we find easily that for the KdV and NLS

$$\frac{\partial_{ij}^2\lambda}{\partial_i\lambda-\partial_j\lambda}=\frac{1}{2(r_i-r_j)}. \qquad (25)$$

The required function f thus satisfies in both cases the same set of equations

$$E_{ij}f=0, \qquad (26)$$

where

$$E_{ij}=\partial_{ij}^2-\frac{\partial_i-\partial_j}{2(r_i-r_j)}$$

is the Euler–Poisson operator.[15]

An important example—the sine-Gordon modulations for which the set (23) reduces to other equations, also without elliptic integrals—is analyzed in the Appendix.

As before, we shall use the KdV equation as an example to further consider the wave-breaking problem. If we use (15) and (21), the boundary conditions for the system (26) take the following form:

$$\begin{aligned} f(0,0,r_3)=f_-(r_3)&=\frac{1}{2}r_3^{-1/2}\int_0^{r_3}x^{-1/2}W_-(x)\,dx\leqslant 0,\\ f(r_1,0,0)=f_+(r_1)&=\frac{1}{2}(-r_1)^{-1/2}\int_0^{-r_1}x^{-1/2}W_+(-x)\,dx\geqslant 0. \end{aligned} \qquad (27)$$

4. GENERAL SOLUTION OF THE GUREVICH–PITAEVSKIĬ PROBLEM

The general solution of the three-dimensional system (26) was found in 1918 by Eisenhart[16] and has the form

$$f=\sum_{i=1}^{3}\int_0^{r_i}\frac{\varphi_i(\mu)}{\left[\prod_{j=1}^{3}(\mu-r_j)\right]^{1/2}}d\mu, \qquad (28)$$

where the $\varphi_i(x)$ $(i=1,2,3)$ are arbitrary functions. The wave-breaking problem, however, has a number of features which require a special study.

The integral representation (28) makes it relatively simple to write down the solution of the problem for the breaking of an antisymmetric profile characterized by a single function $W(x)$. In that case we have [see (27)]

$$f_-(z)=-f_+(-z)=\frac{1}{2}z^{-1/2}\int_0^z x^{-1/2}W(x)\,dx. \qquad (29)$$

The solution has the symmetric form

$$f(r_1,r_2,r_3)=\int_{r_2}^{r_3}\frac{\Phi(\tau)\,d\tau}{\left[-\prod_{j=1}^{3}(\tau-r_j)\right]^{1/2}}+\int_{r_1}^{0}\frac{\Phi(-\tau)\,d\tau}{\left[\prod_{j=1}^{3}(\tau-r_j)\right]^{1/2}}, \qquad (30)$$

where

$$\Phi(z)=\frac{1}{2\pi}\int_0^z\frac{W(x)}{(z-x)^{1/2}}dx. \qquad (31)$$

There is, however, an important difficulty: can a solution of the Whitham equations accomplish the matching of two different hydrodynamic regimes, $r_+(x,t)$ and $r_-(x,t)$ [i.e., is there a solution of the wave-breaking problem with initial data of the general form (4)]? The answer is: it is possible, this matching is guaranteed by the continuity of the normal derivative $\partial_2 f(r_2=0)$ and in this case the $r_2=0$ plane is a singularity and the solution on different sides of it has a different form.

We first find $f(r_1,0,r_3)$ and the Goursat problems (Fig. 5a)

$$E_{31}f=0,\ f(0,0,r_3)=f_-(r_3),\ f(r_1,0,0)=f_+(r_1), \qquad (32)$$

where the $f_\pm(z)$ are given by (27). The solution of the problem (32) has the form

$$f(r_1,0,r_3)=\int_0^{r_3}\frac{\Phi_-(\tau)\,d\tau}{[\tau(r_3-\tau)(\tau-r_1)]^{1/2}}+\int_{r_1}^{0}\frac{\Phi_+(-\tau)\,d\tau}{[-\tau(r_3-\tau)(\tau-r_1)]^{1/2}}=G(r_1,r_3), \qquad (33)$$

where

$$\begin{aligned}\Phi_\pm(z)&=\frac{1}{2\pi}\int_0^z\frac{W_\pm(x)}{(z-x)^{1/2}}dx,\qquad z>0,\\ W_+(x)&=W_+(-x).\end{aligned} \qquad (34)$$

To establish the r_2 dependence of the solution we consider the function $G(r_1,r_3)$ as (Goursat) data for the corresponding boundary-value problems in the $r_1=$ const $(r_2>0)$ and $r_3=$ const $(r_2<0)$ planes. For instance, in each $r_1=r_{10}=$ const plane (Fig. 5b; see also the hatched region in Fig. 3) we have the following problem:

$$\begin{aligned}&E_{23}f(r_{10},r_2,r_3)=0,\\ &f(r_{10},0,r_3)=G(r_{10},r_3),\ f(r_1,r,r)\ \text{– is bounded.}\end{aligned} \qquad (35)$$

The problem (35) has a unique solution:

$$f(r_{10},r_2,r_3)=\int_{r_2}^{r_3}\frac{\Phi_-(\tau)\,d\tau}{[(\tau-r_{10})(\tau-r_2)(r_3-\tau)]^{1/2}}+\int_{r_1}^{0}\frac{\Phi_+(-\tau)\,d\tau}{[(\tau-r_{10})(r_2-\tau)(r_3-\tau)]^{1/2}}.$$

One can similarly construct the solution on the other side of the $r_2=0$ plane (Fig. 5c).

Finally, the required solution has the form (we bear in mind that $r_3\geqslant 0$, $r_1\leqslant 0$)

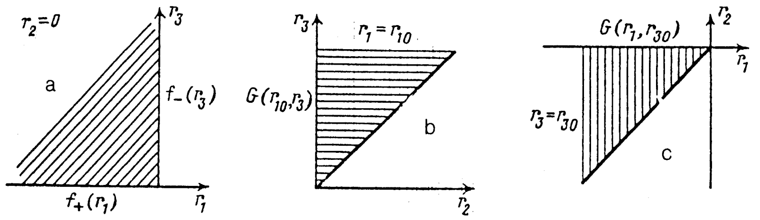

FIG. 5. Integration domains in the planes: a: $r_2=0$; b: $r_1=\text{const}$; c: $r_3=\text{const}$.

$$f(r_1,r_2,r_3)=\begin{cases} \displaystyle\int_{r_2}^{r_3}\frac{\Phi_-(\tau)\,d\tau}{\left[-\prod_{j=1}^{3}(\tau-r_j)\right]^{1/2}}+\int_{r_1}^{0}\frac{\Phi_+(-\tau)\,d\tau}{\left[\prod_{j=1}^{3}(\tau-r_j)\right]^{1/2}}, & r_2\geqslant 0, \quad (36)\\ \displaystyle\int_{r_1}^{r_2}\frac{\Phi_+(-\tau)\,d\tau}{\left[\prod_{j=1}^{3}(\tau-r_j)\right]^{1/2}}+\int_{0}^{r_3}\frac{\Phi_-(\tau)\,d\tau}{\left[-\prod_{j=1}^{3}(\tau-r_j)\right]^{1/2}} & r_2\leqslant 0. \quad (37)\end{cases}$$

One checks easily that $\partial_2 f(r_2=0)$ is continuous [this requirement follows from the continuity of $r_i(x,t)$ and Eqs. (12) and (21)]. Nonetheless, the $r_2=0$ plane is clearly a singularity (cf. Ref. 7) since the solutions on the two sides of it are described by different formulae. Such a weak discontinuity occurs always when the initial profile is nonanalytical at the wave-breaking point. Of course, if $W(r)$ is an analytical monotonic function, the solutions (36) and (37) are the same and turn into the solution (30) obtained earlier. Note that the self-similar solution of the problem of "cubic" wave-breaking which was considered numerically in Ref. 7, in fact, does not contain singularities because of the analyticity of the initial data.

We give also another form of the solution of the Gurevich–Pitaevskiĭ problem. Substituting the expressions (34) for $\Phi_\pm(\tau)$ into (36) and (37) and changing the order of integration we easily find the normal representation in the form of single integrals

$$f(r_1,r_2,r_3)=\frac{1}{\pi(r_3-r_2)^{1/2}}\int_{r_2}^{r_3}\frac{W(x)}{(x-r_1)^{1/2}}K(z)\,dx+\frac{1}{\pi(r_2-r_1)^{1/2}}\times\int_{r_1}^{r_2}\frac{W(x)}{(r_3-x)^{1/2}}K\left(\frac{1}{z}\right)dx, \quad (38)$$

where

$$z=\left[\frac{(r_2-r_1)(r_3-x)}{(r_3-r_2)(x-r_1)}\right]^{1/2},$$

and $K(m)$, the complete elliptic integral of the first kind, is the Riemann function for the Euler–Poisson equation.[17]

The $x^\pm(t)$ curves which bound the DSW region in the physical plane are multiple characteristics and can be found from the solution (12) the boundaries together with the conditions

$$\frac{dx^\pm}{dt}=V^\pm$$

(here $V^\pm$ are the multiple characteristic velocities on the fronts).

We consider the important special case of the breaking of an antisymmetric profile

$$r_0(x)=-|x|^{1/q}\operatorname{sgn}x,$$

where $q>1$ is an arbitrary number. The required solutions $r_i(x,t)$ are self-similar

$$r_i=t^\gamma l_i\left(\frac{x}{t^{\gamma+1}}\right),\quad \gamma=\frac{1}{1-q}.$$

The solution in the r-plane can be found using Eqs. (36), (37), and (34), but it is more convenient to obtain it directly, using the homogeneous solutions of the system (26) of the form

$$f=r_i^q\Phi(-q,\ {}^1/_2;\ {}^1/_2-q;\ r_j/r_i),$$

where $\Phi(a,b;c;z)$ is the solution of the appropriate hypergeometric equation (for details see Ref. 11).

For odd integral values $q=M$ (analytic profile) the solution takes a polynomial form:

$$f(r)=P_M(r)=-\frac{2^M M!}{(2M-1)!!(2M+1)}\times\sum_{m+n+l=M}\frac{\left(\frac{1}{2}\right)_m\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_l}{m!n!l!}r_1^m r_2^n r_3^l. \quad (39)$$

For $M=3$ we have the solution of the "cubic" wave-breaking problem, which was obtained in Ref. 18 by using methods from algebraic geometry. If $q=N$ is even (nonanalytic profile) Eq. (39), of course, does not describe a wave-breaking problem. The solution is then given by the formulae

$$f(r)=\begin{cases} \displaystyle P_N(r)-A_N\int_{r_2}^{r_3}\frac{\tau^{N+1/2}\,d\tau}{[(\tau-r_1)(\tau-r_2)(r_3-\tau)]^{1/2}}, & r_2\geqslant 0,\\ \displaystyle -P_N(r)+A_N\int_{r_1}^{r_2}\frac{(-\tau)^{N+1/2}\,d\tau}{[(\tau-r_1)(\tau-r_2)(\tau-r_3)]^{1/2}}, & r_2\leqslant 0,\end{cases} \quad (40)$$

where

$$A_N=\frac{2}{\pi}\sum_{k=1}^{N}\frac{(-1)^k}{2N-2k+1}\frac{N!}{k!(N-k)!}$$

and has a weak discontinuity with continuous derivative for $r_2 = 0$.

5. QUASISIMPLE WAVES

The quasisimple wave concept was introduced by Gurevich and Krylov in Ref. 8. It was used in Ref. 9 to describe the breaking of a Riemann wave propagating in an unperturbed gas which corresponds to $r_0^+(x) \equiv 0$. In that case the DSW is described by two Whitham equations for r_2 and r_3 and one must put the invariant r_1 equal to zero to guarantee the matching at the leading front. The modulational system, reduced in this way, can be studied using the usual hodograph transformation $(x,t) \to (r_2,r_3)$ which is a special case of the generalized hodograph method. The required solutions can be obtained from Eqs. (36), (37), and (34), which describe the general situation of the breaking of a monotonic profile.

Putting $r_1 \equiv 0$ in the general solution (36), (37) we find at once the integral representation $(r_2 \geqslant r_1)$:

$$f(0, r_2, r_3) = \int_{r_2}^{r_3} \frac{\Phi(\tau)\,d\tau}{[\tau(r_3-\tau)(\tau-r_2)]^{1/2}}, \tag{41}$$

where $\Phi(\tau)$ is as before given by Eq. (34).

The homogeneous solution describing the breaking of the profile

$$r_0^-(x) = (-x)^{1/q}, \quad q>1,$$

has the form

$$f(0, r_2, r_3) = -\frac{r_3^q \pi^{1/2} \Gamma(1+q)}{2\Gamma(q+3/2)} F(-q, 1/2; 1; 1-r_2/r_3), \tag{42}$$

where $F(a,b;c;z)$ is the hypergeometric function.

For integral $q = M$ the hypergeometric series terminates and the solution turns into a polynomial:

$$f(0, r_2, r_3) = -\frac{2^M M!}{(2M-1)!!(2M+1)} \times \sum_{m+n=M} \frac{\left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_n}{m!n!} r_2^m r_3^n. \tag{43}$$

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APPENDIX

Equations for the sine–Gordon modulation without elliptic integrals

The slow modulation of the sine–Gordon equation

$$\partial_{tt}^2 \varphi - \partial_{xx}^2 \varphi + \sin\varphi = 0 \tag{A1}$$

are described by a second-order Whitham system which can be written in Riemann form.[5] We use the potential representation (6), (7) in which

$$\lambda(r) = \frac{16(r_1r_2)^{1/2}}{16(r_1r_2)^{1/2}-1} \int_{r_2}^{0} \frac{d\tau}{[-\tau(\tau-r_1)(\tau-r_2)]^{1/2}}, \tag{A2}$$

$$U(r) = \frac{16(r_1r_2)^{1/2}+1}{16(r_1r_2)^{1/2}-1}, \quad r_1<r_2<0. \tag{A3}$$

Substitution of (A2) and (A3) into the general system (23) leads to a second-order scalar equation

$$\partial_{12}^2 f = \frac{1}{2(1-16(r_1r_2)^{1/2})} \left(\frac{\partial_1 f}{r_2} + \frac{\partial_2 f}{r_1}\right) + \frac{1}{2(r_1-r_2)} (\partial_1 f - \partial_2 f), \tag{A4}$$

which does not contain elliptic integrals. We note that the variables in (A4) separate if we introduce

$$z = \frac{1}{1-16(r_1r_2)^{1/2}}, \quad y = \frac{r_1+r_2}{2(r_1r_2)^{1/2}}$$

instead of r_1 and r_2 (cf. Ref. 19).

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