Critical dynamics of dilute magnetic materials

V. V. Prudnikov and A. N. Vakilov

Omsk State University

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A field-theoretical description is offered for the critical dynamics of magnetic systems with frozen nonmagnetic impurities. Values of the dynamic critical exponent are derived directly for three-dimensional systems in the two-loop approximation. The Padé–Borel summation technique is used. The results are compared with the results found through the use of an ε-expansion and with the values calculated for the dynamic exponent for homogeneous systems in the three-loop approximation. The results are also compared with results found through a Monte Carlo numerical simulation. The effects of the impurity on the critical behavior of two-dimensional magnetic materials are discussed.

Phase transitions in homogeneous magnetic materials are altered when randomly distributed frozen impurities are added to the system or in the case of Ising magnetic materials.[1] The ε-expansion method makes it possible to calculate the critical indices for dilute magnetic materials.[2] However, the asymptotic convergence of the ε-expansion series is even poorer in this case than for homogeneous magnetic materials.[3] The renormalization-group approach to the description of dilute magnetic materials, which was applied directly to three-dimensional systems in Ref. 4, has led to values of the static critical indices which agree with experimental results.[5]

In the present paper we offer a renormalization-group description of the critical dynamics of dilute magnetic materials directly for $d = 3$. Our model is a classical spin system with nonmagnetic impurity atoms frozen at lattice sites. This system is described by the Hamiltonian

$$H = -\frac{1}{2}\sum_{ij} J_{ij} p_i p_j \mathbf{S}_i \mathbf{S}_j,$$

where, as usual, $\mathbf{S}_i$ is an m-component spin variable; $J_{i,j}$ are the constants of the translationally invariant, short-range, ferromagnetic exchange interaction; and p_i is a random variable which is described by the distribution function

$$P(p_i) = p\delta(p_i - 1) + (1-p)\delta(p_i)$$

with $p = 1 - c$, where c is the concentration of the nonmagnetic impurity atoms. This model is thermodynamically equivalent to the $O(m)$-symmetric Ginzburg–Landau–Wilson model, determined by the effective Hamiltonian

$$H[\varphi, V] = \int d^d x \left\{ \frac{1}{2}[|\nabla\varphi|^2 + r_0\varphi^2 + V(x)\varphi^2] + \frac{g_0}{4!}\varphi^4 \right\}. \quad (1)$$

Here $\varphi(x,t)$ is an m-component order parameter, $V(x)$ is the potential of the random impurity field, $r_0 \sim T - T_{0c}(p)$, T_{0c} is the critical temperature of the dilute magnetic material as given by the mean-field theory, g_0 is a positive constant, and d is the dimension of the system. We specify the impurity potential by means of the Gaussian distribution

$$P_V = A_V \exp\left[-(8\delta_0)^{-1} \int d^d x V^2(x) \right],$$

where A_v is a normalization constant, and the positive constant δ_0 is proportional to the impurity concentration and to the square of the impurity potential. Deviations of the impurity distribution from a Gaussian distribution are inconsequential near the critical temperature.[6]

The dynamic behavior of a magnetic material in the relaxation regime near the critical temperature can be described by a kinetic equation of the Langevin type for the order parameter:

$$\frac{\partial\varphi}{\partial t} = -\lambda_0 \frac{\delta H}{\delta\varphi} + \eta + \lambda_0 \mathbf{h}, \quad (2)$$

where λ_0 is a kinetic coefficient, and $\eta(x,t)$ is a Gaussian random force. This force characterizes the effect of the heat reservoir and is specified by the distribution function

$$P_\eta = A_\eta \exp\left[-(4\lambda_0)^{-1} \int d^d x\, dt \eta^2(x,t) \right]$$

with the normalization constant A_η. Here $\mathbf{h}(t)$ is the external field, which is the thermodynamic conjugate of the order parameter. The temporal correlation function $G(x,t)$ of the order parameter is found by solving Eq. (2) with $H[\varphi, V]$ as given in (1) for $\varphi[\eta, \mathbf{h}, V]$, by then taking an average over the Gaussian random force η with the help of P_η, by taking an average over the random potential of the impurity field $V(x)$ with the help of P_v, and by identifying the part of the solution which is linear in $\mathbf{h}(0)$. In other words, we write

$$G(x,t) = \frac{\delta}{\delta \mathbf{h}(0)} [\langle \varphi(x,t) \rangle]_{imp}|_{\mathbf{h}=0},$$

with

$$[\langle \varphi(x,t) \rangle]_{imp} = B^{-1} \int D\{\eta\} D\{V\} \varphi(x,t) P_\eta P_V,$$

$$B = \int D\{\eta\} D\{V\} P_\eta P_V.$$

When we attempt to apply the standard renormalization-group technique to this dynamic model, we run into some serious difficulties. However, it has been shown[7] that for homogeneous systems without any disorder caused by impurities the critical dynamic model based on the Langevin equation is completely equivalent to an ordinary Lagrangian system[8] with the Lagrangian

$$L = \int d^d x\, dt \left\{ \lambda_0^{-1} \varphi^2 + i\varphi^* \left(\lambda_0^{-1} \frac{\partial\varphi}{\partial t} + \frac{\delta H}{\delta\varphi} \right) \right\}.$$

The correlation function of the order parameter, $G(x,t)$, for a homogeneous system is given by

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$$G(x,t)=\langle\varphi(0,0)\varphi(x,t)\rangle=\Omega^{-1}\int D\{\varphi\}$$

$$\times D\{\varphi^*\}\varphi(0,0)\varphi(x,t)\exp(-L[\varphi,\varphi^*]),$$

with

$$\Omega=\int D\{\varphi\}D\{\varphi^*\}\exp(-L[\varphi,\varphi^*]).$$

A generalization of this group-theoretical approach was given by one of the present authors in a previous paper,[9] which also gave the details of the application of this approach to the critical dynamics of dilute magnetic materials with frozen point impurities and extended defects on the basis of an ε-expansion.

The Feynman diagrams which determine the contributions to the correlation function of the order parameter and the 4-tail vertices contain a d-dimensional integration over momenta. Near the critical point, they are characterized by an ultraviolet divergence of the pole type at large momenta q. To eliminate these poles, we use a dimensional regularization technique involving the introduction of renormalized quantities.[10] We define the renormalized order parameter as $\varphi=Z^{-1/2}\varphi_0$. The renormalized vertex functions then have the generalized form

$$\Gamma^{(n)}(q,\omega;\ r,\ g,\ \delta,\ \lambda,\ \mu)=Z^{n/2}\Gamma_0^{(n)}(q,\ \omega;\ r_0,\ g_0,\ \delta_0,\ \lambda_0) \qquad (3)$$

with renormalized coupling constants g and δ, renormalized temperature r, and renormalized kinetic coefficient λ, given by

$$g_0=\mu^{4-d}Z_g g,\quad \delta_0=\mu^{4-d}Z_\delta\delta,$$
$$r_0=\mu^2 Z_r r,\quad \lambda_0^{-1}=\mu^2 Z_\lambda\lambda^{-1}. \qquad (4)$$

The scale factor μ is introduced to put the quantities in dimensionless form. The quantity $\Gamma^{(2)}$ in (3) corresponds to the inverse correlation function of the order parameter, $G(q,\omega)$, and $\Gamma^{(4)}$ corresponds to 4-tail vertex functions $\Gamma_g^{(4)}$ and $\Gamma_\delta^{(4)}$ for the coupling constants g and δ, respectively. The Z-factors are determined in each successive order of the diagram expansion of the vertex functions in g and δ, on the basis of the requirement that the renormalized vertex functions be regular. We have implemented this scheme for regularizing the vertex functions for dilute magnetic materials in the two-loop approximation (Fig. 1). The next step in the field-theory approach is to determine the scaling functions $\beta_1(g,\delta)$, $\beta_2(g,\delta)$, $\gamma_r(g,\delta)$, $\gamma_\varphi(g,\delta)$, and $\gamma_\lambda(g,\delta)$, which specify a differential renormalization-group equation for the vertex functions:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_1\frac{\partial}{\partial g}+\beta_2\frac{\partial}{\partial\delta}-\gamma_r r\frac{\partial}{\partial r}+g\lambda\frac{\partial}{\partial\lambda}-\gamma_\varphi\frac{n}{2}\mu\frac{\partial\ln Z}{\partial\mu}\right]$$
$$\times\Gamma_0^{(n)}(q,\omega;r,g,\delta,\lambda,\mu)=0.$$

To pursue the discussion of the dynamic model, we need explicit expressions for only the functions β_1 and β_2 and for the dynamic scaling function γ_λ, which are derived in the two-loop approximation. Using the notation $v_1=(m+8)J_1 g/6$ and $v_2=16J_1\delta$, we thus find

$$\beta_1(v_1,v_2)=-v_1\left\{1-v_1+\frac{3}{2}v_2\right.$$
$$-v_1^2\left[\frac{8(5m+22)}{(m+8)^2}f(d)-\frac{8(m+2)}{(m+8)^2}h(d)\right]$$
$$-v_1v_2\left[\frac{12(m+5)}{(m+8)}f(d)-\frac{2(m+2)}{(m+8)}h(d)\right]$$
$$\left.+v_2^2\left[\frac{21}{4}f(d)-\frac{1}{4}h(d)\right]\right\}+o(v^4),$$

$$\beta_2(v_1,v_2)=-v_2\left\{1+v_2-\frac{2(m+2)}{(m+8)}v_1\right.$$
$$+v_2^2\left[\frac{11}{4}f(d)-\frac{1}{4}h(d)\right]$$
$$+v_1^2\left[\frac{24(m+2)}{(m+8)^2}f(d)-\frac{8(m+2)}{(m+8)^2}h(d)\right]$$
$$\left.-v_1v_2\left[\frac{12(m+2)}{(m+8)}f(d)-\frac{2(m+2)}{(m+8)}h(d)\right]\right\}+o(v^4), \qquad (5)$$

$$\gamma_\lambda(v_1,v_2)=(4-d)\left\{\frac{v_2}{4}+\frac{v_2^2}{8}[3f(d)-h(d)+\varphi(d)]\right.$$
$$-v_1v_2\frac{(m+2)}{(m+8)}[f(d)-h(d)]$$
$$\left.+v_1^2\frac{4(m+2)}{(m+8)^2}[\psi(d)-h(d)]\right+o(v^3).$$

Here

$$h(d)=D_1'/J_1^2,\ f(d)=D_2/J_1^{2-1/2},\ \psi(d)=D_3/J_1^2,\ \varphi(d)=D_4/J_1^2$$

are combinations of the single-loop integral

$$J_1=\int d^dq/(q^2+1)^2=\frac{S_d}{2}\Gamma(d/2)\Gamma(2-d/2),$$

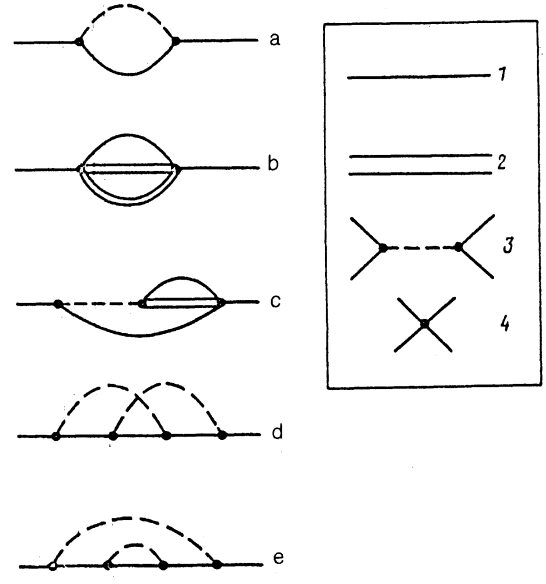

FIG. 1. Feynman-diagram representation of the contributions to the vertex function $\Gamma^{(2)}(q,\omega)=G^{-1}(q,\omega)$ in the two-loop approximation with corresponding weight factors. a—(-4δ), b—$(-(m+2)g^2/18)$, c—$4(m+2)g\delta/3$, d,e—$(-16\delta^2)$. In the inset, line 1 corresponds to $G_0(q,\omega)=(-i\omega/\lambda+q^2)^{-1}$, line 2 corresponds to $C_0(q,\omega)=2\lambda^{-1}(\omega^2/\lambda^2)+q^4)^{-1}$, vertex 3 corresponds to $2\delta\delta(\omega)$, and vertex 4 corresponds to g.

with $Sd = 2\pi^{d/2}/(2\pi)^d\,\Gamma(d/2)$ [$\Gamma(x)$ is the gamma function] and of the two-loop integrals

$$D_1' = \frac{\partial}{\partial k^2}\int \frac{d^dq_1\,d^dq_2}{(q_1^2+1)(q_2^2+1)[(q_1+q_2+k)^2+1)}\Big|_{k=0},$$

$$D_2 = \int \frac{d^dq_1\,d^dq_2}{(q_1^2+1)(q_2^2+1)[(q_1+q_2)^2+1]},$$

$$D_3 = 3\int \frac{d^dq_1\,d^dq_1}{(q_1^2+1)(q_2^2+1)[q_1^2+q_2^2+(q_1+q_2)^2+1]^2}, \qquad (6)$$

$$D_4 = 2\int \frac{d^dq_1\,d^dq_2}{(q_1^2+1)(q_2^2+1)^3} = J_1^2\,\frac{2-d/2}{1-d/2}.$$

The expressions for the functions $\beta_1(v_1,v_2)$ and $\beta_2(v_1,v_2)$ are the same as the corresponding functions which were derived in Ref. 4 in a description of the equilibrium critical behavior of dilute magnetic materials. A numerical calculation of the integrals has yielded the values $h(d=3) = 0.07408$, $f(d=3) = 0.16665$, $\psi(d=3) = 0.40619$, and $\varphi(d=3) = -1$.

The nature of the critical point for each value of m and for each value of d is determined completely by the stable fixed point for the coupling constants (v_1^*, v_2^*). This point is found from the requirement that the functions $\beta_1(v_1,v_2)$ and $\beta_2(v_1,v_2)$ vanish: $\beta_1(v_1^*,v_2^*) = 0$, $\beta_1(v_1^*,v_2^*) = 0$. It can be seen from the expressions for these functions that v_1^* and v_2^* are quantities of order $4-d$, so the expansions in v_1 and v_2 for the functions β_1, β_2, and γ_λ with $d=3$ are asymptotically converging. To sum them we use the Padé–Borel method,[11] which we have also used for the functions β_1, β_2, and γ_λ. Numerical analysis of the equations for determining the fixed points and their stability conditions shows that in this case, in contrast with the ε-expansion approach, there is no accidental degeneracy of the fixed points with $m=1$ (Ref. 2). Of the four fixed points, only two are of interest: the fixed point for homogeneous systems $(v_1^*\neq 0, v_2^* = 0)$ and the impurity fixed point $(v_1^*\neq 0, v_2^*\neq 0)$, which specifies new critical properties of dilute magnetic materials. The impurity fixed point is stable for $m < m_c$. An estimate yields $m_c = 2.0114$ for $d=3$ (Ref. 4). The homogeneous fixed point is not stable for $m < m_c$. The opposite situation prevails for $m > m_c$.

A calculation of the exponent for the specific heat, α_{xy}, for the three-dimensional xy model ($m=2$; Ref. 12), which is the most accurate calculation to date, has shown that the condition $\alpha_{xy} < 0$ holds and that, in accordance with the Harris criterion,[1] impurities do not lead to new critical behavior. With regard to the value $m_c = 2.0114$, one can hope that incorporating higher orders of the expansion in v_1 and v_2 for the functions β_1 and β_2 will reduce m_c, pulling it below 2. The only system in which impurities cause new critical behavior is thus the Ising-like impurity with $m=1$. The impurity fixed point for a three-dimensional Ising model is specified by the values $v_1^* = 2.39631$, $v_2^* = 0.60509$, while the homogeneous point is specified by the values $v_1^* = 1.59661$, $v_2^* = 0$.

Substitution of the values of the coupling constants at the fixed point into the scaling function $\gamma_\lambda(v_1,v_2)$ reveals the dynamic critical exponent z: $z = 2 + \gamma_\lambda(v_1^*, v_2^*)$. Applying the Padé–Borel summation method to the asymptotic series of the expansion of γ_λ in powers of v_1 and v_2, we find the following expression for the exponent z:

$$z = 2 - \frac{\alpha_1}{\beta} - \frac{1}{\beta}\left(\alpha_2 + \frac{\alpha_1}{\beta}\right)[1 - {}_2F_0(1,1;\beta)], \qquad (7)$$

where

$$\alpha_1 = 2(4-d)(m+2)[\psi(d) - h(d)]\,(v_1^*)^2/(m+8)^2,$$

$$\alpha_2 = (4-d)\,v_2^*/4,$$

$$\beta = [3f(d) - h(d) + \mu(d)]\,v_2^*/4 - 4(m+2)[f(d) - h(d)]\,v_1^*/(m+8),$$

and ${}_2F_0(1,1;\beta)$ is the confluent hypergeometric function.

When the values of the coupling constants at an impurity fixed point are used, we find the following values of the exponent z for the three-dimensional Ising model: $z_{\text{imp}}^{(2)}(d=3) = 2.237$. The value found for z_{imp} in the same two-loop approximation by means of the ε-expansion, on the other hand, is $z_{\text{imp}}^{(2)} = 2.336$.

Evaluation of the exponent z for the homogeneous Ising model in the two-loop approximation does not require the use of a summation technique for γ_λ, since in this case γ_λ contains only a single term, proportional to v_1^2. As a result we find $z_{\text{pure}}^{(2)}(d=3) = 2.125$, while an ε-expansion calculation leads to $z_{\text{pure}}^{(2)} = 2.011$. To refine the effect of the method for summing the asymptotic Padé–Borel series, we calculated the function γ_λ for homogeneous magnetic materials in the three-loop approximation:

$$\gamma_\lambda = 4(4-d)\frac{(m+2)}{(m+8)^2}[\psi(d) - h(d)]\,v_1^2\left\{1 - \frac{3}{2}v_1\left[\frac{c(d)-b(d)}{\psi(d)-h(d)} - \frac{4}{3}\right]\right\}, \qquad (8)$$

where $b(d) = D_5'/J_1^3$, $c(d) = D_6/J_1^3$, and D_5' and D_6 are the three-loop integrals

$$D_5' = -\frac{\partial}{\partial k^2}\int d^dq_1\,d^dq_2\,d^dq_3/\{[(q_1+k)^2+1] \times (q_2^2+1)(q_3^2+1)[(q_1+q_2)^2+1][(q_1+q_3)^2+1]\}|_{k=0},$$

$$D_6 = 8\int d^dq_1\,d^dq_2\,d^dq_3/\{(q_1^2+1)(q_2^2+1) \times (q_3^2+1)[q_1^2+q_2^2+(q_1+q_2)^2+1][q_1^2+q_3^2+(q_1+q_3)^2+1]^2\}.$$

Numerical calculations of the integrals yield the values $b(d=3) = 0.09466$ and $c(d=3) = 2.12991$. The Padé–Borel method yields the following expression for γ_λ:

$$\gamma_\lambda = 8(m+2)(4-d)[\psi(d) - h(d)]\frac{-1+\alpha(d)v_1/2+{}_2F_0(1,1;-\alpha(d)v_1/2)}{[\alpha(d)(m+8)]^2}, \qquad (9)$$

with

$$\alpha(d) = \frac{c(d)-b(d)}{\psi(d)-h(d)} - \frac{4}{3}.$$

As a result, we find $z_{\text{pure}}^{(3)} = 2.014$ for the Ising model, while the result calculated in the three-loop approximation on the basis of an ε-expansion is $z_{\text{pure}}^{(3)} = 2.025$.

Let us compare these results with those of other studies. In the theoretical work on the dynamics of dilute magnetic materials of which we are aware,[9,12,13] the ε-expansion has been used. These results are automatically reproduced by our own results in the case $d = 4 - \varepsilon$. Unfortunately, we do not know of any papers reporting a numerical simulation of the dynamics of dilute magnetic materials. We are presently pursuing this line of research.

With regard to the values of z_{pure} for homogeneous magnetic materials, we do find enough data in the literature to make a comparison. Numerical Monte Carlo simulations have yielded $z_{\text{pure}} = 2.17 \mp 0.06$ (Ref. 14), $\simeq 2.08$ (Ref. 15), 2.11 ± 0.03 (Ref. 16), and 1.99 ± 0.03 (Ref. 17). The field-theory approach in the two-loop approximation, through an interpolation of the results of $1 + \varepsilon$ and $4 - \varepsilon$ expansions, has led to the value 2.02 (Ref. 18). We thus see that the value which we have found for the dynamic exponent, $z_{\text{pure}}(d = 3) = 2.014$, agrees with only the results of Refs. 17 and 18. Systems of very large dimensions—128^3, 256^3, and 512^3—were simulated in Ref. 17 by a block procedure. There is reason to believe that the result of Ref. 17 is the best of the results which we have cited. We have carried out a simulation of the critical dynamics of a system with dimensions of 60^3 by Monte Carlo methods with a subsequent block partitioning. We found $z = 2.07 \pm 0.05$. This result agrees well (at the lower limit) with the theoretical value and is surpassed only by the result of Ref. 17.

Summarizing this analysis of the critical dynamics of dilute magnetic materials, we can say that an impurity-related disorder causes a substantial change in the critical behavior of the three-dimensional Ising model. The behavior is characterized by values of the dynamic exponent z_{imp} which are higher than the exponent of the homogeneous model. This circumstance is reflected in the anomalously long relaxation times for the magnetization near the critical point: $\tau_{\text{rel}} \propto |T - T_c|^{-z\nu}$, where ν is the critical exponent for the correlation length. As a result, there are changes in the kinetic properties of the magnetic materials.

Of particular interest to researchers are dilute low-dimension magnetic materials which can be described by the two-dimensional Ising model. Since the exponent for the specific heat in the homogeneous model is zero, the effect of disorder caused by an impurity becomes a secondary factor. A numerical analysis of the stability of the fixed points has shown[4] that the critical dimension of the order parameter for two-dimensional systems is $m_c = 1.1937$. A new critical behavior could thus be induced by impurities only in the Ising model. However, a more detailed inspection of this case[19,20] has led to the conclusion that the impurity affects only the behavior of the specific heat, $C(T) \propto \ln|\ln|T - T_c||$; the other thermodynamic and correlation functions undergo no change in critical behavior. As a result, the critical dynamics of dilute two-dimensional Ising-like magnetic materials in the relaxation regime is the same as the dynamics of the homogeneous model. We can use expression (8) for the function γ_λ in the three-loop approximation. With $v_1^* = 2.42438$, $\psi = 1.18991$, $h = 0.11464$, $b = 0.15740$, and $c = 4.01356$ (these figures correspond to $d = 2$ and $m = 1$), we can determine the dynamic critical exponent. We thus find $z_{\text{pure}}^{(3)}(d = 2) = 2.277$.

Let us compare this result with the results of other studies.

Numerical simulations by Monte Carlo methods have yielded $z = 2.13 \pm 0.03$ (Ref. 20), 2.22 ± 0.13 (Ref. 22), $\simeq 2.23$ (Ref. 23), 2.10 ± 0.10 (Ref. 24), and 2.14 ± 0.02 (Ref. 17).

The high-temperature expansion has led to $z = 2.125 \pm 0.010$ (Ref. 25).

The field-theory approach in the two-loop approximation, with interpolation of the results of $1 + \varepsilon$ and $4 - \varepsilon$ expansions, has led to $z = 2.126$ (Ref. 18).

The renormalization-group approach in real space with a decimation procedure has yielded $z = 2.22$ (Ref. 25).

The results of Ref. 17 and those of the present paper can be regarded as the best of those which we have cited. In Ref. 17, systems of very large dimensions, up to 8192^2, were simulated. We have carried out a Monte Carlo simulation of the critical dynamics of systems with dimensions of 320^2, 360^2, and 400^2; we found the value $z = 2.13 \pm 0.05$ for the exponent. The procedure which we have used to simulate the critical dynamics and to analyze the results leads to good values of the exponents in calculations on personal computers of the IBM PC AT type.

The set of results presented here suggests that the dynamic exponent z which we have found by a field-theory approach for the two-dimensional Ising model is on the high side. The reason for this result is the weaker asymptotic convergence of the expansion series for the scaling functions in the $d = 2$ case. In order to find more reliable values for the critical exponents, it will be necessary to calculate the scaling functions in higher orders of the expansion (in the four-loop and five-loop approximations) and to use the Padé–Borel summation method.

Particularly noteworthy in the critical behavior of dilute magnetic materials is the region of high impurity concentrations, approaching the percolation threshold. It has been suggested in several places[25–27] that scaling is violated when there is an impurity concentration in the percolation region. In particular, it has been suggested that the dynamic exponent for such a concentration would be of the form $z = A \log \xi + B$, where ξ is a correlation length, and A and B are some nonuniversal coefficients. This form of the exponent z leads to an explanation of the anomalously large value which has been measured for this exponent in $Rb_2(Mg_{0.41}Co_{0.59})F_4$ during neutron scattering.[29]

There are several experimental methods for checking the theoretical prediction regarding the effect of an impurity on the dynamic critical behavior of magnetic materials [the value of $z_{\text{imp}}(d = 3)$ would be higher than $z_{\text{pure}}(d = 3)$]: inelastic neutron scattering (the linewidth would be $\omega_\varphi \propto |T - T_c|^{z\nu}$ in the $q = 0$ case and $\omega_\varphi \propto q^z$ at $T = T_c$), the ESR and NMR magnetic-resonance methods (the linewidth of the resonance would be $\Delta\omega \propto |T - T_c|^{(d-2+\eta-z)\nu}$, where η is the Fisher index); measurements of the dynamic susceptibility in the case of a high-frequency external magnetic field [$\chi(\omega) \propto \omega^{-\gamma/z\nu}$ at $T = T_c$, where γ is the susceptibility exponent]; ultrasonic experiments [the acoustic attenuation coefficient would be $\alpha\omega \propto |T - T_c|^{-(\alpha+z\nu)} w^2 g(\omega/|T - T_c|^{z\nu})$]; and acoustic dispersion [$C^2(\omega) - C^2(0) \propto |T - T_c|^{-\alpha} f(\omega/|T - T_c|^{z\nu})$].

[1]A. B. Harris, J. Phys. C 7, 1671 (1974).
[2]D. E. Khmel'nitskiĭ, Zh. Eksp. Teor. Fiz. **68**, 1960 (1975) [Sov. Phys. JETP **41**, 981 (1975)].

[3]C. Jayaprakash and H. J. Katz, Phys. Rev. B **16**, 3987 (1977).
[4]G. Jug, Phys. Rev. B **27**, 609 (1983).
[5]R. J. Birgeneau *et al.*, Phys. Rev. B **27**, 6747 (1983).
[6]T. C. Lubensky, Phys. Rev. B **11**, 3573 (1975).
[7]C. De Dominicis, Nuovo Cim. Lett. **12**, 567 (1975).
[8]E. Brezin *et al.*, Phys. Rev. D **8**, 434 (1973); Phys. Rev. D **8**, 2418 (1973).
[9]V. V. Prudnikov and I. D. Lawrie, J. Phys. C **17**, 1655 (1984).
[10]N. N. Bogolyubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Nauka, Moscow, 1980 (Wiley-Interscience, New York, 1980).
[11]G. A. Baker, B. G. Nickel, and D. I. Meiron, Phys. Rev. B **17**, 1365 (1978).
[12]U. Krey, Z. Phys. B **26**, 355 (1977).
[13]G. Grinstein, S-k. Ma, and G. F. Mazenko, Phys. Rev. B **15**, 258 (1977).
[14]C. K. Chakrabarti, H. J. Baumgaertel, and B. Stauffer, Z. Phys. B **44**, 333 (1981).
[15]H. C. Yalabik and J. D. Gunton, Phys. Rev. B **25**, 534 (1982).
[16]N. Jan, L. L. Mosely, and D. Stauffer, J. Stat. Phys. **33**, 1 (1983).
[17]C. Kalle, J. Phys. A **17**, L801 (1984).
[18]R. Bausch, V. Dohm, H. K. Janssen, and R. K. P. Zia, Phys. Rev. Lett. **47**, 1837 (1981).
[19]G. Jug, Phys. Rev. B **27**, 4518 (1983).
[20]V. S. Dotsenko and V. S. Dotsenko, J. Phys. C **15**, 495 (1982); J. Phys. C **15**, L557 (1982).
[21]J. K. Williams, J. Phys. A **18**, 49 (1985).
[22]J. Tobochnik, S. Sarker, and R. Cordery, Phys. Rev. Lett. **46**, 1417 (1981).
[23]S. L. Katz, J. D. Gunton, and C. P. Liu, Phys. Rev. B **25**, 6008 (1982).
[24]H. Takano, Progr. Theor. Phys. **68**, 493 (1982).
[25]Z. Racz and M. F. Collins, Phys. Rev. B **13**, 3074 (1976).
[26]E. J. S. Lage, J. Phys. C **19**, L91 (1986).
[27]C. K. Henley, Phys. Rev. Lett. **54**, 2030 (1985).
[28]C. K. Harris and R. B. Stinchcombe, Phys. Rev. Lett. **56**, 869 (1986).
[29]G. Aeppli, H. Guggenheim, and Y. J. Uemura, Phys. Rev. Lett. **52**, 942 (1984).

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