Phase transitions with a change of symmetry of the mixed state in superconductors with anisotropic pairing

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A symmetry analysis of the normal-metal-superconductor transition in a magnetic field is developed by means of the Landau approach. All possible irreducible projective representations for this transition are found. The symmetry of the vortex lattice near H_{c2} is discussed. For superconductors with anisotropic pairing a situation in which the critical fields of two irreducible representations have close values is characteristic. In this case, within the mixed state, a phase transition from a one-quantum hexagonal lattice of vortices to a lattice that has either a doubled or tripled number of flux quanta per unit cell or a reduced number of rotational elements should arise. This theory offers the possibility of explaining the observed phase diagram of UPt_3.

1. INTRODUCTION

The first data (published a few years ago[1,2]) giving evidence of a phase transition in superconducting UPt_3 at $H_{cl} < H^* H_{c2}$ stimulated a flood of investigations, which, by now, have provided a reasonably complete study of the experimental picture of this phenomenon (see Refs. 3–5 and the literature cited therein). According to these papers, the phase diagram of the mixed state of UPt_3 in the (H,T) plane contains at least three different superconducting phases (Fig. 1a). Besides the nonexponential temperature dependence of the thermodynamic quantities at $T\to 0$, this is one of the basic proofs that strongly correlated spin states in compounds with heavy fermions can lead to mechanisms and types of electron pairing that differ from those proposed in the BCS theory.

From a symmetry point of view, we must associate with the phase transition to the superconducting state an order parameter $\Delta_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2) = \langle\hat{\Psi}_\alpha(\mathbf{r}_1)\hat{\Psi}_\beta(\mathbf{r}_2)\rangle$, or $\Delta_{\alpha\beta}(\mathbf{k},\mathbf{r})$ if we take the Fourier transform with respect to $\mathbf{r}_1 - \mathbf{r}_2$ and go over to $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$. We speak of anisotropic pairing of electrons whenever, in the homogeneous state, $\Delta_{\alpha\beta}(\mathbf{k})$ possesses symmetry lower than $G\times R$, where G is the point group of the crystal and R is time reversal (strong spin-orbit interaction is assumed). Near T_c, the order parameter transforms according to one of the irreducible representations of the group G (Ref. 6). Therefore, in most of the theoretical papers in which the phase diagram of UPt_3 has been interpreted[7–9] the authors have attempted to associate with the split superconducting-transition temperature one of the irreducible degenerate representations of the group D_{6h} (the point group of UPt_3) that is split because this symmetry is weakly broken. While it explains the transition in zero field, this approach fails to elucidate the nature of the phase transition in the mixed state and the accompanying change of symmetry of the vortex lattice.

In this paper we develop a different method, which makes it possible to determine immediately the symmetry of the nonuniform superconducting order parameter $\Delta_{\alpha\beta}(\mathbf{k},\mathbf{r})$. In Sec. 2 we shall give a symmetry analysis of the superconducting transition in a magnetic field, by generalizing the results of Ref. 10. In Sec. 3 we discuss the symmetry of the superconducting state near H_{c2}, and in Sec. 4 the possible ways in which this symmetry can be broken are determined. These scenarios for transitions at $H_{c1} < H^* < H_{c2}$ are illustrated in Sec. 5 for the example of a two-component Ginzburg-Landau functional.

2. IRREDUCIBLE REPRESENTATIONS FOR NUCLEI OF THE SUPERCONDUCTING PHASE AT $H=H_{c2}$

The classic problem of the theory of type-II superconductivity is the problem of the phase transition in a magnetic field from the normal state of a metal to the superconducting state.[11,12] There are two basic ways of investigating this problem—phenomenological (on the basis of a Ginzburg–Landau functional), and microscopic (on the basis of the Gor'kov equations). In both cases the procedure for calculating $H_{c2}(T)$ reduces to the determination of the eigenvalues and eigenfunctions of a certain linear operator $\hat{\mathfrak{L}}$ that possesses a certain symmetry in $\mathbf{k}$-space and in $\mathbf{r}$-space.

On the other hand, this transition, which is a second-order phase transition, can be considered from a purely symmetry point of view in the framework of the Landau classification of phase transitions. For this it is necessary to know the complete symmetry group of the system in a uniform magnetic field.

Since the sizes of the Cooper pairs that are formed in the superconducting state are large in comparison with the interatomic spacing, the system is invariant under the group of continuous translations. The crystal lattice changes only the symmetry of the directions for the interacting electrons. At first, for simplicity, we shall assume that there is also symmetry under rotations through an arbitrary angle about the

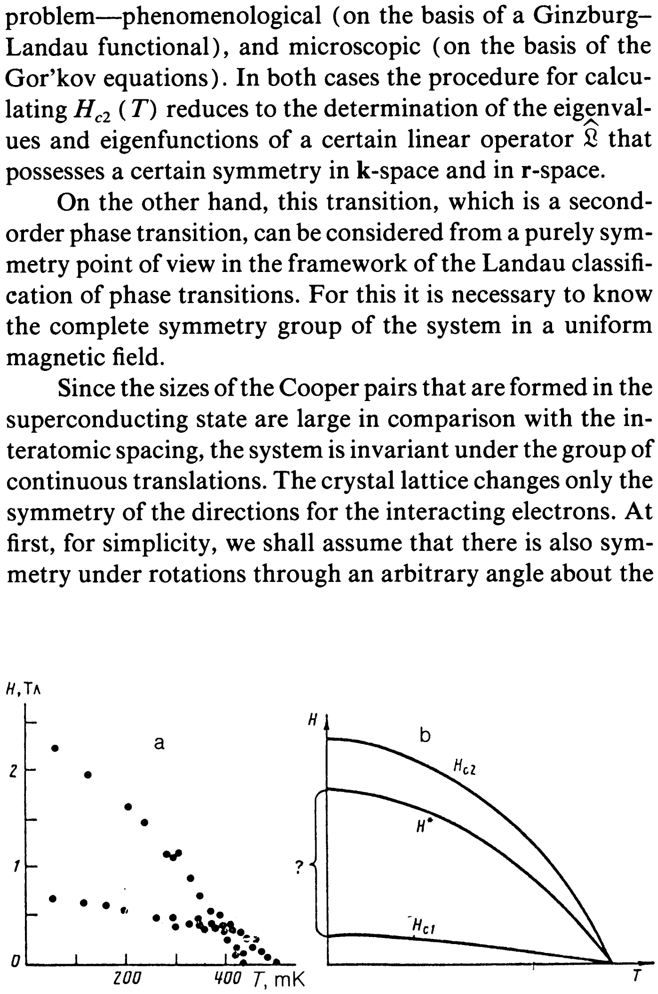

FIG. 1. (a) Phase diagram of the superconductor UPt_3 from Ref. 4 ($\mathbf{H}\perp\mathbf{c}$); (b) phase diagram of a uniaxial superconductor with anisotropic pairing in the absence of effects that split T_c.

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direction of the magnetic field **H**. Then the symmetry group of the system, and, consequently, of the operator $\hat{\mathfrak{L}}$, is the group $G_0 = T \times D_{\infty h}(C_{\infty h}) \times U(1)$, where T is the group of translations, $U(1)$ is the gauge group, and $D_{\infty h}(C_{\infty h})$ is the group containing both the subgroup of rotations about **H** with the inversion P and the combined symmetry elements RU_2 and $R\sigma_v$; here, U_2 are rotations through angle π about axes perpendicular to **H**, and σ_v are reflections in planes containing **H**.

The action of the symmetry transformations on the order parameter $\Delta(\mathbf{k},\mathbf{r})$ (we temporarily omit the spin indices) includes not only a change of the coordinates but also the gauge transformations that return our chosen vector potential $\mathbf{A}(\mathbf{r})$ to its original form. The operators of magnetic translations and magnetic rotations have the form (the axis $\mathbf{z} = \mathbf{x}_3$ is parallel to **H**):

$$T_{\mathbf{a}}=\exp\left\{-i\frac{2e}{\hbar c}\int_0^{\mathbf{x}}[A_k(\mathbf{y}+\mathbf{a})-A_k(\mathbf{y})]\,dy_k\right\}\exp(\mathbf{a}\nabla), \tag{1a}$$

$$L_\varphi=\exp\left\{-i\frac{2e}{\hbar c}\int_0^{\mathbf{x}}[s_{pk}A_p(s_{jq}y_q)-A_k(y_j)]\,dy_k\right\}\exp[i\varphi_z(l_z+l_k)], \tag{1b}$$

where

$$l_z=-i(x_1\partial_2-x_2\partial_1),\quad s_{pk}=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix},$$

the l_k are the generators of rotations in momentum space, and $\partial_i \equiv \partial/\partial x_i$. For the operators (1) the following relations are fulfilled:

$$T_{\mathbf{a}}T_{\mathbf{b}}=T_{\mathbf{a}+\mathbf{b}}\exp\left\{-i\frac{2e}{\hbar c}\int_0^{\mathbf{a}}[A_k(\mathbf{y}+\mathbf{b})-A_k(\mathbf{y})]\,dy_k\right\},$$

$$T_{\mathbf{a}}T_{\mathbf{b}}=T_{\mathbf{b}}T_{\mathbf{a}}\exp\left(i\frac{2e}{\hbar c}\mathbf{H}[\mathbf{ab}]\right), \tag{2}$$

$$L_{\varphi_1}L_{\varphi_2}=L_{\varphi_1+\varphi_2}.$$

As can be seen from (2), the operators $T_{\mathbf{a}}$ form not an ordinary vector representation but a projective representation of the group of commutative translations. For this group there exist many different projective representations. They arise, for example, with different choices of gauge. However, we shall be interested in the decomposition of only one projective representation (corresponding to the given vector potential) into irreducible representations.

For this, by considering the limits

$$T_{\mathbf{a}}\to 1+ia_1t_1+ia_2t_2 \quad \text{as} \quad \mathbf{a}\to 0,$$
$$L_\varphi\to 1+i\varphi L_z \quad \text{as} \quad \varphi\to 0,$$

we introduce the analog of "the generators of infinitesimal transformations" for the projective representations:

$$t_1=-i\partial_1-\frac{2e}{\hbar c}\int_0^{\mathbf{x}}\partial_1A_k\,dy_k\equiv-i\partial_1-\frac{2e}{\hbar c}[A_1(\mathbf{x})+Hx_2],$$

$$t_2=-i\partial_2-\frac{2e}{\hbar c}\int_0^{\mathbf{x}}\partial_2A_k\,dy_k\equiv-i\partial_2-\frac{2e}{\hbar c}[A_2(\mathbf{x})-Hx_1], \tag{3}$$

$$L_z=l_k+l_z-\frac{2e}{\hbar c}\int_0^{\mathbf{x}}(y_1\partial_2-y_2\partial_1)A_k\,dy_k-\frac{2e}{\hbar c}\int_0^{\mathbf{x}}(A_2\,dy_1-A_1\,dy_2)$$
$$\equiv l_k+l_z-\frac{2e}{\hbar c}\left[x_1A_2-x_2A_1-\frac{H}{2}(x_1^2+x_2^2)\right],$$

$$[t_1,t_2]=-\frac{2e}{\hbar c}Hi,\quad [t_1,L_z]=-it_2,\quad [t_2,L_z]=it_1.$$

By virtue of (3), the operators t_1, t_2, and L_z do not form an algebra. Nevertheless, we can use these operators to introduce the analog of the Casimir operator, viz., an operator $\hat{\mathfrak{J}}$ that commutes with the operators (2) and with the operator $\hat{\mathfrak{L}}$ and makes it possible to enumerate all the irreducible representations within a given projective representation:

$$\hat{\mathfrak{J}}=\frac{\hbar c}{4|e|H}(t_1^2+t_2^2)+L_z-\frac{1}{2}. \tag{4}$$

A proof of the commutation relations for $\hat{\mathfrak{J}}$ is given in the Appendix. Defining the operators

$$a^{(1)}=t_1+\frac{2e}{\hbar c}Hx_2,\quad a=\left(\frac{\hbar c}{4|e|H}\right)^{1/2}(a^{(1)}-ia^{(2)}),$$

$$a^{(2)}=t_2-\frac{2e}{\hbar c}Hx_1,\quad [a,a^+]=1,\quad [a^+,L_z]=-a^+,\quad [a,L_z]=a,$$

we obtain expressions for $\hat{\mathfrak{J}}$ in terms of creation and annihilation operators:

$$\hat{\mathfrak{J}}=a^+a+l_k.$$

The eigenfunctions of the operator $\hat{\mathfrak{J}}$ (combinations of the wave functions of the infinitely degenerate Landau levels and the eigenfunctions of the internal angular momentum l_k) are characterized by two indices n and m:

$$\Delta_{n,m}(\mathbf{k},\mathbf{r})=f_n(\mathbf{r})\Psi_m(\mathbf{k})\qquad \mathfrak{J}=n+m. \tag{5}$$

Functions with different n or m form linear subspaces on which the action of the magnetic operators is irreducible. But on each of the subspaces constructed on the eigenfunctions $\Delta_{n,m}$ with $n+m=N$ one and the same irreducible representation of the operators (2) is realized. In addition, the discrete symmetry of the system makes it possible to introduce one further quantum number for the functions (5)—parity under reflection in the plane perpendicular to **H**. As a result, in a superconducting transition on the irreducible representation with $\hat{\mathfrak{J}} = N$ the linear operator $\hat{\mathfrak{L}}$ should contain symmetry-allowed terms that mix the specified $\Delta_{n,m}$ with the same parity. Thus, the eigenfunctions of $\hat{\mathfrak{L}}$ are

$$\Delta_N^{\pm}(\mathbf{k},\mathbf{r})=\sum_{n+m=N}C_nf_n(\mathbf{r})\Psi_m^{\pm}(\mathbf{k}) \tag{6}$$

with certain constants $C_n(H,T)$ that now depend on the concrete construction of the operator $\hat{\mathfrak{L}}$.

As regards the spin state of the superconducting nuclei, in contrast to the case $H=0$ (for which Δ does not depend on **r**), when, by means of inversion, it is possible to separate the singlet and triplet states by symmetry,[6] when order parameter is nonuniform this is no longer possible. The action of inversion on the variables **r** mixes the basis functions on each Landau level. Therefore, the functions (5) do not possess any parity under inversion. This leads to the result that, at H_{c2}, simultaneously with the singlet states those triplet states on which the same projective representation of the group G_0 is realized should arise (strong spin-orbit interaction is assumed). Confining ourselves to the s, p, and d harmonics, we write out all the possible representations:

$$\overset{+}{\Delta}_{\alpha\beta N}(\mathbf{k},\mathbf{r})=i\sigma_y\{[C_1+C_2(2k_z^2-k_x^2-k_y^2)]f_N(\mathbf{r}) +C_3(k_x-ik_y)^2f_{N+2}(\mathbf{r}) +C_4(k_x+ik_y)^2f_{N-2}(\mathbf{r})+\sigma_z[C_5(k_x-ik_y)f_{N+1}(\mathbf{r}) +C_6(k_x+ik_y)f_{N-1}(\mathbf{r})] +C_7(\sigma_x-i\sigma_y)k_zf_{N+1}(\mathbf{r})+C_8(\sigma_x+i\sigma_y)k_zf_{N-1}(\mathbf{r})\}, \tag{7}$$

$$\overset{-}{\Delta}_{\alpha\beta N}(\mathbf{k},\mathbf{r})=i\sigma_y\{D_1(k_x-ik_y)k_zf_{N+1}(\mathbf{r})+D_2(k_x+ik_y)k_zf_{N-1}(\mathbf{r}) +f_N(\mathbf{r})[D_3\sigma_zk_z+D_4(\sigma_x-i\sigma_y)(k_x+ik_y)+D_5(\sigma_x+i\sigma_y)(k_x-ik_y)] +D_6(\sigma_x-i\sigma_y)(k_x-ik_y)f_{N+2}(\mathbf{r})+D_7(\sigma_x+i\sigma_y)(k_x+ik_y)f_{N-2}(\mathbf{r})\},$$

where σ_x, σ_y, and σ_z are the Pauli matrices.

Near T_c, where Ginzburg–Landau theory is applicable, the prinicipal role in (7) is played by just one (e.g., singlet) representation of the group G. All the other harmonics, including the triplet states, are small in comparison with the principal representation by virtue of the closeness to the critical temperature. However, moving along the line $H_{c2}(T)$, near $T=0$ we can obtain an important contribution from p harmonics. Thus, the problem of the paramagnetic limit for superconductivity appears in a form somewhat different from that customarily assumed.[1)] The paramagnetic limit is associated not with the singlet type of electron pairing that occurs for $H=0$ but with the distinctive features of the spin-orbit interaction for the given compound, which suppress the appearance of the triplet states at $H=H_{c2}$.

The reason for this could be, for example, the complete absence of spin-orbit effects. A more nontrivial situation arises in the case when the spin-orbit interaction is small in the basal plane of a uniaxial crystal, i.e., rotations about the **z** axis in spin space and coordinate space are performed independently, while the rotations U_2 act on all variables. In this case, in Eqs. (7) only spinors of the form $i\sigma_y\sigma_z$ are mixed into the singlet states. Such triplet states correspond to pairing of electrons with opposite spins (the spins are directed along **z**) and do not give a contribution to the spin susceptibility; this leads to the existence of a paramagnetic limit for them. For $\mathbf{H}\perp\mathbf{z}$ these same states correspond to pairs with a parallel orientation of spin along the field, and thus do not have a paramagnetic limit for this direction of the field.

One further type of discrete transformation (RU_2) leads to the result that in the representations (6) and (7) all the constants C_n and D_n are real.

For a rotation axis of finite order p, one operator that commutes with $\hat{\mathfrak{L}}$ is $\exp[(2\pi i/p)\hat{\mathfrak{J}}]$, and the classification of the superconducting nuclei $\Delta_N(\mathbf{k},\mathbf{r})$ differs from the previous one only in that it is necessary to take N modulo p (Ref. 10).

Up to now we have considered translations parallel to the magnetic field in the direction of the **z** axis. In taking these into account, we should introduce one further Casimir operator, $\hat{\mathfrak{J}}'=t_3^2=[-i\partial_3-(2e/\hbar c)A_3(\mathbf{x})]^2$, in order to complete the classification of the irreducible representations of the group G_0. The eigenfunctions of this Casimir operator for $\mathfrak{J}'\neq 0$ are characterized not by the parity $\pm$ but by another quantum number p_z instead ($\mathfrak{J}'=p_z^2$); a σ_h reflection carries functions with p_z and $-p_z$ into each other. The necessity of discarding such irreducible representations is obvious for a one-component Ginzburg–Landau functional. In this case, we can solve the corresponding linear problem for $p_z\neq 0$ and convince ourselves that it only decreases the value of H_{c2}. Even for just a two-component Ginzburg–Landau functional the analogous assertion is not so obvious.[13,14] For these irreducible representations, however, the Lifshitz criterion (absence of invariants linear in the gradients) is violated: An antisymmetric combination of functions with p_z and $-p_z$ transforms in the same way as ∂_z. Consequently, superconductivity cannot arise continuously on representations with $\hat{\mathfrak{J}}\neq 0$.

The intersecting lines $H_{c2}(T)$ on the phase diagram of Fig. 1a should correspond to different irreducible representations, characterized[10] either by different values of N or by different signs ($+$ or $-$). In addition, they can correspond, in general, to different projective representations, this being even more probable in view of the "isotropy" of the diagram for different directions of **H**. For a given $\mathbf{A}(\mathbf{r})$, different values of the charge e in (1) lead to different projective representations. And values of e that differ from the electron charge arise, e.g., for a superconducting glass. In this case the charge is equal to twice the electron charge.[9]

3. VORTEX LATTICES NEAR H_{c2}

In the nonlinear region below H_{c2} the symmetry G_0 of the normal state is only partly preserved. The possible symmetry groups of the superconducting state can be found by enumerating all the subgroups of the group G_0.

Near H_{c2} that symmetry is selected which corresponds to the minimum of the fourth-order terms in the Landau functional for the representation under consideration. But, because the representations of interest to us are infinite-dimensional and admit an infinite number of different fourth-degree invariants, this problem in its general form appears to be insoluble.

We shall assume that in the superconducting transition the symmetry of the normal state is broken not completely but only to a discrete subgroup. By choosing translations through **a** and **b** as its generators, we formulate the condition of periodicity of $\Delta(\mathbf{k},\mathbf{r})$ under $T_\mathbf{a}$ and $T_\mathbf{b}$:

$$T_\mathbf{a}\Delta(\mathbf{k},\mathbf{r})=\exp(i\varphi_1)\Delta(\mathbf{k},\mathbf{r}),\quad T_\mathbf{b}\Delta(\mathbf{k},\mathbf{r})=\exp(i\varphi_2)\Delta(\mathbf{k},\mathbf{r}). \tag{8}$$

It follows from this that magnetic translations for this subgroup should commute. Thus, any translationally symmetric solution possesses an integer number of flux quanta per unit cell.[15]

Suppose that distances are measured in units of the magnetic length $l_H^2=\hbar c/|e|H$, and the vector potential is given in the Landau gauge. Then the wave function of the nth Landau level,

$$\psi_n(\mathbf{r},\tau)=\sum_m \exp\left\{-\pi i\rho m^2+\pi im(\rho+1)+\frac{2\pi i}{a}\left(m-\frac{1}{2}\right)x -\left[y-\left(m-\frac{1}{2}\right)b\sin\alpha\right]^2\right\}H_n\left[y-\left(m-\frac{1}{2}\right)b\sin\alpha\right], \tag{9}$$

satisfies the condition (8) with $\mathbf{a}=(a,0)$, $\mathbf{b}=(b\cos\alpha, b\sin\alpha)$ and $\varphi_1=\pi$, $\varphi_2=\pi(\rho+1)$, possessing one quantum of flux per cell. We have introduced the notation

$\rho = (b/a)\cos\alpha, \sigma = (b/a)\sin\alpha, \tau = \rho + i\sigma$. The functions (9) can be rewritten in the form

$$\psi_0 = \theta_{11}(x - iy, \tau)\exp(-y^2), \quad \psi_n \sim (a^+)^n \psi_0,$$

where $\theta_{11}(z,\tau)$ is a Jacobi theta function. In mathematics it is well known that the theta function is the only entire function with given periods **a**, **b** and phases φ_1, φ_2. From this follows the uniqueness of $\psi_n(\mathbf{r},\tau)$, which, of course, does not depend on the choice of gauge. Functions with different φ_1 and φ_2 go over into each other under the action of translations that do not appear in the symmetry group of the order parameter, and are therefore equivalent. Substituting (9) into (6), we obtain all possible one-quantum vortex lattices. We shall be interested primarily in the solutions (see Sec. 5)

$$\Delta_{-1}(\mathbf{k},\mathbf{r}) = \psi_0(\mathbf{r})(k_x - ik_y)k_z, \qquad (10a)$$

$$\Delta_{+1}(\mathbf{k},\mathbf{r}) = \psi_2(\mathbf{r})(k_x - ik_y)k_z + \psi_0(\mathbf{r})(k_x + ik_y)k_z, \qquad (10b)$$

although all the following discussions are independent of the concrete form of $\Delta(\mathbf{k},\mathbf{r})$.

Originally, the form of the vortex lattice was chosen by direct numerical comparison of the energies of the different configurations in the framework of the Ginzburg-Landau model, using $\psi_0(\mathbf{r},\tau)$ (Ref. 11). In the more complete analysis in Ref. 12 it is proposed that one consider the energy parameter $\gamma = \langle|\psi_0|^4\rangle/\langle|\psi_0|^2\rangle^2$ of the same model as a function of ρ and σ, thereby establishing a number of its symmetry properties. We shall attempt to generalize this approach. This is necessary, since the parameter γ for exotic superconductors is not expressed explicitly in terms of $\Delta(\mathbf{k},\mathbf{r})$, and the symmetry properties for it do not follow so obviously from its analytical form.[14]

We imagine a lattice constructed on the vectors **a** and **b**. The choice of basis for the lattice is not unique. Instead of **a** and **b**, the basis vectors can be any integer combinations of them ($k\mathbf{a} + l\mathbf{b}$ and $m\mathbf{a} + n\mathbf{b}$) such that $kn - lm = \pm 1$. The matrices $\binom{k\ l}{m\ n}$, which do not change the orientation of the basis, form the group $SL(2,\mathbb{Z})$. In combination with the replacement $\mathbf{a} \to -\mathbf{a}$, $\mathbf{b} \to -\mathbf{b}$ they form all possible parametrizations of the original lattice. In a physical problem, various "symmetrical objects" can form the lattice. The state of the system in this case is specified not only by the lattice but also by the additional parameters describing the orientation of the "object" in relation to the basis vectors. Therefore, in the space of the parameters a transformation that consists only of a different choice of the basis vectors for the lattice changes the initial state of the system. Thus, we are dealing with the presence or absence of a new symmetry, different from translations and rotations—symmetry under the action of the group $SL(2,\mathbb{Z}) \times \mathbb{Z}_2$ in the space of the parameters of the system. The functions $\psi_n(\mathbf{r},\tau)$ are uniquely determined by their lattice, and, consequently, they are obliged to be symmetric under $SL(2,\mathbb{Z}) \times \mathbb{Z}_2$. We shall verify this.

In complex coordinates on a plane the two new basis vectors $e_l' = (k + l\tau)a$ and $e_2' = (m + n\tau)a$ define a new lattice parameter $\tau' = e_2'/e_l' = (n\tau + m)/(l\tau + k)$, with $a' = |e_1'| = (\pi/\sigma')^{1/2}$ and $b' = |e_2'| = a'\sigma'$. The function $\psi_n(\mathbf{r},\tau')$ is periodic under translations lying at the sites of the lattice constructed on the vectors a' and $b'\tau'$. In order to obtain a function that is periodic with respect to the initial lattice, it is necessary to rotate the system, carrying a' into e_1'. Thus, for an arbitrary element $\hat{M} \in SL(2,\mathbb{Z})$ we have

$$\hat{M}\psi_n(\mathbf{r},\tau) = L_\varphi \psi_n\left(\mathbf{r}, \frac{n\tau + m}{l\tau + k}\right), \qquad (11)$$

where the angle φ satisfies

$$\exp(i\varphi) = \left(\frac{l\tau^* + k}{l\tau + k}\right)^{1/2}.$$

As a consequence of the $SL(2,\mathbb{Z})$ symmetry, the transformed function should either coincide with the original function or go over into it under translations, rotations, and gauge transformations. We shall establish this only for three elements of the group $SL(2,\mathbb{Z}) \times \mathbb{Z}_2$, combining which we can obtain the whole group. From (9) it follows that

$$\psi_n(\mathbf{r},\tau + 1) = \psi_n(\mathbf{r},\tau), \qquad (12a)$$

and with the formula of Poisson sums,

$$\psi_n(\mathbf{r},\tau) = \left(\frac{\rho + i\sigma}{\rho - i\sigma}\right)^{n/2} (\rho - i\sigma)^{-1/2} \times \exp\left[\frac{\pi i}{4}\left(\rho - 3 + \frac{\rho}{\rho^2 + \sigma^2}\right)\right] L_\varphi \psi_n(\mathbf{r}, -1/\tau), \qquad (12b)$$

where

$$\exp(i\varphi) = \left(\frac{\rho - i\sigma}{\rho + i\sigma}\right)^{1/2}.$$

Since rotations of the **k** and **r** spaces must be performed simultaneously, the same symmetry is possessed by the one-quantum lattice $\Delta(\mathbf{k},\mathbf{r})$ for any representation from Sec. 2:

$$\Delta_N\left(\mathbf{k},\mathbf{r},-\frac{1}{\tau}\right) = \exp(iN\varphi)(\rho - i\sigma)^{1/2} \times \exp\left[\frac{\pi i}{4}\left(\rho - 3 - \frac{\rho}{\rho^2 + \sigma^2}\right)\right] L_{-\varphi}\Delta_N(\mathbf{k},\mathbf{r},\tau).$$

In addition, the replacement of the lattice base orientation $\rho \to -\rho$ ($\tau \to -\tau^*$) gives:

$$\psi_n(\mathbf{r}, -\tau^*) = RU_{2y}\psi_n(\mathbf{r},\tau). \qquad (12c)$$

By substituting such an order parameter into an arbitrary functional that is invariant under translations, rotations, and gauge transformations, we obtain the energy of the system as a function of τ—an energy that does not change under transformations from the group $SL(2,\mathbb{Z}) \times \mathbb{Z}_2$. In this case, we need not consider the energy in the entire complex plane of the parameter τ, but can confine ourselves to the fundamental region in Fig. 2. The point Q in the figure

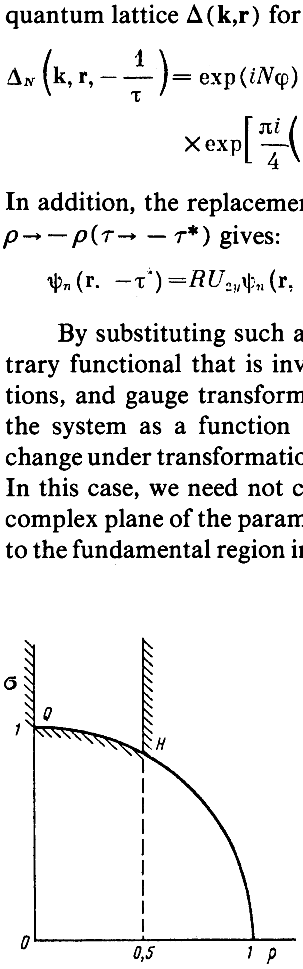

FIG. 2. Fundamental region for the energy of a one-quantum vortex lattice in the plane of the complex parameter $\tau = \rho + i\sigma$.

corresponds to a square lattice, and the point H to a hexagonal lattice. Near these corner points transformations from $SL(2,\mathbb{Z})$ act as rotation axes—second-order at Q, and third-order at H. This implies that the energy as a function of ρ and σ is bound to have extrema at the corner points. Whereas at Q there can be an extremum of any type, at H there can be only a maximum or a minimum. Also extremal is the point $\sigma=\infty$; the absolute maximum corresponds to it. If we assume that other extrema, not due to symmetry, are absent, we find that the absolute minimum of the nonlinear problem is bound to be situated at H.

Thus, the one-quantum hexagonal vortex lattice is extremal for any of the representations from Sec. 2—in particular, for the solutions (10a) and (10b) (see also Sec. 5.2). Therefore, the statement in a number of papers[16,17] that a hexagonal lattice of vortices is distorted in exotic superconductors is incorrect.

In addition to the translational symmetry, the lattices corresponding to a given Landau level also possess rotational symmetry. From (12b), for a hexagonal lattice with $\tau_0=\exp(\pi i/3)$, it follows that

$$L_{\pi/3}\psi_n(\mathbf{r},\tau_0)=\exp[-\pi i(1-n)/3]\psi_n(\mathbf{r},\tau_0).$$

Therefore, the point symmetry group of $\psi_n(\mathbf{r},\tau_0)$ includes the elements $\exp[\pi i(1-n)/3]L_{\pi/3}$, RU_{2y}, $(-1)^{n+1}RU_{2x}$. The presence of combined symmetry elements—products of rotations and gauge transformations—leads to the existence of zeros of the functions $\psi_n(\mathbf{r},\tau_0)$ at all the positions of such rotation axes.

Far from the line of phase transitions the order parameter no longer corresponds to some particular representation. There remains only the symmetry of the order parameter—the subgroup G_0. However, it may be asserted that the symmetry of the minimum of the nonlinear problem found above for one representation will remain a symmetry of the minimum of the general nonlinear problem in a certain neighborhood of $H_{c2}(T)$. In this region, all the other representations on which it is possible to construct a lattice with the same symmetry are mixed into the initial representation. For example, for an ordinary hexagonal Abrikosov lattice, in the case of s-pairing, starting from H_{c2} all Landau levels with n equal to a multiple of six will be mixed into $\psi_0(\mathbf{r},\tau_0)$ (this result was obtained numerically in Ref. 18). From the same arguments applied to $\Delta(\mathbf{k},\mathbf{r})$ it follows that the representations (10a) and (10b) are not mixed near H_{c2}. Since the critical fields corresponding to them can approach very close to each other (see Sec. 5.2), below H_{cz} a second-order phase transition should arise, when (10a) is mixed into (10b) or vice versa.

4. CLASSIFICATION OF THE PHASE TRANSITIONS IN A LATTICE OF SUPERCONDUCTING VORTICES

Following the conclusions of the preceding section, we assume that the superconducting nuclei that have arisen on the representation $\Delta_N^{\pm}$ form, in a certain neighborhood of H_{c2}, an Abrikosov lattice with hexagonal symmetry:

$$G_N^{\pm}=\{\exp(-i\varphi_1)T_{a_1};\ \exp(-i\varphi_2)T_{a_2};\ \exp[\pi i(1-N)/3]L_{\pi/3};\ \pm\sigma_h;\ RU_{2y}\}. \qquad (13)$$

The elements listed are the generators of the symmetry group $G_N^{\pm}$ of the lattice. Below H_{c2} the magnetic field in the sample is nonuniform. However, Eqs. (1) make it possible to introduce discrete magnetic symmetry operators in this case too; the integrals in (1) have meaning under the condition that $\operatorname{curl}\mathbf{A}(\mathbf{r}+\mathbf{a})=\operatorname{curl}\mathbf{A}(\mathbf{r})$. Thus, the symmetry analysis given below is valid in the entire range of fields $H_{c1}<H<H_{c2}$. We represent the group (13) in the form of a product of three groups:

$$G_N^{\pm}=T\times C_{6h}(E)\times\{E,RU_{2y}\}.$$

The first group is the commutative group of discrete translations. The magnetic operators of the translations (1) are unitary. The group $C_{6h}(E)$ is also a unitary group, isomorphous to the group C_{6h}. The phase factors that accompany the rotations, translations, and reflections do not change the rules for multiplication of these elements, or, correspondingly, the classification of the irreducible representations. The third group $\{E,RU_2\}$ contains an anti-unitary operator. Therefore, we first determine the irreducible representations of the group $T\times C_{6h}(E)$ and then we consider how this classification is changed by the presence of the symmetry operator RU_2.

The irreducible representations of the space group $T\times C_{6h}(E)$ are found in the same way as for a two-dimensional crystal with an isomorphous group. Since this group is symmorphous, its irreducible representations are determined by the star of a wave vector lying in the Brillouin zone (BZ) and by the label of the small representation. Following the Lifshitz criterion, we consider only the most symmetrically positioned vectors (points in the BZ) for which there are no terms linear in the gradients in the Landau expansion. In the case of a regular hexagonal BZ these points are the point Γ—the center of the BZ (the star consists of one vector, and the little group $P_\Gamma=C_{6h}$), the point M—a vertex of the hexagon (the star consists of two vectors, and $P_{\rm M}=C_{3h}$), and the point X—a midpoint of a side of the hexagon (the star consists of three vectors, and $P_{\rm X}=C_{2h}$).

We consider the irreducible representations that are realized by the functions corresponding to the center of the BZ. In transitions on such representations the volume of the unit cell does not change. The group $P_\Gamma=C_{6h}$ has twelve one-dimensional irreducible representations $\Gamma_n^{\pm}$ $(n=0,...,5)$. The label n determines the phase $\pi in/3$ that the basis functions acquire under the action of the rotational symmetry element $\exp[\pi i(1-N)/3]L_{\pi/3}$, and the sign $\pm$ denotes the parity under reflections in the plane σ_h. In the presence of a magnetic field the symmetry under time reversal is violated, and there are no requirements that basis functions of an irreducible representation be real. In fact, the operator RU_2 carries complex basic functions of these one-dimensional representations over into themselves.

The classification of the irreducible representations has a different distinctive feature, associated with the presence of the operator RU_2. The gauge symmetry of the system is already broken after the first transition from normal metal to superconductor. The gauge transformations do not appear separately in the group $G_N^{\pm}$, but appear only in combinations with translations and rotations. Therefore, the requirement that arises is not that the basis functions be real, but that the coefficients multiplying them be real (or imaginary). This is so, of course, only in the case when the basis functions do not acquire complex factors under rotations or translations. It follows from this that the representations with $n = 0$ and $n = 3$ should be classified by one further parity—their parity under the action of the operator RU_2.

Thus, for the center of the BZ there are eight one-dimensional real irreducible representations with $n = 0, 3$ (the coefficients of the expansion of the new order parameter in these basis functions are real), and eight complex one-dimensional representations with $n = 1, 2, 4, 5$. For the complex representations the phase with which new nuclei appear is determined by minimizing the fourth-degree terms in the Landau expansion, while for the real representations the phase already appears explicitly in the quadractic terms (see Sec. 5). The possible changes of symmetry in phase transitions on these representations are a decrease in the number of rotational elements and violation of the mirror symmetry and Ru_2 symmetry; the latter symmetry can disappear completely, but it can appear in combination with other elements. Phase transitions on the representations Γ_2^+ and Γ_4^+ can only be first-order, since a cubic invariant exists for them. Transitions on all the other representations are second-order.

The star of the wave vector for the point M of the Brillouin zone contains two rays, which can be expressed as follows in terms of the basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ of the reciprocal lattice: $\mathbf{k}_1 = (2\mathbf{b}_1 + \mathbf{b}_2)/3$ and $\mathbf{k}_2 = -\mathbf{k}_1$. The proper symmetry group of these vectors is isomorphous to C_{3h}. The transformation RU_2 carries the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ into themselves, but, because of the complex factors that are acquired by the functions under translations, for the basis functions it is not possible to introduce a parity under RU_2. The small representations for this star are one-dimensional and complex ($\Gamma_n^{\pm}$, where $n = 0, 1, 2$, and $\pm$ denotes the parity of the basis functions under σ_h). For the point M there are, in all, six two-dimensional complex representations. As a result of the onset of instability on any of them, the volume of the unit cell increased by a factor of three, and at the phase-transition point the system becomes periodic under a hexagonal lattice of translations through $3\mathbf{a}_1$ and $\mathbf{a}_1 + \mathbf{a}_2$, or, if we choose the cell in a more symmetrical form, through $\mathbf{a}_1 + \mathbf{a}_2$ and $2\mathbf{a}_2 - \mathbf{a}_1$. For the representations with $n = 0$ the rotational symmetry either does not change or is decreased to C_{3h}, while for $n \neq 0$ it is broken completely. Transitions on the representations Γ_n^+ are first-order, and the other transitions are second-order.

The star of the wave vector for the point X of the Brillouin zone contains three rays: $\mathbf{p}_1 = \mathbf{b}_1/2$, $\mathbf{p}_2 = \mathbf{b}_2/2$, and $\mathbf{p}_3 = -(\mathbf{b}_1 + \mathbf{b}_2)/2$. In this case the volume of the unit cell is doubled, if one basis function arises (for $\mathbf{p}_1$ the periods are $2\mathbf{a}_1$ and $\mathbf{a}_2$), or increases by a factor of four, if two or three functions arise (the periods are $2\mathbf{a}_1$ and $2\mathbf{a}_2$). The group $P_X = C_{2h}$ has representations $\Gamma_n^{\pm}$ $(n = 0,1)$. The presence of the RU_2 symmetry leads to the result that the representations with $n = 0$ should be characterized by a parity under the RU_2 transformation. For the representations with $n = 1$ the basis functions with real and imaginary phases also transform differently under RU_{2y}, although it is not possible to introduce a parity for these. The result is that for the point X there are eight three-dimensional real representations. For the representation Γ_0^{++} there exists an invariant cube (since $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$), and, therefore, the transition for it is only first-order. If in the transition one of the parities is violated, combined symmetry elements of the type $T_{\mathbf{a}_1}\sigma_h$ or $T_{\mathbf{a}_2}RU_{2y}$ arise.

5. PHASE TRANSITIONS IN A MODEL WITH A TWO-COMPONENT GINZBURG–LANDAU FUNCTIONAL

It is well known that in the case of ordinary isotropic superconductors a one-quantum hexagonal lattice of vortices is stable in the entire range of fields $H_{c1} < H < H_{c2}$. One can convince oneself of this by investigating, e.g., the standard one-component Ginzburg–Landau (GL) functional. In the determination of H_{c2} (the solution of the corresponding linear problem) a set of Landau levels arises. Amongst these, the principal role is played by the zeroth level, for which the critical field is a maximum. The next critical field, for the first Landau level, is already three times smaller. As the external field decreases no additional transitions on the other (see the end of Sec. 3) representations (Landau levels) arise, since the superconducting order parameter that has arisen on the zeroth Landau level [and has, in this range of fields, a squared magnitude of order $\sim (H_{c2} - H)/H_{c2} \sim 1$] renormalizes the critical fields for the other Landau levels, making the appearance of these impossible.

An investigation of the mixed state of superconductors with anisotropic pairing near the critical temperature can also be performed on the basis of a GL functional. The order parameter in this case transforms according to one of the irreducible representations of the point group of the crystal; these can be one- and two-dimensional for uniaxial crystals, and also three-dimensional for the cubic group. Let us consider the simplest case—the E_1 representation of the group D_6. Then the free energy has the following form:

$$F = \int dV[(T-T_c)\eta_i^*\eta_i+\beta_1(\eta_i^*\eta_i)^2+\beta_2|\eta_i\eta_i|^2+K_1p_i^*\eta_j^*p_i\eta_j$$
$$+K_2p_i^*\eta_i^*p_j\eta_j+K_3p_i^*\eta_j^*p_j\eta_i+K_4p_z^*\eta_i^*p_z\eta_i].$$
$$p_i = -\frac{i}{\varkappa}\partial_i+A_i, \quad i=x,y, \tag{14}$$

where the η_i are the coefficients of the expansion of $\Delta(\mathbf{k})$ in basis functions that transform under rotations as a two-dimensional vector, and $\varkappa$ is the GL parameter. The solutions of the linear problem for this functional were found for $\mathbf{H}\perp\mathbf{c}$ in Ref. 13 and for $\mathbf{H}\|\mathbf{c}$ in Refs. 16 and 17. They possess the following distinctive feature: For each of the directions of $\mathbf{H}$ two solutions have critical fields, which, for a sufficiently general choice of the constants in the GL functional, lie close to each other. In this case, below H_{c2} one further transition, corresponding to the second solution, arises. Because of the proximity to H_{c2}, this transition cannot be suppressed by the already existing superconductivity. The possible types of symmetry breaking were found above in Sec. 4. We now consider this phenomenon in more detail.

5.1. H⊥c

Let $\mathbf{H}\|\mathbf{x}$; then the GL equations for the energy (14) must be solved by setting $p_x\eta_i = 0$. The solutions corresponding to the maximum critical field are (as an example, we use d-pairing)[13]

$$\Delta^-(\mathbf{k},\mathbf{r})=k_xk_z\eta_x(\mathbf{r}), \quad [(K_1/K_4)^{1/2}p_y-ip_z]\eta_x=0 \tag{15a}$$

for

$$K_{23}>0,\ H_{c2}=H_1=\varkappa(T_c-T)(K_1K_4)^{-1/2}$$

and

$$\Delta^+(\mathbf{k},\mathbf{r}) = k_y k_z \eta_y(\mathbf{r}), \quad [(K_{123}/K_4)^{1/2} p_y - i p_z]\eta_y = 0 \qquad (15b)$$

for

$$K_{23} < 0, \quad H_{c2} = H_2 = \varkappa (T_c - T)(K_{123}K_4)^{-1/2},$$

where $K_{i...j} = K_i + ... + K_j$.

According to the classification in Sec. 2, for the case of a magnetic field oriented along a second-order axis the irreducible representations of the superconducting nuclei can be distinguished either by their parity under reflection in the σ_x plane or by the label N, which takes the values 0 and 1. The solutions found differ in their parity. In addition, their position dependence is described by the deformed zeroth Landau levels that arise because of the special form of the gradient terms in the GL expansion. They are a superposition of undeformed Landau levels with $n = 0,2,...$. Therefore, when the dependence on $\mathbf{k}$ is taken into account, the solutions (15a) and (15b) correspond to $N = 0$ and $N = 1$. Since the lattices of nuclei for Δ^+ and Δ^- possess different parity under σ_x and different rotational symmetry, near H_{c2} they do not mix.

The magnetic field of the superconducting currents, which is found from Eqs. (15), is directed along the $\mathbf{x}$ axis and has the form

$$h_{s1} = -\frac{(K_1K_4)^{1/2}}{2\varkappa^2}|\eta_x|^2, \quad h_{s2} = -\frac{(K_{123}K_4)^{1/2}}{2\varkappa^2}|\eta_y|^2.$$

Taking it into account in the terms of second order in $|\eta|$ and neglecting it in the fourth-order terms ($\varkappa \gg 1$), we obtain the free energy in the form

$$F = \frac{(K_1K_4)^{1/2}}{\varkappa}(H - H_1)\langle|\eta_x|^2\rangle|\lambda_1|^2 + \frac{(K_{123}K_4)^{1/2}}{\varkappa}(H - H_2)\langle|\eta_y|^2\rangle|\lambda_2|^2 + (\beta_1+\beta_2)(\langle|\eta_x|^4\rangle|\lambda_1|^4 + \langle|\eta_y|^4\rangle|\lambda_2|^4) + 2\beta_1\langle|\eta_x|^2|\eta_y|^2\rangle|\lambda_1|^2|\lambda_2|^2 + \beta_2\langle(\eta_x^{*2}\eta_y^2\lambda_1^{*2}\lambda_2^2 + \eta_y^{*2}\eta_x^2\lambda_2^{*2}\lambda_1^2)\rangle,$$

$$\Delta(\mathbf{k},\mathbf{r}) = \lambda_1\eta_x(\mathbf{r})k_xk_z + \lambda_2\eta_y(\mathbf{r})k_yk_z,$$

where $\langle...\rangle$ denotes averaging over the volume. This dependence on H of the terms quadratic in λ_1 and λ_2 could have been predicted, in the Landau theory of phase transitions, beforehand. However, their coefficients, and also the form of the coupling of the representations in the fourth-order terms, depend entirely on the GL expansion used. We note that, even with a split critical temperature in (14), we would have obtained an energy of the same form near H_{c2}.

Assuming for definiteness that $K_{23} > 0$, we obtain the following sequence of phase transitions:

$$H > H_{c2} = H_1: \quad \Delta(\mathbf{k},\mathbf{r}) = 0, \qquad (16)$$

$$H^* < H < H_1: \quad \Delta^-(\mathbf{k},\mathbf{r}) = \lambda_1\eta_x(\mathbf{r})k_xk_z,$$

$$F_1 = -\frac{K_1K_4}{4\varkappa^2(\beta_1+\beta_2)\gamma_1}(H_1 - H)^2,$$

$$H < H^*: \quad \Delta(\mathbf{k},\mathbf{r}) = \lambda_1\eta_xk_xk_z + \lambda_2\eta_yk_yk_z,$$

$$F_2 = F_1 - \frac{K_4(H^* - H)^2}{4\varkappa^2(\beta_1+\beta_2)\gamma_1}\frac{(\gamma_1K_{123}^{1/2} - \delta K_1^{1/2})^2}{\gamma_1\gamma_2 - \delta^2}.$$

where

$$H^* = H_1 - \frac{(H_1 - H_2)K_{123}^{1/2}\gamma_1}{\gamma_1K_{123}^{1/2} - \delta K_1^{1/2}}, \quad \gamma_1 = \frac{\langle|\eta_x|^4\rangle}{\langle|\eta_x|^2\rangle^2},$$

$$\delta = \frac{\langle 2\beta_1|\eta_x|^2|\eta_y|^2 + \beta_2(\eta_x^{*2}\eta_y^2 + \eta_y^{*2}\eta_x^2)\rangle}{2(\beta_1+\beta_2)\langle|\eta_x|^2\rangle\langle|\eta_y|^2\rangle}.$$

In the region $H^* < H < H_1$ a one-quantum hexagonal vortex lattice, deformed in the ratio $K_1^{1/2}:K_4^{1/2}$ along the $\mathbf{y}$ and $\mathbf{z}$ axes, is stable. In order to find the field H^* (the field of the second transition in the vortex lattice) it is necessary to minimize the parameter δ with respect to the distribution of the functions η_y. To order $(H_1 - H_2)/H_1$, which we assume to be small, in place of η_x and η_y in the expression for δ we can substitute the corresponding solutions of the linear equations (15). For the phase η_x this solution is a rhombic lattice ($A_y = -Hz$, $\alpha^2\sigma = 2\pi/\varkappa H_1$, $\sigma = (3K_4/4K_1)^{1/2}$):

$$\eta_x(\mathbf{r}) = \sum_m \exp\left[-\frac{1}{2}\pi i m^2 + \frac{2\pi i}{a}my - \frac{\varkappa H_1}{2}\left(\frac{K_1}{K_4}\right)^{1/2}(z - mb\sin\alpha)^2\right]. \qquad (17a)$$

For the phase η_y this solution is that linear combination of superconducting nuclei that transforms according to some particular irreducible representation of the symmetry group of the pre-existing lattice (see Sec. 4). To the wave function of the irreducible representation (a wave function corresponding to the wave vector $\mathbf{k}$ from the BZ) there corresponds a lattice displaced relative to the original lattice by $y_0 = k_z/\varkappa H$, $z_0 = -k_y/\varkappa H$:

$$\eta_y(\mathbf{r}) = \sum_m \exp\left[i\theta - \frac{1}{2}\pi i m^2 + \frac{2\pi i}{a}\left(m - \frac{z_0}{b\sin\alpha}\right)(y + y_0)\right] \times \exp\left[-\frac{\varkappa H_2}{2}\left(\frac{K_{123}}{K_4}\right)^{1/2}(z + z_0 - mb\sin\alpha)^2\right]. \qquad (17b)$$

The points of interest to us on the boundary of the BZ correspond to displacements lying on the boundary of the Wigner–Seitz cell.

The maximum H^* must be sought among the displacements to symmetry points: 1) the undisplaced lattice ($y_0 = z_0 = 0$; the point Γ in the BZ); 2) displacement through half of the smallest period ($y_0 = a/2$, $z_0 = 0$; the point X in the BZ); 3) displacement to an interstice ($y_0 = a/2$, $z_0 = (b\sin\alpha)/3$; the point M in the BZ). By substituting (17a) and (17b) into (16) and assuming, in the first approximation, the $K_{23}/K_1 \ll 1$, we obtain the following values of the parameter δ for these three cases (for the procedure of the calculation, see Ref. 12):

$$\delta_1 = 1{,}16(\beta_1 - |\beta_2|)/(\beta_1+\beta_2), \quad \delta_2 = (0{,}95\beta_1 - 0{,}67|\beta_2|)/(\beta_1+\beta_2), \quad \delta_3 = 0{,}92\beta_1/(\beta_1+\beta_2).$$

The average $\langle\eta_x^{*2}\eta_y^2 + \eta_y^{*2}\eta_x^2\rangle \equiv 0$ for all irreducible representations except two ($k = 0$ and $k = \pi/a$).

The condition that the phase-transition point of the transition to the superconducting state at $H = 0$ be stable imposes restrictions on the coefficients β: $\beta_1 > 0$ and $\beta_2 > -\beta_1$. In this region we have

$$\delta_3 < \delta_2 < \delta_1 \quad \text{for} \quad |\beta_2| < 0{,}04\beta_1,$$

$$\delta_2 < \delta_3,\ \delta_2 < \delta_1 \quad \text{for} \quad 0{,}04\beta_1 < |\beta_2| < 0{,}43\beta_1,$$

$$\delta_1 < \delta_2 < \delta_3 \quad \text{for} \quad |\beta_2| > 0{,}43\beta_1.$$

The transitions under consideration for $\mathbf{H}\perp\mathbf{c}$ occur with violation of the parity under reflection in the σ_x plane, and, consequently, they should all be second-order phase transitions. The change of the lattice symmetry can be found by investigating the energy below H^*. For the three cases indicated above we obtain

$$G_1=\{T_{a_1};\ T_{a_2};\ -L_\pi\sigma_x;\ -RU_{2y}(\beta_2>0),\ -RU_{2z}(\beta_2<0)\},$$

$$G_2=\{T_{a_1};\ -\sigma_x T_{a_2};\ L_\pi;\ RU_{2y}\sigma_x(\beta_2>0),\ -RU_{2z}(\beta_2<0)\},$$

$$G_3=\{T_{3a_1};\ T_{a_1+a_2};\ RU_{2z}\sigma_z \ \text{or}\ RU_{2z}\}.$$

In the weak-coupling approximation there is a definite relationship between the coefficients in (14): $\beta_2=\beta_1/2$. In addition, the comparable magnitudes of the specific-heat discontinuities in UPt_3 at $H=0$ indicate that $\beta_2\sim\beta_1$. Thus, if we have UPt_3 in mind, the most probable variants of the transition are a transition with doubling of the lattice constant and a transition with no change in the size of the cell but with violation of the parity under σ_x.

From Eqs. (16) we also obtain the temperature dependence of the transition field and the ratio of the specific heats:

$$\frac{dH^*}{dT}=\frac{dH_1}{dT}\,\frac{(\gamma_1-\delta)K_1^{1/2}}{\gamma_1 K_{123}^{1/2}-\delta K_1^{1/2}},\quad \frac{\Delta C_2}{\Delta C_1}=\frac{\gamma_1-\delta}{\gamma_1\gamma_2-\delta^2}.$$

5.2. H||c

For **H**||**z** the solutions of the linearized GL equations for the functional (14) can also be found exactly:[16,17]

$$H_{c2}=H_1=\varkappa(T-T_c)/\Lambda_1,$$

$$\Lambda_1=3(K_1+K_{123})/2-[2K_{23}^2+(4K_1+K_2+3K_3)^2/4]^{1/2}, \quad (18a)$$

if $(K_2-K_3)<K_{23}^2/(2K_1+K_{23})$, and

$$\Delta_1=\varphi_2(k_x-ik_y)k_z+\omega\varphi_0(k_x+ik_y)k_z,\quad \omega=K_{23}2^{1/2}/(\Lambda_1-K_{12}),$$

$$H_{c2}=H_2=\varkappa(T-T_c)/\Lambda_2,\quad \Lambda_2=K_1+K_3, \quad (18b)$$

if $(K_2-K_3)>K_{23}^2/(2K_1+K_{23})$, $\Delta_2=f_0(k_x-ik_y)k_z$, where φ_2, φ_0, and f_0 are normalized Landau-level functions. In the classification of Sec. 2, the solutions (18) belong to irreducible projective representations that differ in the labels $N=1$ and $N=-1$, respectively. For both solutions a hexagonal lattice of vortices is energetically favored (see below). However, as shown in Sec. 3, these hexagonal symmetries are different, and the solutions Δ_1 and Δ_2 do not mix near H_{c2}. A natural restriction on the coefficients in (14), associated with the approximate particle-hole symmetry near the Fermi surface,[9] is the condition $K_2-K_3\sim(T_c/\varepsilon_F)^2\ll 1$. Therefore, for a two-component GL functional the appearance of nuclei of the phase Δ_1 is the most probable at $H=H_{c2}$. In order to rewrite the GL functional in the form of a Landau expansion about H_{c2}, it is necessary, as in Sec. 5.1., to find the magnetic field of the superconducting currents. By making use of the results of Ref. 14 and assuming that $\varkappa\gg 1$, we arrive at a free energy of the form

$$F=\frac{H-H_1}{2\varkappa}\Lambda_1|\lambda_1|^2(1+\omega^2)\langle|\varphi_0|^2\rangle+\frac{H-H_2}{2\varkappa}\Lambda_2|\lambda_2|^2\langle|f_0|^2\rangle$$
$$+\frac{\beta_1}{4}\langle|f_0|^4\rangle|\lambda_2|^4+|\lambda_1|^4\Big\langle\frac{\beta_1}{4}(|\varphi_2|^4+\omega^4|\varphi_0|^4)$$
$$+\frac{\beta_1+2\beta_2}{2}\omega^2|\varphi_2|^2|\varphi_0|^2\Big\rangle$$
$$+|\lambda_1|^2|\lambda_2|^2\Big\langle\beta_1|f_0|^2|\varphi_2|^2+\frac{\beta_1+2\beta_2}{2}\omega^2|f_0|^2|\varphi_0|^2\Big\rangle.$$
$$\Delta=\lambda_1\Delta_1+\lambda_2\Delta_2.$$

This functional contains the following sequence of phase transitions:

$$H>H_{c2}=H_1:\quad \Delta(\mathbf{k},\mathbf{r})=0,$$

$$H^*<H<H_1:\quad \Delta(\mathbf{k},\mathbf{r})=\lambda_1\Delta_1,\qquad F_1=-\frac{(H_1-H)^2\Lambda_1^2}{4\varkappa^2\beta_1\gamma_1},$$

$$H<H^*:\quad \Delta(\mathbf{k},\mathbf{r})=\lambda_1\Delta_1+\lambda_2\Delta_2,$$

$$F_2=F_1-\frac{(H^*-H)^2}{4\varkappa^2\beta_1\gamma_1}\,\frac{(\gamma_1\Lambda_2-\delta\Lambda_1)^2}{\gamma_1\gamma_2-\delta^2},$$

where

$$H^*=H_1+\frac{(H_1-H_2)\Lambda_2\gamma_1}{\delta\Lambda_1-\Lambda_2\gamma_1},\quad \gamma_2=\frac{\langle|f_0|\rangle^4}{\langle|f_0|^2\rangle^2},$$

$$\gamma_1=\frac{\langle\beta_1(|\varphi_2|^4+\omega^4|\varphi_0|^4)+2(\beta_1+2\beta_2)\omega^2|\varphi_2|^2|\varphi_0|^2\rangle}{\beta_1(1+\omega^2)^2\langle|\varphi_0|^2\rangle^2}.$$

$$\delta=\frac{\langle 2\beta_1|f_0|^2|\varphi_2|^2+(\beta_1+2\beta_2)\omega^2|f_0|^2|\varphi_0|^2\rangle}{\beta_1(1+\omega^2)\langle|f_0|^2\rangle\langle|\varphi_0|^2\rangle}.$$

In the region $H^*<H<H_1$ the form of the lattice is found from minimization of the parameter γ_1. As has been proved, the one-quantum hexagonal lattice is an extremal for this problem. In addition, for small $K_{23}/2K_1$, or, equivalently, for large ω, we find that $\gamma_1\to\langle|\varphi_0|^4\rangle/\langle|\varphi_0|^2\rangle^2$ (the usual Abrikosov parameter), for which a hexagonal lattice is favored. The corrections, which are small in $1/\omega^2$ and have the same symmetry with respect to the parameter τ, cannot change the position of the absolute minimum of the nonlinear problem. If we take into account that $1/\omega^2\sim 0.1$ even in the weak-coupling approximation, when $K_{23}/2K_1=1$, in a real system (UPt_3) we must expect that for $H^*<H<H_1$ the lattice will have a hexagonal form that becomes distorted below H^*. In order to find this symmetry, and also the field H^*, it is necessary to minimize the parameter δ with respect to the function f_0. The desired combinations of the basis functions f_0 will again be periodic solutions, displaced to symmetric positions in relation to the original lattice (see Sec. 5.1). The calculation of the parameter δ in these three cases gives the following results:

For $\beta_2/\beta_1>-1/2-0.23/\omega^2$ the instability corresponds to the point M of the BZ (small representation Γ_1^+), and

$$\delta=\delta_3=\left(1{,}957+0{,}920\,\frac{\beta_1+2\beta_2}{\beta_1}\omega^2\right)\Big/(1+\omega^2),$$

for $-1/2-0{,}23/\omega^2>\beta_2/\beta_1>-1/2-0{,}7/\omega^2$, point X ($\Gamma_1^+$),

$$\delta=\delta_2=\left(1.963+0.946\,\frac{\beta_1+2\beta_2}{\beta_1}\omega^2\right)\Big/(1+\omega^2), \quad (19)$$

for $-1/2-0{,}7/\omega^2>\beta_2/\beta_1>-1$, point Γ (Γ_2^+),

$$\delta=\delta_1=\left(2.113+1.160\,\frac{\beta_1+2\beta_2}{\beta_1}\omega^2\right)\Big/(1+\omega^2).$$

The transition corresponding to the point X of the BZ is second-order. Minimization of the energy below H^* gives for it a doubling of the lattice constant. The points M and Γ correspond to first-order transitions, since in the expansion in powers of λ_2 cubic terms not forbidden by symmetry are present. In our expression for the energy there are no such terms, since they appear only when sixth-order invariants are included in the functional (14). The coefficient of λ_1^3 in the cubic term in this case is of order $[(H_1-H^*)/H_1]^{3/2}$ and is small in the region under consideration. Thus, we are concerned with weak first-order transitions. For such transitions the displacement of the critical field from H^* is of order $[(H_1-H^*)/H_1]^3$, the discontinuity of the order parameter is of order $[(H_1-H^*)/H_1]^{3/2}$, and the latent heat of the transition is of order $[(H_1-H^*)/H_1]^3$. In calorimetric investigations with a temperature step much greater than $[(H_1-H^*)/H_1]^3[T_c(H)-T^*(H)]$ it is not possible to detect the heat of the transition, and the transition looks like an ordinary second-order transition.

The symmetry groups of the superconducting order parameter in these three cases have the form

$$G_1=\{T_{a_1};\ T_{a_2};\ L_\pi;\ \sigma_h;\ RU_2\},$$

$$G_2=\{T_{2a_1};\ T_{a_2};\ \sigma_h;\ RU_{2y}\},$$

$$G_3=\{T_{a_1+a_2};\ T_{2a_2-a_1};\ \exp(4\pi i/3)L_{2\pi/3};\ \sigma_h;\ RU_2\}.$$

According to (19), for UPt_3 the transition should occur with a tripling of the lattice constant.

6. CONCLUSION

Thus, at least for certain symmetry directions of the magnetic field, the phase diagram of superconductor with anisotropic pairing should have the form shown in Fig. 1b. The generalization to the case of an arbitrary orientation of **H** must be performed as a supplement to this. Our theory does not impose restrictions on the existence of transitions in the region of fields $H_{c1}<H<H^*$. Comparison of the symmetries found for the Abrikosov lattices below H^* with the symmetries of the lattices of isolated vortices near H_{c1} makes it possible to reach a rigorous conclusion about the possiblity of these additional transitions.

The symmetry of the mixed state of superconductors with anisotropic pairing below H^* differs from that in ordinary superconductors by the large number of spontaneously broken symmetries. This should lead to diversity of the physical properties of the mixed state, in analogy with the diversity found in rotating superfluid ^{3}He (see Ref. 19). Possible observable effects of this kind are dipole and magnetic moments of the cores, spontaneous current along the vortex axis, etc. In experiments on neutron scattering by a vortex lattice a phase transitions with a change of the number of flux quanta per unit cell should signaled by the appearance of additional Bragg peaks, corresponding to the wave vectors in Sec. 4 and with an amplitude that increases as $[(H^*-H)/H^*]^2$ below the transition point. A transition without a change of the translational symmetry is accompanied by deformation of the vortex lattice, and this can also be observed by means of neutron diffraction.

It is of interest to compare the results of our theory with the numerical investigation of the functional (14) that was performed for the case $\mathbf{H}\|\mathbf{c}$ in Ref. 20. There is a discrepancy between these results, which is evidently due to the fact that the method applied in Ref. 20 permits one to find only the structure of a vortex for $H=H_{c1}$, leaving the structure of the lattice unelucidated.

After this article had been completed, we become aware of Ref. 21, in which the author considered transitions for the functional (14) for $\mathbf{H}\perp\mathbf{c}$. The formulas obtained in Ref. 21 are analgous to ours from Sec. 5.1, but, nevertheless, in Ref. 21 the complete change of the symmetry of the superconducting order parameter in the transition was not elucidated and an analysis of which of the different vortex structures is energetically favored when the values of the coefficients in the GL expansion change was not carried out.

In conclusion, we should like to thank G. E. Volovik, V. G. Marikhin, and V. P. Mineev for useful discussions.

APPENDIX

We shall prove (3). Let

$$f=\int_0^{\mathbf{x}}\partial_1 A_k\,dy_k-A_1(\mathbf{x}).$$

Then

$$\partial_1 f=\partial_1 A_1-\partial_1 A_1=0,\quad \partial_2 f=\partial_1 A_2-\partial_2 A_1=H,$$

and, consequently,

$$f=Hx_2.$$

The other formulas in (3) are derived analogously.

Any magnetic-translation operator can be represented, to within a numerical factor, in the form of a product of operators from two one-parameter commutative groups (translations along the axes $\mathbf{x}_1$ and $\mathbf{x}_2$): $T_\mathbf{a}\sim T_{\mathbf{a}_1}T_{\mathbf{a}_2}$.

For an arbitrary gauge the operator $T_{\mathbf{a}_1}$ ($T_{\mathbf{a}_2}$) is not an exponential function of the translation generator t_1 (t_2), but, nevertheless,

$$[T_{\mathbf{a}_1},t_1]=[T_{\mathbf{a}_2},t_2]=0.$$

By virtue of (3), $[\hat{\mathfrak{T}},t_1]=0$. We represent $T_{\mathbf{a}_1}=(T_{\mathbf{a}_1/N})^N$; then, as $N\to\infty$,

$$T_{\mathbf{a}_1/N}\to 1+i\frac{a_1}{N}t_1+\frac{a_1^2}{N^2}B,$$

$$[\hat{\mathfrak{T}},T_{\mathbf{a}_1/N}]=\frac{a_1^2}{N^2}[\hat{\mathfrak{T}},B]=\frac{a_1^2}{N^2}\hat{C},$$

$$[\hat{\mathfrak{T}},T_{\mathbf{a}_1}]=[\hat{\mathfrak{T}},(T_{\mathbf{a}_1/N})^N]=\frac{a_1^2}{N^2}\sum_{k=0}^{N-1}T^k_{\mathbf{a}_1/N}\hat{C}T^{N-k-1}_{\mathbf{a}_1/N}.$$

Since the sum, which contains N terms, increase in proportion to N, the entire right-hand side tends to zero as $N\to\infty$, and, consequently, $[\hat{\mathfrak{T}},T_{\mathbf{a}_1}]=0$. Analogously, $[\hat{\mathfrak{T}},T_{\mathbf{a}_2}]=0$, and, finally,

$$[\hat{\mathfrak{T}},T_\mathbf{a}]=0.$$

In exactly the same way, from $[\hat{\mathfrak{T}},L_z]=0$ it follows that

$$[\hat{\mathfrak{T}},L_\varphi]=0.$$

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