Extended superconformal current algebras and finite-dimensional Manin triples

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Conditions for $N = 2,4,8,...$ supersymmetric extensions of the $N = 1$ superconformal Kac–Moody algebras are investigated. A correspondence is established between $N = 2,4$ superconformal Kac–Moody algebras and finite-dimensional Manin triples.

1. INTRODUCTION

The extended superconformal field theories in two dimensions have recently attracted attention for two reasons. First, in connection with compactification of strings with space-time supersymmetry.[1–4] Second, as was noted in Ref. 5, the so-called twisted $N = 2$ minimal models introduced in Ref. 6 describe topological matter which can interact with topological gravity. In Refs. 7 and 8 Wess–Zumino–Novikov–Witten (WZNW) models were studied, which allow extended supersymmetry, and conditions were formulated that the compact Lie group must satisfy so that its WZNW model would have extended supersymmetry.

In the present note a more general formulation is given of the conditions under which the $N = 1$ Kac–Moody superalgebra admits $N = 2,4,...$ supersymmetric extensions.

2. N=1 SUPERCONFORMAL KAC–MOODY ALGEBRAS

Following Ref. 9 we construct the $N = 1$ supersymmetric extension of the Kac–Moody algebra. Let g be a finite-dimensional Lie algebra, which we take for simplicity to be a simple Lie algebra, and let B be its Killing form. We fix an orthonormal (relative to B) basis E_i, $i = 1,...,D = \dim g$. Let $J^i(z)$, $\psi^i(z)$ be the corresponding bosonic and fermionic currents with operator expansions of the form:

$$J^i(z)J^j(w) = -(z-w)^{-2}\frac{q}{2}\delta^{ij} + (z-w)^{-1}f^{ijk}J^k(w),$$
$$\psi^i(z)\psi^j(w) = (z-w)^{-1}\frac{k}{2}\delta^{ij}, \qquad (1)$$

where f^{ijk} are the structure constants of the algebra in the basis E_i, q is the level of the Kac–Moody algebra, $k = q + h_g$, h_g is the dual Coxeter number.

We define new bosonic currents:

$$I^j(z) = J^j(z) - \frac{i}{k}f^{nmj}\psi^n\psi^m(z). \qquad (2)$$

The currents $I^j(z)$ form the Kac–Moody algebra of level $k = q + h_g$. Further, the fermionic currents $\psi^j(z)$ belong to the adjoint representation of the algebra of currents $I^j(z)$. We shall refer to the current algebra with generators $I^j(z)$, $\psi^j(z)$, $j = 1,...,D$, as the current superalgebra.

From the bilinear combinations of the current I^i and ψ^i we construct the spin 2 current T and spin 3/2 current G:

$$G = \frac{2}{k}\left(\psi^i J^i - \frac{i}{3k}f^{nml}\psi^n\psi^m\psi^l\right),$$
$$T = \frac{1}{k}(J^iJ^i - \psi^i\partial\psi^i). \qquad (3)$$

The currents T and G form the $N = 1$ Virasoro superalgebra with central charge $c = D/2 + qD/(q + h_g)$. The algebra generated by the currents ψ^j, I^j, T and G is called the $N = 1$ superconformal current algebra.

3. N=2 SUPERCONFORMAL CURRENT ALGEBRA AND MANIN TRIPLES

Let G be a compact Lie group with Lie algebra g. Let us consider the corresponding $N = 1$ superconformal current algebra (SCCA). Is it possible to construct with the help of the currents J^i and ψ^i one more spin 3/2 current Q, so that the currents T, G and Q would generate the $N = 2$ Virasoro superalgebra? When such a possibility exists we shall say that the $N = 1$ SCCA admits an $N = 2$ extension. The necessary and sufficient conditions for $N = 2$ extension were formulated in Ref. 7:

Assertion 1. $N = 1$ SCCA admits $N = 2$ extension if and only if the Lie algebra g has the following properties: there exists on g an invariant nondegenerate symmetric bilinear form B and an integrable complex structure I skew-symmetric relative to B, i.e.:

a) $I^2 = 1$;

b) for arbitrary vectors x, y of the algebra g the relation $[Ix,Iy] - I[Ix,y] - I[x,Iy] = [x,y]$ should hold;

c) $B(Ix,y) + B(x,Iy) = 0$.

We now give two assertions from which follows that an arbitrary Lie algebra g of a compact group G, which satisfies the requirements of assertion 1, is a compact form of a certain complex Manin triple.

Assertion 2. Let g be a real Lie algebra, let B be a bilinear symmetric nondegenerate invariant form on g, and let the endomorphism $I: g \to g$ satisfy the conditions a), b), and c). Let g_{C} be the complexification of g and let g^+, g^- be the proper subspaces of I in g_{C}. Then (g,g^+,g^-) is a Manin triple and the automorphism of complex conjugation $\sigma: g_{\mathrm{C}} \to g_{\mathrm{C}}$ establishes the antilinear isomorphism between g^+ and g^-.

Assertion 3. Let (g,g^+,g^-) be the Manin triple of the complex Lie algebra g. Let $\sigma: g \to g$ be the antilinear involutive automorphism that maps g^+ on g^-. Then its manifold of

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fixed points g_σ forms a Lie algebra with an integrable complex structure skew-symmetric with respect to the restriction to g_σ of the bilinear form on g.

If there is given a finite-dimensional triple (g,g^+,g^-) and E^a, E_a $(a=1,\ldots,\dim g/2)$, an orthonormal basis in g such that f^{ab}_c, f^c_{ab} are the structure constants of the subalgebras g^+, g^-, then the two spin 3/2 currents

$$G^+=\alpha(S^b\psi_b+\beta f^{ab}{}_c\psi_a\psi_b\psi^c),$$
$$G^-=\alpha(S_b\psi^b-\beta f_{ab}{}^c\psi^a\psi^b\psi_c). \quad (4)$$

where α,β are normalization constants, and the spin 1 and 1/2 currents S^a, ψ^a, S_a, ψ_a corresponding to the chosen basis, generate the $N=2$ Virasoro superalgebra.

Example. $Sl(2,\mathbb{C})$. Let $n_+\oplus h\oplus n_-$ be a certain Gaussian decomposition of the $sl(2,\mathbb{C})$ algebra and b_+ and b_- its Borel subalgebras. Then a Lie algebra structure can be introduced on the space $p=b_+\oplus b_-$ such that (p,b_+,b_-) is a Manin triple. In the orthonormal basis E^0,E^1, E_0,E_1 on the space p the structure constants are as follows:

$$[E^0,E^1]=E^1,\quad [E_0,E_1]=-E_1,$$
$$[E^0,E_1]=-E_1,\quad [E^1,E_0]=-E^1,$$
$$[E^1,E_1]=E^0+E_0$$

(E^0,E^1 is the basis in b_+; E_0, E_1 is the basis in b_-). Let S^a, ψ^a, S_a, ψ_a $(a=0,1)$ be the bosonic and fermionic currents with operator expansions:

$$S^0(z)S^1(w)=(z-w)^{-1}S^1(w),$$
$$S_0(z)S_1(w)=-(z-w)^{-1}S^1(w),$$
$$S^0(z)S^0(w)=-(z-w)^{-2},\quad S_0(z)S_0(w)=-(z-w)^{-2},$$
$$S^0(z)S_1(w)=-(z-w)^{-1}S_1(w), \quad (5)$$
$$S_0(z)S^1(w)=-(z-w)^{-1}S^1(w),$$
$$S^1(z)S_1(w)=(z-w)^{-2}k+(z-w)^{-1}(S^0+S_0)(w),$$
$$S^0(z)S_0(w)=(z-w)^{-2}(k+2),$$
$$\psi^a(z)\psi_b(w)=(z-w)^{-1}\delta_b{}^a.$$

Then the currents

$$T=\frac{1}{2}(\partial\psi_a\psi^a-\psi_a\partial\psi^a)+\frac{S_aS^a+S^aS_a}{2(k+2)},$$
$$G^+=\left(\frac{2}{k+2}\right)^{1/2}(\psi^aS_a+\psi^0\psi^1\psi_1),$$
$$G^-=\left(\frac{2}{k+2}\right)^{1/2}(\psi_aS^a-\psi_0\psi_1\psi^1), \quad (6)$$
$$K=\psi_0\psi^0+\frac{k}{k+2}\psi_1\psi^1+\frac{S^0+S_0}{k+2}$$

generate the $N=2$ Virasoro superalgebra with central charge $c=3+3k/(k+2)$.

4. $N=4$ SUPERCONFORMAL CURRENT ALGEBRA

It was shown in Ref. 6 that the $N=2$ SCCA admits the $N=4$ extension if in addition to the conditions enumerated in assertion 1 the following is true: there exists on the Lie algebra one more integrable complex structure J skew-symmetric relative to the form B and anticommuting with the first structure. In that case the product of the complex structures IJ is again an integrable complex structure. Let us formulate the analogs of assertions 2 and 3.

Assertion 4. Let g be a real Lie algebra, B a bilinear symmetric nondegenerate invariant form on g, I_1, I_2 anticommuting integrable complex structures skew-symmetric relative to B. Let $g_{\mathbb{C}}$ be the complexification of g. Let g_1^+,g_1^- (g_2^+,g_2^-) be the proper subspaces of the operator $I_1(I_2)$ in $g_{\mathbb{C}}$. Then $(g_{\mathbb{C}},g_1^+,g_1^-)$ $((g_{\mathbb{C}},g_2^+,g_2^-))$ is a Manin triple, there exist on the subalgebras g_1^+,g_1^- (g_2^+,g_2^-) nondegenerate 2-cocycles, and the complex conjugation automorphism σ establishes an isomorphism between $g_1^+,(g_2^+)$ and $g_1^-,(g_2^-)$ and conjugates the 2-cocycles.

Assertion 5. Let (g,g^+,g^-) be the Manin triple of the complex Lie algebra g. There exist on the subalgebras g^+, g^- nondegenerate 2-cocycles r_+, r_-. Let σ be the antilinear involutive automorphism that maps g^+ on g^- such that $\sigma r_+\sigma^{-1}=-r_-$, then the manifold of its fixed points g_σ is a Lie algebra with two integrable anticommuting complex structures skew-symmetric relative to the restriction to g_σ of the form B.

If the Manin triple (g,g^+,g^-) is given and there exist on the subalgebras g^+, g^- nondegenerate 2-cocycles r_+, r_-, which are represented in the orthonormal basis by mutually inverse matrices, then it is possible to construct 4 spin 3/2 currents G^+, G^-, G_+, G_-, which generate the $N=4$ Virasoro superalgebra. In the orthonormal basis the currents G^+, G^- are the same as in (4) while

$$G_+=\gamma(r^{ab}\psi_aS_b+\mu f^{ab}{}_cr_{ad}r_{be}r^{cg}\psi^d\psi^e\psi_g),$$
$$G_-=\gamma(r_{ab}\psi^aS^b-\mu f_{ab}{}^cr^{ad}r^{be}r_{cg}\psi_d\psi_e\psi^g). \quad (7)$$

where r^{ab}, r_{ab} are the matrices of the cocycles: $r^{ab}r_{bc}=\delta^a_c$, and γ, μ are some renormalization constants.

5. THE POISSON BRACKETS ON LIE GROUPS AND $N=4$ SCCA

As is well known, each Manin triple is in a one-to-one correspondence with a Lie bialgebra. In turn, the tangent space of unity of the Poisson–Lie group is a Lie bialgebra. To Lie bialgebras, which arise in the case of $N=4$, there correspond Poisson brackets that somewhat generalize the brackets of the Poisson–Lie groups.

Let G be a real Lie group on which is given a certain Poisson bracket w. Then there is defined on the space of 1-forms T^*G a Lie-algebra structure.[10] Let g be the Lie algebra of the group G, g^* the space of linear functions on g, ε: $g^*\to T^*G$ the operation that associates with each vector from g^* a left-invariant 1-form. If there is given on g^* a Lie-algebra structure such that ε is a homomorphism, then we will call the Lie algebra g a bialgebra and the Poisson bracket w will be called consistent with the bialgebra g. The Poisson–Lie groups introduced in Ref. 11 arise as a special case of the situation described above, when group multiplication for the Poisson bracket consistent with the bialgebra is the Poisson operation.

Let (g,g^+,g^-) be a real Manin triple and let there exist on the subalgebras g^+, g^- nondegenerate 2-cocycles r_+, r_-

such that in the orthonormal basis on g they are represented by matrices that are each others inverses. G^+ is a connected, simply connected Lie group and g^+ is its Lie algebra. Let us fix a basis of left-invariant vector fields D_i, $i = 1,...,\dim g^+$ on G^+. Then the bivector field $r_-^{ij} D_i \wedge D_j$ defines a nondegenerate Poisson bracket on the group. It is easy to show that for any real a $w(a) = w + ar_-$ the Poisson bracket is consistent with the bialgebra g^+.

6. $N=8$ SUPERCONFORMAL CURRENT ALGEBRAS

$N = 4$ SCCA admits $N = 8$ extension if there is present on g a third, anticommuting with the first two, integrable complex structure I_3 skew-symmetric with respect to B (I_3 not being the product of the first two complex structures). But in that case it can be shown that g_C is the direct sum of Lie algebras, each of which satisfies the conditions for $N = 4$ extension.

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