Setting in of Debye screening in a Maxwell plasma

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We discuss the onset of screening of an electric charge introduced into a classical isothermal plasma. Within the framework of the formalism of a linear response and on the basis of the analytic properties of the dielectric constant, we obtain a general expression for the electric field strength in the plasma as a function of coordinates and time for an arbitrary mode of introduction of an external charge into the plasma. With numerical calculations we show that for a Maxwell plasma Debye screening sets in over a time interval equal to many periods of plasma oscillations, and the number of these periods grows as the distance from the charge increases.

1. INTRODUCTION

The solution of the problem of the time-independent potential of an electric field generated by a small point external charge at rest in a plasma is well-known.[1] The respective coordinate function is the Debye screened potential:

$$\varphi(r)=(e_1/r)\exp(-r/a). \tag{1}$$

where e_1 is the magnitude of the external charge and a the electron Debye radius defined as

$$a=(T/4\pi Ne^2)^{1/2}, \tag{2}$$

with T the plasma temperature, and N the electron number density (for the sake of simplicity the ions are assumed to be infinitely heavy). This result refers to a collisionless Maxwell plasma and is caused by the fact that the electrons in the plasma are attracted to a specified (positive, for the sake of definiteness) external charge screening the charge's Coulomb field.

This paper analyzes how such equilibrium sets in with the passage of time as an external electric charge is introduced into the plasma. The introduction process is assumed spherically symmetric and is modeled by a charge density in the form

$$\rho(\mathbf{r}, t)=e_1 f(t)\delta(\mathbf{r}), \tag{3}$$

which corresponds to a current

$$\mathbf{j}(r,t)=-\frac{1}{4\pi}\frac{df(t)}{dt}\frac{e_1}{r^2}\frac{\mathbf{r}}{r}, \tag{4}$$

where $f(t)$ is the "switch-on" function, arbitrary except for the following limits: $f(-\infty)=0$ and $f(+\infty)=1$. The current is longitudinal, and so the problem deals only with the longitudinal component of the electric induction vector.

The goal of our study was to determine the time it takes the equilibrium potential (1) to set-in in units of the period of electronic plasma oscillations, which justifies the choice of a simple spatial pattern in (3). The above mode of introduction of a charge may be impact ionization of an atom by plasma electrons.

As is clear from general physical considerations and subsequent calculations, introduction of a charge is inevitably accompanied by excitation of Langmuir waves. Long-wave modes, as is known, decay weakly; hence, the concept of the setting-in time for screening requires clarification. We will interpret it as the time of formation of a stationary charged cloud with a charge practically equal to $-e_1$ and with dimensions of approximately $10a$. The case where the charge is introduced very slowly is trivial, and the time it takes the screening to set in is defined as the time in which the switch-on function $f(t)$ changes substantially. Hence, in calculations involving a smooth switch-on function [see Eq. (28) below] we assume that the constant of its variation, λ, is equal to ω_p.

A qualitative answer to the posed problem would be obvious if the problem amounted to the response of a single linear oscillator with low damping to an external impact (sharp or smooth). In our problem, however, screening is the response not of a single oscillator but of a continuum set of oscillators with essentially different damping constants, up to $\gamma\sim\omega_p$, and it cannot be said that that any one oscillator plays the dominant role.

Such reasoning formulates the physical meaning of the problem and, as we see, the variation in time of the test charge does not contradict the Maxwell equation used in the time-independent case.

The linear response approximation is valid when the magnitude of the introduced charge is small compared to the total charge of the electrons in the plasma's Debye volume, $e_1 \ll Na^3e$. In practice $e_1\sim e$, so that this condition is reduced to that of a rarefied plasma, $Na^3\gg 1$, which we assume valid. In the linear response approximation, the electric induction vector and the electric field vector are related in a linear manner, and the proportionality factor is the longitudinal dielectric constant of the plasma.

2. GENERAL SOLUTION OF THE MAXWELL EQUATIONS

Let us write the Maxwell equation for the Fourier component $\varphi(k,\omega)$ of the scalar potential in coordinates and time:

$$\varphi(k,\omega)=4\pi\rho(\omega)/k^2\varepsilon(k,\omega). \tag{5}$$

where $\varepsilon(k,\omega)$ is the Fourier component of the longitudinal dielectric constant, and $\rho(\omega)$ the Fourier component of the density of the introduced charge,

$$\rho(\omega)=e_1 f(\omega), \tag{6}$$

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with $f(\omega)$ a Fourier component of the switch-on function for the external charge.

It is convenient to first perform the inverse Fourier transformation for the scalar potential φ in the coordinates. Since $\varphi(k,\omega)$ is independent of the angles of the wave vector **k**, integration with respect to the angles of this vector can be done explicitly. The result is

$$\varphi(r,\omega)=\frac{2e_1 f(\omega)}{\pi r}\int_0^\infty \frac{\sin(kr)\,dk}{k\varepsilon(k,\omega)}. \quad (7)$$

We can now easily find the Fourier component in time of the electric field strength, which has only a radial component in a spherical system of coordinates:

$$E(r,\omega)=-d\varphi(r,\omega)/dr=Q(r,\omega)/r^2, \quad (8)$$

with

$$Q(r,\omega)=\frac{2e_1 f(\omega)}{\pi}\int_0^\infty \frac{\sin(kr)-(kr)\cos(kr)}{k\varepsilon(k,\omega)}\,dk. \quad (9)$$

The physical meaning of $Q(r,\omega)$ is that it is the temporal Fourier component of the total charge inside a sphere of radius r, which follows directly from Gauss's law.

Now let us turn to the inverse transformation of charge $Q(r,t)$ with respect to time. For an arbitrary switch-on function $f(t)$, using the properties of the convolution of functions, we calculate the charge $Q(r,t)$ in general form:

$$Q(r,t)=\int_{-\infty}^{t} f(t')Q_1(r,t-t')\,dt', \quad (10)$$

where we have introduced the notation

$$Q_1(r,t)=\frac{2e_1}{\pi}\int_0^\infty [\sin(kr)-(kr)\cos(kr)]I(k,t)\frac{dk}{k} \quad (11)$$

with

$$I(k,t)=i\int_{-\infty}^{\infty}\frac{\exp(-i\omega t)\,d\omega}{2\pi(\omega+i0)\varepsilon(k,\omega)}. \quad (12)$$

Clearly, $Q_1(r,t)$ is the charge inside a sphere of radius r at time t when the charge e_1 introduced into the plasma is "switched on" in a step-like manner at time $t=0$.

As is known, causality requires that the dielectric constant $\varepsilon(k,\omega)$ be analytic in the upper half-plane of the complex variable ω and tend to unity as $\omega\to\infty$ in this half-plane. Hence, for $t<0$ integral (12) is zero, that is,

$$Q_1(r,t)=0 \quad \text{for} \quad t<0. \quad (13)$$

Up to this point we have not specified the concrete form of the temporal and spatial dispersion laws for the dielectric constant. We now consider the case of a Maxwell plasma. According to Ref. 1 we have

$$\varepsilon(k,\omega)=1+(ka)^{-1}\left[1+F\left(\frac{\omega}{2^{1/2}ka\omega_p}\right)\right], \quad (14)$$

where the function $F(x)$ is defined by the integral[2]

$$F(x)=\frac{x}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{\exp(-z^2)\,dz}{z-x-i0}, \quad (15)$$

and we have introduced the parameter

$$\omega_p=(4\pi Ne^2/m)^{1/2}, \quad (16)$$

known as the electronic-plasma or Langmuir frequency (m is the electron mass). Let us now calculate the electric field strength in the case of a Maxwell plasma.

3. THE ELECTRIC FIELD IN A MAXWELL PLASMA

Function (15) is a Cauchy type integral. Hence, the dielectric constant $\varepsilon(k,\omega)$ as a function of ω can be continued analytically into the lower half-plane of the complex variable ω. This half-plane is known to contain the roots

$$z=\pm\omega(k)-i\gamma(k) \quad (17)$$

of the dispersion equation

$$\varepsilon(k,z)=0. \quad (18)$$

These roots correspond to a pair of poles in the expression (12) for the function $I(k,t)$, which also has a pole at $\omega=0$ corresponding to the perturbation's switch-on function.

If we consider instants $t>0$, the integration contour in (12) can be closed in the lower half-plane of ω. The contour is shown in Fig. 1, which also indicates the motion of poles as k grows. Thus, evaluating integral (12) is reduced to calculating the residues at the simple poles.

In what follows it proves expedient to operate with dimensionless variables for time and coordinate,

$$\tau=\omega_p t, \quad \rho=r/a, \quad (19)$$

for the real and imaginary parts of the frequency,

$$w=\omega/\omega_p, \quad g=\gamma/\omega_p, \quad (20)$$

and, finally, for the wave number,

$$\xi=ka. \quad (21)$$

Evaluating the integral (12) by the theory of residues, we get

$$I(\xi,\tau)=\xi^2/(1+\xi^2)+2\xi^2 R(\xi,\tau). \quad (22)$$

where we have introduced the notation

$$R(\xi,\tau)=\mathrm{Re}\left\{\frac{\exp[(iw-g)\tau]}{(w+ig)^2-(1+\xi^2)}\right\}. \quad (23)$$

Here, in accordance with (17), we have $w=w(\xi)$ and $g=g(\xi)$. Substituting (22) and (23) into (11) and performing integration with the first term on the right-hand side of (22) explicitly, we get

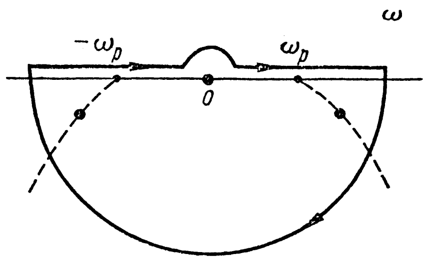

FIG. 1. The contour of integration for (12).

$$Q_1(\rho,\tau)=e_1\Big\{(1+\rho)\exp(-\rho)$$
$$+(4/\pi)\int_0^\infty[\sin(\xi\rho)-(\xi\rho)\cos(\xi\rho)]R(\xi,\tau)\xi\,d\xi\Big\}. \quad (24)$$

We can now substitute this expression into (10). Performing the integration explicitly with respect to time with the first term, we arrive at a formula for determining the charge $Q(\rho,t)$ concentrated inside a sphere of radius $\rho = r/a$:

$$Q(\rho,t)=e_1\Big\{(1+\rho)\exp(-\rho)f(t)$$
$$+(4/\pi)\int_0^\infty[\sin(\xi\rho)-\xi\rho\cos(\xi\rho)]\Phi(\xi,t)\xi\,d\xi\Big\}. \quad (25)$$

where we have introduced the notation

$$\Phi(\xi,t)=\mathrm{Re}\left[\frac{\int_{-\infty}^{t}f(t')\exp[iw(\xi)-g(\xi)\omega_p(t-t')]dt'}{[w(\xi)+ig(\xi)]^2-(1+\xi^2)}\right]. \quad (26)$$

The first term on the right-hand side of (25) corresponds to time-independent, or stationary, Debye screening, while the second term corresponds to the decaying perturbation caused by introduction of the charge into the plasma.

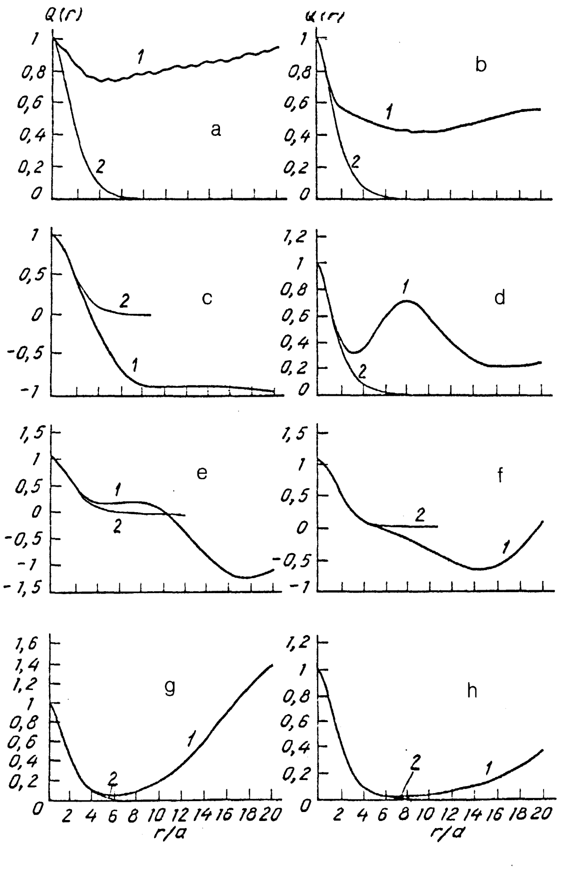

FIG. 2. The charge inside a sphere of radius r for the case where an external charge is instantaneously introduced into the plasma, as a function of r/a at different instants of dimensionless time $\omega_p t = 0.3$ (a), 1.0 (b), 3.0 (c), 5.0 (d), 10.0 (e), 20.0 (f), 30.0 (g), and 50.0 (h). Curves *1* correspond to the results of numerical modeling, curves *2* to the Debye contour.

Thus, in terms of the spectra of plasma waves we have obtained an explicit expression for the electric field created by the charge introduced into the plasma.

The numerical calculation of $Q(\rho,t)$ was performed with two switch-on functions, a step function,

$$f_1(t)=\theta(t), \quad (27)$$

and a smooth function,

$$f_2(t)={}^1/_2[1+\mathrm{th}(\omega_p t/2)]. \quad (28)$$

4. DISCUSSION

The results of the numerical calculation of function $Q(\rho)$ for various values of the dimensionless time $\omega_p t$ are represented in Fig. 2 for the step function (27) and in Fig. 3 for the smooth switch-on function (28). All diagrams also show the Debye result for the same function, corresponding to infinite time, $\omega_p t\to\infty$, where Eq. (1) holds true, that is,

$$Q(\rho,t)\to e_1\exp(-\rho)(1+\rho)f(t). \quad (29)$$

For the sake of simplicity we have set $e_1 = 1$ everywhere.

We start by discussing the numerical results for the case of instantaneous introduction of the charge into the plasma. Figure 2 shows that at $\tau = 0.3$ screening is practically nil. For $r < 6a$ screening is achieved only at $\tau = 20$, that is, very slowly. As for great distances $r > 10a$, Debye screening is not attained even at $\tau = 50$, as shown by Fig. 2; more precisely, at great distances from the introduced charge there exists a large charge oscillating in time and alternating in sign.

Figure 3 depicts the situation with the charge inside a sphere of radius r for different moments when the external charge is introduced smoothly in accordance with the switch-on function (28). On the whole, Debye screening is achieved faster. Already at $\tau = 0.3$ and, more than that, at $\tau = 1$ the electric field is close to the one described by the Debye formula (1). Nevertheless, if we again turn to the case of great distances, $r > 6a$, the charge proves to rapidly oscillate owing to propagation of the perturbation via plas-

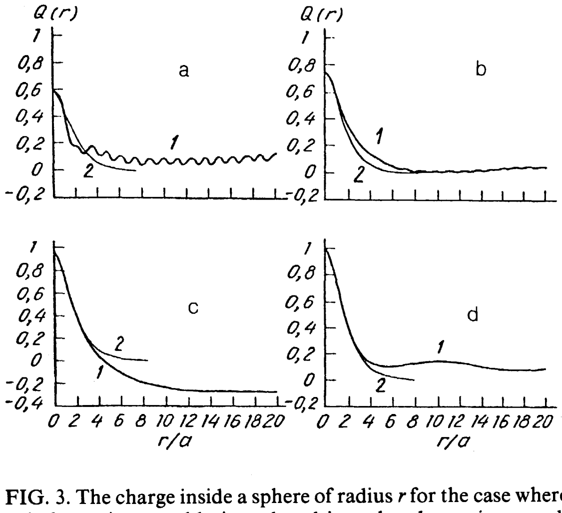

FIG. 3. The charge inside a sphere of radius r for the case where an external charge is smoothly introduced into the plasma in accordance with (28), as a function of r/a at different instants of dimensionless time $\omega_p t = 0.3$ (a), 1.0 (b), 3.0 (c), and 5.0 (a). Curves *1* correspond to the results of numerical modeling, curves *2* to the Debye contour.

ma oscillations. The damping of the charge oscillations is due to the damping of the plasma oscillations (see Refs. 3–5).

Thus, we can conclude that the greater the distance from a given point in space to the introduced charge, the slower Debye stationary screening sets in. The above results demonstrate numerically the extent of this approach to the state of equilibrium. Note that because of the self-similarity achieved by introducing the dimensionless quantities (19)–(21) corresponding to the main characteristics of the problem, the obtained patterns are extremely general and solve the problem completely.

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