Ion-acoustic turbulence in a plasma with two species of ions

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We develop an ion-acoustic turbulence theory for a plasma of hot electrons and two species of cold ions and establish the distribution of ion-acoustic waves over frequencies and wave-vector angles. We show that because of absorption of energy by ions in the process of induced scattering of waves on them, an anisotropic two-temperature Maxwellian distribution sets in for the majority of the ions. Explicit dependences of the longitudinal and transverse ion temperatures on time, plasma parameters, and the strength of the electric field that generates the instability are determined.

The theory of ion-acoustic turbulence (IAT) can be regarded as well developed (e.g., Refs. 1 and 2). At the same time, the theory has not allowed for the peculiar condition that arises in a plasma containing several species of ion. The interest in such a plasma is due both to the situation in which the presence of a large number of impurities is a characteristic feature (see the reviews in Refs. 3 and 4) and to such an object as a plasma with negative ions[5] or a dusty plasma,[6] where there are generally fewer electrons than ions. We also note two papers, Refs. 7 and 8, that consider the properties of a plasma with cold heavy and hot light ions, where there may be an additional purely ionic mode of vibrations similar to ordinary ion sound. Both papers focus on establishing the plasma properties that are responsible for the existence of the additional vibrational branch. At the same time, the specific properties caused by the well-known ion-acoustic wave are yet to be established of a multicomponent plasma with hot electrons.

The aim of this paper is to fill the existing gap and offer a theory of IAT for a plasma that contains two species of ions in addition to hot electrons. The difference between such a plasma and a plasma considered earlier with a single species of ion manifests itself in the difference in probability of induced scattering of ion-acoustic waves on ions. The result is a change in the functional dependence on the wave vector of the short-wave spectrum of turbulent pulsations and a variation in the dependence of the intensity of such pulsations on plasma parameters, which, among other things, manifests itself in a new expression for the Knudsen turbulence number. Finally, the peculiar nature of induced scattering in a plasma with two species of ions manifests itself in the transformation of the distribution function for the bulk of the ions. In contrast to a plasma with ions of a single species, in which a slowly decreasing power-law distribution of thermal ions forms,[9,10] here an anisotropic two-temperature Maxwellian distribution sets in. Such a distribution varies in time primarily because of changes in the longitudinal and transverse temperatures of the ions. Below we establish explicitly the dependence of the temperatures on time and plasma parameters. We demonstrate that the transverse temperature of the ions grows much faster than the longitudinal, especially in the limit of low Knudsen turbulence numbers, when the angular distribution of pulsations is highly anisotropic.

Let us consider a plasma consisting of electrons and two species of ions. In such a plasma the dispersion law of ion-acoustic waves with a phase velocity much lower than the root-mean-square velocity of electrons but much higher than the root-mean-square velocity of ions has the form

$$\omega=\frac{\omega_L k r_{De}}{(1+k^2 r_{De}^2)^{1/2}}, \qquad (1)$$

where r_{De} is the Debye electron radius and $\omega_L^2=\omega_{L1}^2+\omega_{L2}^2$, with $\omega_{L\alpha}=(4\pi e_\alpha^2 n_\alpha/m_\alpha)^{1/2}$ the Langmuir frequency of ions with charge e_α, mass m_α, and number density n_α ($\alpha=1,2$). The smallness of ω/k in comparison to the velocities of the majority of electrons is realized if ω_L is lower than the Langmuir electron frequency. In turn the phase velocity of ion-acoustic waves for $kr_{\mathrm{De}}<1$ is $v_s=\omega_L r_{\mathrm{De}}$ and exceeds the velocity of the majority of ions if the Debye ion radiuses are small: $r_{\mathrm{D}\alpha}\ll r_{\mathrm{De}}\omega_L/\omega_{L\alpha}$.

In such a plasma a fairly strong uniform time-independent electric field $\mathbf{E}=(0,0,E)$ generates an ion-acoustic instability. If the instability threshold is considerably exceeded, to determine the electron distribution function f_e we can employ a quasilinear equation in which ordinary Coulomb collisions are completely ignored:[11]

$$\frac{\partial}{\partial t}f_e+\frac{e}{m_e}E\left(\xi\frac{\partial}{\partial v}+\frac{1-\xi^2}{v}\frac{\partial}{\partial\xi}\right)f_e$$
$$=\frac{\partial}{\partial\xi}\left[\frac{1-\xi^2}{v^2}D_{\xi\xi}\frac{\partial}{\partial\xi}f_e+\frac{(1-\xi^2)^{1/2}}{v}D_{v\xi}\frac{\partial}{\partial v}f_e\right]$$
$$+\frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2D_{vv}\frac{\partial}{\partial v}f_e+v(1-\xi^2)^{1/2}D_{v\xi}\frac{\partial}{\partial\xi}f_e\right], \qquad (2)$$

where e and m_e are the electron charge and mass, $D_{\alpha\beta}$ the components of the quasilinear diffusion tensor in the spherical system of coordinates, and $\xi=\cos\theta$, with θ the angle between vectors $\mathbf{v}$ and $-\mathbf{E}$. Since for the majority of electrons $v\sim v_0=\omega_{\mathrm{Le}}r_{\mathrm{De}}\gg v_s\gg\omega/k$ (here ω_{Le} is the Langmuir electron frequency), we have $D_{\xi\xi}\gg D_{v\xi}\gg D_{vv}$, and the components proper are

$$D_{\xi\xi}=\frac{v_0^3}{v}\nu_2((1-\xi^2)^{1/2}),$$
$$D_{v\xi}=\frac{v_s v_0^3}{v^2}\nu_1((1-\xi^2)^{1/2}), \qquad (3)$$
$$D_{vv}=\frac{v_s^2 v_0^3}{v^3}\nu_0((1-\xi^2)^{1/2}).$$

The turbulence frequencies $\nu_n(y)$ in Eqs. (3) are determined by the axially symmetric distribution of the ion-acoustic waves over the wave vectors, $N(\mathbf{k})=N(k,x)$, where $x=\cos\theta_k$, with θ_k the angle between vectors $\mathbf{k}$ and

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$-$ **E**, and are given by the following formula:

$$\nu_n(y)=\int_{k_{\min}}^{k_{\max}}\frac{k^3dk}{4\pi^2}\left(\frac{\omega}{kv_s}\right)^{4-n}\int_{-y}^{y}\frac{dx}{(y^2-x^2)^{1/2}}\left(\frac{x}{y}\right)^n\frac{\omega N(k,x)}{n_e m_e v_0}, \tag{4}$$

where $k_{\min}$ and $k_{\max}$ are the boundaries of the turbulence region in k, n_e is the electron number density, and $n = 0, 1, 2$.

Allowing for Eqs. (3), we can seek the solution to Eq. (2) in the form of the sum $f_e = f_{0e} + \delta f_e$ of a large isotropic term f_{0e} and a small anisotropic correction δf_e. Over large intervals of time, when $\nu_2(\sin\theta)t \gg 1$, this correction can be unambiguously expressed in terms of f_{0e}. This allows, among other things, writing the electron growth rate of the ion-acoustic instability in the form

$$\gamma_e(k,\theta_k)=\gamma_s(k)\left(\frac{\omega}{kv_s}\right)^3\left[\frac{2}{\pi}\cos\theta_k\int_0^{\sin\theta_k}\frac{d\xi}{(\sin^2\theta_k-\xi^2)^{1/2}}\times\frac{\nu_E+\nu_1((1-\xi^2)^{1/2})/(1-\xi^2)^{1/2}}{\nu_2((1-\xi^2)^{1/2})}-\frac{\omega}{kv_s}\right], \tag{5}$$

where $\gamma_s(k) = \sqrt{\pi/8}\,Ukv_s\omega_L/\omega_{Le}$, and $\nu_E = \sqrt{9\pi/8}\,eE/m_e v_s U > 0$, with $U = (2\pi)^{3/2}v_0^3 n_e^{-1} f_{0e}(v_s)$.

The electron growth rate (5) gives the quantity by which the number of ion-acoustic waves increases because of the Cherenkov radiation given off by moving electrons. On the other hand, the Cherenkov interaction of the waves with ions drives this number down. Since the phase velocities of the waves considerably exceed the root-mean-square velocities of ions, we can conclude that $n_{\alpha h}$, the number density of ions with $v \gtrsim \omega/k$ that resonantly interact with the waves, is low, $n_{\alpha h} \ll n_\alpha$. Because of wave absorption a small group of the resonance ions with $v \gtrsim \omega/k$ gets rapidly heated. Already at moments of time at which

$$\left(\frac{e_\alpha m_e}{em_\alpha}\right)^2\frac{\omega_{Le}^3}{\omega_L^3}\nu_2(\sin\theta)t\gg 1$$

the root-mean-square velocities of resonance ions, $v_{\alpha h}$, considerably exceed the speed of sound v_s. This makes it possible to describe resonance ions in the same way as electrons.[12] For this in Eq. (2) we need only replace f_e with f_α and D_{ij} with $D_{ij}^{(\alpha)} = (e_\alpha m_e/em_\alpha)^2 D_{ij}$ and drop the term containing E. The fact that we can ignore the effect of the electric field on f_α in the subsequent description of wave damping is due to the smallness of the parameter $|e_\alpha n_{\alpha h}/en_e|$, that is, $|e_\alpha n_{\alpha h}/en_e| \ll 1$. In these conditions the rate of damping of the waves on hot resonance ions is

$$\gamma_\alpha(k,\theta_k)=\gamma_s(k)\left(\frac{\omega}{kv_s}\right)^3\delta_\alpha\left[\frac{2}{\pi}\cos\theta_k\int_0^{\sin\theta_k}\frac{d\xi}{(\sin^2\theta_k-\xi^2)^{1/2}}\times\frac{\nu_1((1-\xi^2)^{1/2})}{\nu_2((1-\xi^2)^{1/2})}\frac{1}{(1-\xi^2)^{1/2}}-\frac{\omega}{kv_s}\right], \tag{6}$$

where the parameter δ_α is the ratio of the rates of wave damping by hot resonance ions and electrons,

$$\delta_\alpha=\frac{n_e\omega_{L\alpha}^2 f_{0\alpha}(v_s)}{n_\alpha\omega_{Le}^2 f_{0e}(v_s)}, \tag{7}$$

with $f_{0\alpha}(v_s)$ the isotropic part of the distribution function for the resonance ions.

As is known,[13] allowing only for the Cherenkov damping on electrons and hot ions does not lead to a quasistationary IAT spectrum. Such a spectrum establishes itself if we also allow for induced scattering of sound on ions. Owing to the Cherenkov interaction of ions with beats of ion-acoustic waves, it becomes possible for the wave energy and momentum to be transferred to the majority of thermal ions, which leads to stabilization of the turbulence noise level. Following Refs. 14 and 15, we can write the rate of damping of waves due to induced scattering on ions as

$$\gamma_{NL}(\mathbf{k})=\sum_{\alpha=1,2}\int\frac{d\mathbf{v}}{2m_\alpha}\int\frac{d\mathbf{k}'}{(2\pi)^3}N(\mathbf{k}')W_\alpha(\mathbf{k},\mathbf{k}',\mathbf{v})\left(\mathbf{k}''\frac{\partial f_\alpha}{\partial\mathbf{v}}\right), \tag{8}$$

where $W_\alpha(\mathbf{k},\mathbf{k}',\mathbf{v})$ is the scattering probability,

$$W_\alpha(\mathbf{k},\mathbf{k}',\mathbf{v})=4(2\pi)^9|\Lambda_\alpha(\mathbf{k},\mathbf{k}',\mathbf{v})|^2\times\left[\omega^2\frac{\partial\varepsilon(\omega,\mathbf{k})}{\partial\omega}\frac{\partial\varepsilon(\omega',\mathbf{k}')}{\partial\omega'}\right]^{-1}\delta(\omega''-\mathbf{k}''\mathbf{v}),$$
$$\mathbf{k}''=\mathbf{k}-\mathbf{k}',\quad\omega''=\omega-\omega', \tag{9}$$
$$\varepsilon(\omega,\mathbf{k})=1+\delta\varepsilon_e(\omega,\mathbf{k})+\delta\varepsilon_1(\omega,\mathbf{k})+\delta\varepsilon_2(\omega,\mathbf{k}),$$

and $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$, with $\omega'' = \omega - \omega'$, $\varepsilon(\omega,\mathbf{k}) = 1 + \delta\varepsilon_e(\omega,\mathbf{k}) + \delta\varepsilon_1(\omega,\mathbf{k}) + \delta\varepsilon_2(\omega,\mathbf{k})$, $\delta\varepsilon_\alpha(\omega,\mathbf{k})$ the partial contribution to the dielectric constant from particles of the α-species, and $\Lambda_\alpha(\mathbf{k},\mathbf{k}',\mathbf{v})$ the scattering amplitude,

$$\Lambda_\alpha(\mathbf{k},\mathbf{k}',\mathbf{v})=(2\pi)^{-3}\omega\left\{\frac{e_\alpha^2}{m_\alpha}\frac{\mathbf{k}\mathbf{k}'}{kk'}\frac{1}{(\omega-\mathbf{k}\mathbf{v})}\frac{1}{(\omega'-\mathbf{k}'\mathbf{v})}+\frac{4\pi e_\alpha}{kk'}\sum_{\beta=e,1,2}\frac{e_\beta^3}{m_\beta^2}\frac{1}{k''^2\varepsilon(\omega'',\mathbf{k}'')}\int\frac{d\mathbf{v}'}{\omega''-\mathbf{k}''\mathbf{v}'}\times\left[\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}'}\right)\frac{1}{\omega'-\mathbf{k}'\mathbf{v}'}\left(\mathbf{k}'\frac{\partial}{\partial\mathbf{v}'}\right)f_\beta(\mathbf{v}')-\left(\mathbf{k}'\frac{\partial}{\partial\mathbf{v}'}\right)\frac{1}{\omega-\mathbf{k}\mathbf{v}'}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}'}\right)f_\beta(\mathbf{v}')\right]\right\}. \tag{10}$$

Let us simplify Eq. (10). We first allow for the fact that the phase velocity of the waves is much higher than the velocity of the majority of ions but much lower than the velocity of the majority of electrons. Next we note that in IAT theory the more important wave numbers are those smaller than or of the order of the inverse of the electron Debye radius. Because of this, if $r_{De} \gg r_{D\alpha}$ for at least one species of ion, the partial contribution to the longitudinal dielectric constant from this species is great: $\delta\varepsilon_\alpha(\omega'',\mathbf{k}'') \gg 1$ and $\delta\varepsilon_\alpha(\omega'',\mathbf{k}'')$. In addition, we assume that $r_{De} \gg r_{D\alpha}\max[1,|e_\alpha n_\alpha/en_e|^{1/2}]$ for at least one species of ion, which allows ignoring completely the electron contribution to the scattering amplitude $\Lambda_\alpha(\mathbf{k},\mathbf{k}',\mathbf{v})$ (10). Taking all this into account, we arrive at the following approximation:

$$\Lambda_\alpha(\mathbf{k},\mathbf{k}',\mathbf{v})\doteq(2\pi)^{-3}\frac{e_\alpha}{\omega'}\frac{\mathbf{k}\mathbf{k}'}{kk'}\left\{\frac{e_\alpha}{m_\alpha}\left(\frac{\mathbf{k}\mathbf{v}}{\omega}+\frac{\mathbf{k}'\mathbf{v}}{\omega'}\right)+[\delta\varepsilon_1(\omega'',\mathbf{k}'')+\delta\varepsilon_2(\omega'',\mathbf{k}'')]^{-1}\sum_{\beta=1,2}\left(\frac{e_\alpha}{m_\alpha}-\frac{e_\beta}{m_\beta}\right)\delta\varepsilon_\beta(\omega'',\mathbf{k}'')\right\}, \tag{11}$$

where retaining $\delta\varepsilon_\alpha$ of a given species of particles makes sense only if $\delta\varepsilon_\alpha \gg 1$ and $\delta\varepsilon_\alpha \gg \delta\varepsilon_e$.

By combining Eqs. (9) and (11) with the expansion $\delta[\omega'' - (\mathbf{k}''\mathbf{v})] \simeq \delta(\omega'') - (\mathbf{k}''\mathbf{v})\partial\delta(\omega'')/\partial\omega$ we can transform the damping rate (8). Assuming the distribution function for thermal ions to be axially symmetric in velocity and introducing the definitions of longitudinal and transverse temperatures,

$$T_{\parallel\alpha} = \frac{m_\alpha}{\varkappa n_\alpha}\int d\mathbf{v}(\mathbf{n}\mathbf{v})^2 f_\alpha(\mathbf{v}),$$
$$T_{\perp\alpha} = \frac{m_\alpha}{2\varkappa n_\alpha}\int d\mathbf{v}[\mathbf{n}\mathbf{v}]^2 f_\alpha(\mathbf{v}), \tag{12}$$

with $\mathbf{n} = -\mathbf{E}/|\mathbf{E}|$, and $\varkappa$ the Boltzmann constant, we obtain

$$\gamma_{NL}(\mathbf{k}) = \int \frac{d\mathbf{k}'}{8\pi} N(\mathbf{k}') \frac{|\mathbf{k}-\mathbf{k}'|^2}{\omega_L^4}\left(\frac{\mathbf{k}\mathbf{k}'}{kk'}\right)^2 \frac{\partial\delta(\omega-\omega')}{\partial\omega}$$
$$\times\Big\{\sum_{\alpha=1,2}\frac{\varkappa\omega_{L\alpha}^4}{4\pi n_\alpha m_\alpha^2}(T_{\parallel\alpha}(\mathbf{n},\mathbf{k}+\mathbf{k}')^2 + T_{\perp\alpha}[\mathbf{n},\mathbf{k}+\mathbf{k}']^2)$$
$$+\omega\omega'\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2[\delta\varepsilon_1(0,\mathbf{k}'')+\delta\varepsilon_2(0,\mathbf{k}'')]^{-2}$$
$$\times[\omega_{L1}^2\delta\varepsilon_2^2(0,\mathbf{k}'')+\omega_{L2}^2\delta\varepsilon_1^2(0,\mathbf{k}'')]\Big\}. \tag{13}$$

In the case of an anisotropic Maxwellian distribution of particles, the DC dielectric constants assume the form

$$\delta\varepsilon_\alpha(0,\mathbf{k}) = \frac{m_\alpha\omega_{L\alpha}^2}{\varkappa(k_z^2T_{\parallel\alpha}+k_\perp^2T_{\perp\alpha})}, \tag{14}$$

where $k_z^2 = (\mathbf{k}\cdot\mathbf{n})^2$, and $k_\perp^2 = k^2 - k_z^2$. For an isotropic Maxwellian distribution, where the longitudinal and transverse temperatures coincide, $\delta\varepsilon_\alpha(0,\mathbf{k}) = 1/k^2r_{D\alpha}^2$.

If the plasma consists of ions of a single species, the last term in the braces on the right-hand side of Eq. (13) vanishes, and the other two terms lead to an expression for γ_{NL} used earlier in IAT theory.[1,2,16] For a plasma with two species of ions with $e_1/m_1 \neq e_2/m_2$ all terms in (13) are generally important. The first two terms stem from a velocity-dependent term in the scattering amplitude $\Lambda_\alpha(\mathbf{k},\mathbf{k}',\mathbf{v})$ given by Eq. (11) and lead to a trivial generalization of the known expression for γ_{NL} (Ref. 16) to the case of a plasma containing two species of ions. The last term in the braces in (13) comes from a contribution, independent of ion velocities, to the scattering amplitude (11) and exists only in a plasma containing ions with different charge-to-mass ratios, $e_1/m_1 \neq e_2/m_2$. In view of this it is interesting to examine the conditions in which this term dominates in γ_{NL} and when one can expect the most striking deviation of the IAT theory from the corresponding theory for a plasma with one species of ion. The appropriate conditions are realized when the plasma parameters are such that

$$\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\frac{r_{De}^2\omega_L^2}{(r_{D1}^2+r_{D2}^2)^2}\sum_{\alpha=1,2}\omega_{L\alpha}^2r_{D\alpha}^4 \gg \sum_{\alpha=1,2}\frac{e_\alpha^2}{m_\alpha^2}r_{D\alpha}^2\omega_{L\alpha}^4, \tag{15}$$

where $r_{D\alpha}^2 = \varkappa T_\alpha/4\pi e_\alpha^2 n_\alpha$, and $T_\alpha = (T_{\parallel\alpha} + 2T_{\perp\alpha})/3$.

Condition (15) can be met, for instance, in a plasma consisting of cold ions with commensurable charges $e_1 \sim e_2 \sim Z|e|$, masses $m_1 \sim m_2$, number densities $n_1 \sim n_2$, and temperatures $T_1 \sim T_2 \sim T$, and hot electrons with a temperature $T_e \gg T/Z$. The only requirements here are that the plasma strongly differ from an isothermal one and that the charge-to-mass ratios for the ions of different species not be too close:

$$\left(1-\frac{e_2m_1}{e_1m_2}\right)^2\frac{ZT_e}{T} \gg 1. \tag{16}$$

But if the number density of one ion component is much lower than that of the other, say $n_2 \ll n_1$, condition (15) assumes the form

$$1 \gg \frac{n_2}{n_1} \gg \frac{T}{ZT_e}\left(1-\frac{e_2m_1}{e_1m_2}\right)^{-2}. \tag{17}$$

Conditions specified by (17) not only guarantee the relative smallness of the contribution to γ_{NL} (13) from the terms that dominated in the IAT theory for a plasma with a single species of ion, but also allow ignoring $\delta\varepsilon_2(0,\mathbf{k}'')$ in (13). The nonlinear damping rate γ_{NL} defined by (13) is then determined by the induced scattering on ions with a lower number density and is described by an expression that does not depend on the explicit form of $\delta\varepsilon_\alpha(0,\mathbf{k}'')$, the partial contributions of ions to the longitudinal dielectric constant:

$$\gamma_{NL}(k,\theta_k) = \frac{\omega_{L2}^2}{2\omega_L^4}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\frac{k\omega}{d\omega/dk}$$
$$\times\frac{\partial}{\partial k}\Big\{\frac{k^3\omega}{d\omega/dk}\int_{-1}^{1}d(\cos\theta_k')\int_0^{2\pi}\frac{d\varphi_k'}{2\pi}\left(1-\frac{\mathbf{k}\mathbf{k}'}{kk'}\right)\left(\frac{\mathbf{k}\mathbf{k}'}{kk'}\right)^2$$
$$\times N(k,\cos\theta_k')\Big\}, \tag{18}$$

where φ_k' is the azimuthal angle of the vector $\mathbf{k}'$, and $\omega_L \approx \omega_{L1}$.

Formula (18) can also be applied when $m_2 \gg m_1$ if instead of (17) the following conditions are realized:

$$1 \gg \frac{m_1}{m_2} \gg \frac{n_2}{n_1} \gg \frac{m_2}{m_1}\frac{T}{ZT_e}, \tag{19}$$

which do not contradict each other in a highly nonisothermal plasma, $ZT_e/T \gg (m_2/m_1)^2$.

As noted earlier, under conditions defined by the inequalities (17) and (19) there proved to be no need to detail the form of $\delta\varepsilon_\alpha(0,\mathbf{k}'')$ when deriving Eq. (18). Explicit expressions for $\delta\varepsilon_\alpha(0,\mathbf{k}'')$ become necessary in conditions defined by the inequality (16) for $n_1 \sim n_2$ and $m_1 \sim m_2$ or for $n_1 \gg n_2 \gg n_1m_1/m_2$ when $m_2 \gg m_1$. To construct a theory of the IAT spectrum in such conditions it has proved sufficient to employ (14) for $\delta\varepsilon_\alpha(0,\mathbf{k}'')$. The resultant expression for the nonlinear damping rate γ_{NL} differs from the damping rate (18). This difference, however, is essential only in the presence of an ion temperature that is anisotropic and manifests itself in a change in the dependence of the IAT spectrum on the angle of the wave vector. Indeed, if the longitudinal and transverse temperatures coincide, $T_{\parallel\alpha} = T_{\perp\alpha}$, then $\delta\varepsilon_\alpha(0,\mathbf{k}'') = (k''r_{D\alpha})^{-2}$, and the dependence of the nonlinear damping rate on $\mathbf{k}$ remains the same as in (18). This means that the IAT theory based on the use of the nonlinear damping rate (18) is applicable also for $n_1 \sim n_2$ and $m_1 \sim m_2$

or for $n_1 \gg n_2 \gg n_1 m_1/m_2$ if in (18) ω_{L2}^2 is replaced with $(\omega_{L1}^2 r_{D1}^4 + \omega_{L2}^2 r_{D2}^4)(r_{D1}^2 + r_{D2}^2)^2$.

Now that we know the electron growth rate (5) and the rates (6) and (18) for wave damping on resonance and thermal ions, respectively, we can determine the quasistationary spectrum of turbulence noise, $N(k,\cos\theta_k)$. The basis for finding $N(k,\cos\theta_k)$ is the following equation:

$$\gamma_e(k,\theta_k)+\gamma_1(k,\theta_k)+\gamma_2(k,\theta_k)+\gamma_{NL}(k,\theta_k)=0. \quad (20)$$

By employing the approximation $\omega/kv_s \equiv 1$ in the last term in the square brackets in Eqs. (5) and (6), which has been tested in IAT theory,[17] we can seek the solution of Eq. (20) in the form $N(k,\cos\theta_k) = N(k)\Phi(\cos\theta_k)$. For the distribution function in the wave numbers we then arrive at

$$N(k)=\left(\frac{\pi}{2}\right)^{1/2}\frac{U\omega_L}{\omega_{Le}\omega_{L2}^2}v_s^5\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^{-2}y(kr_{De}), \quad (21)$$

with $y(x)$ defined as

$$y(x)=\frac{1}{x^4(1+x^2)}\left[\ln\frac{(1+x^2)^{1/2}+1}{x}-\frac{1}{(1+x^2)^{1/2}}-\frac{1}{3(1+x^2)^{3/2}}\right], \quad (22)$$

which yields the following asymptotic behavior:

$$y(kr_{De})\approx\begin{cases}(kr_{De})^{-4}\left(\ln\dfrac{2}{kr_{De}}-\dfrac{4}{3}\right), & kr_{De}\ll 1,\\ {}^1/_5(kr_{De})^{-11}, & kr_{De}\gg 1.\end{cases}$$

The distribution of fluctuations over the wave numbers in the long-wave region, where $kr_{De} \ll 1$, follows the Kadomtsev–Petviashvili scaling,[18] as it does in a plasma with a single species of ion. In the short-wave region, where $kr_{De} \gg 1$, the level of fluctuations decreases slower than it would according to the Galeev-Sagdeev formula $N(k) \propto k^{-13}$ (Ref. 19), which describes a plasma with ions of a single species.

For determining the angular distribution of turbulent pulsations we have, via Eq. (20), the following integral equation:

$$\frac{2}{\pi}x\int_0^{(1-x^2)^{1/2}}\frac{d\xi}{(1-x^2-\xi^2)^{1/2}}\frac{\nu_E+(1+\delta)\nu_1((1-\xi^2)^{1/2})/(1-\xi^2)^{1/2}}{\nu_2((1-\xi^2)^{1/2})}$$
$$=1+\delta+M_2x^2+{}^1/_2(M_0-M_2)(1-x^2)-M_3x^3-{}^3/_2(M_1-M_3)(x-x^3). \quad (23)$$

In this equation $\delta=\delta_1+\delta_2$, $\nu_n((1-\xi^2)^{1/2}) = \nu_N\lambda_n x_n))1-\xi^2)^{1/2})$,

$$\nu_N=\frac{U}{2(2\pi)^{1/2}}\frac{\omega_L^7}{n_e m_e\omega_{Le}^2\omega_{L2}^2}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^{-2},$$

$$M_n=\int_0^1 dx\,\Phi(x)x^n,$$

$$\chi_n(y)=\int_0^y\frac{dx\Phi(x)}{(y^2-x^2)^{1/2}}\left(\frac{x}{y}\right)^n,$$

$$\lambda_n=\int_{u_{min}}^{u_{max}} du\; u^4y(u)(1+u^2)^{n/2-5/2},\quad n=0,1,2,$$

$$u_{max}=k_{max}r_{De}\gg 1,\; u_{min}=k_{min}r_{De}\ll 1.$$

with $u_{max} = k_{max}r_{De} \gg 1$ and $u_{min} = k_{min}r_{De} \ll 1$. Asymptotically the λ_n differ little: $\lambda_1 = 3G/4 - 121/840 \approx 0.5429$ and $\lambda_2 = 17\pi/96 \approx 0.5563$, where $G = 0.91596$ is Catalan's constant.

Using the Knudsen turbulence number

$$K_N=\frac{\nu_E}{\nu_N}=\frac{6\pi^2}{U^2}\frac{en_eE}{v_s}\frac{\omega_{Le}^2\omega_{L2}^2}{\omega_L^7}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2, \quad (24)$$

assuming that $K_N \gg (\lambda_2-\lambda_1)(1+\delta)^2 = 0.013(1+\delta)^2$, and ignoring the small difference between λ_1 and λ_2, we obtain from Eq. (23)

$$\int_0^x\frac{t\Phi(t)\,dt}{(x^2-t^2)^{1/2}}\left\{\frac{t}{x^2}\left[1+\frac{\varphi(x)}{1+\delta}\right]-1\right\}=\frac{K_Nx^2}{\lambda(1+\delta)}, \quad (25)$$

where $\lambda \approx \lambda_1 \approx \lambda_2$, and also

$$\varphi(x)={}^1/_2(M_0-M_2)+{}^1/_2(-M_0-6M_1+3M_2+8M_3)x^2$$
$$+(3M_1-5M_3)x^4+{}^1/_2(M_0-3M_2)x^2(1-x^2)^{1/2}\ln\frac{1+(1-x^2)^{1/2}}{x}. \quad (26)$$

Equation (25) has simple solutions in the limits of small and large values of K_N.

Following Ref. 20, we find that in the limit of small K_N, that is, $K_N \ll \lambda(1+\delta)^2$,

$$\Phi(x)=\frac{4K_N}{3\pi\lambda(1+\delta)x}\frac{d}{dx}\frac{x^4}{(1+\varepsilon-x)^{1-\alpha}}, \quad (27)$$

where ε and α are determined by the system of equations

$$\varepsilon=\frac{\varphi(1)}{1+\delta}=\frac{M_2-M_3}{1+\delta}\approx\frac{4K_N}{3\pi\lambda(1+\delta)^2}\frac{1}{\alpha}(1-\varepsilon^\alpha),$$

$$\alpha=-\frac{\varphi'(1)}{2(1+\delta)}=\frac{1}{1+\delta}(M_0-3M_1-3M_2+6M_3)$$
$$\approx\frac{4K_N}{3\pi\lambda(1+\delta)^2}\varepsilon^{\alpha-1},$$

whose approximate solution yields

$$\alpha=\frac{\ln 2}{\ln[3\pi\lambda(1+\delta)^2\ln 2/2K_N]}\ll 1, \quad (28)$$

$$\varepsilon=\frac{2K_N}{3\pi\lambda(1+\delta)^2\ln 2}\ln\frac{3\pi\lambda(1+\delta)^2\ln 2}{2K_N}\ll 1. \quad (29)$$

In the limit of large K_N, that is, $K_N \gg \lambda(1+\delta)^2$, the solution to Eq. (25) has the form

$$\Phi(x)=\frac{2K_N}{\pi\lambda x^2}\frac{d}{dx}\int_0^x\frac{t^5\,dt}{\varphi(t)(x^2-t^2)^{1/2}}, \quad (30)$$

where the moments that determine the function $\varphi(t)$ via (26) can be found from the following system of equations

$$M_0=\frac{2K_N}{\pi\lambda}\int_0^1\frac{t^2\,dt}{\varphi(t)}\left[\frac{t}{(1-t^2)^{1/2}}+\arccos t\right]=\mathscr{M}_0\left(\frac{K_N}{\lambda}\right)^{1/2},$$

$$M_1=\frac{2K_N}{\pi\lambda}\int_0^1\frac{t^3dt}{\varphi(t)(1-t^2)^{1/2}}=\mathscr{M}_1\left(\frac{K_N}{\lambda}\right)^{1/2},$$

$$M_2=\frac{2K_N}{\pi\lambda}\int_0^1\frac{t^5\,dt}{\varphi(t)(1-t^2)^{1/2}}\equiv\mathcal{M}_2\left(\frac{K_N}{\lambda}\right)^{1/2},$$

$$M_3=\frac{2K_N}{\pi\lambda}\int_0^1\frac{t^5\,dt}{\varphi(t)}\left[\frac{1}{(1-t^2)^{1/2}}-\ln\frac{1+(1-t^2)^{1/2}}{t}\right]\equiv\mathcal{M}_3\left(\frac{K_N}{\lambda}\right)^{1/2}.$$

Numerical solution of this system yields $\mathcal{M}_0=2.47$, $\mathcal{M}_1=1.84$, $\mathcal{M}_2=1.44$, and $\mathcal{M}_3=1.17$. Thus, Eq. (30) determines the angular distribution in the form of an integral. Figure 1 shows the graph of the function $\psi(x)=\sqrt{\lambda/K_N}\,\Phi(x)$.

The established distribution of pulsations over the wave numbers makes it possible to describe a fairly broad spectrum of properties of a turbulent plasma. Below we discuss the ion distribution. We are interested in the distribution function $f_\alpha=f_\alpha(\mathbf{v},t)$ for the majority of ions, which is described by the following kinetic equation (cf. Ref. 9):

$$\frac{\partial f_\alpha}{\partial t}+\frac{e_\alpha}{m_\alpha}\mathbf{E}\frac{\partial f_\alpha}{\partial\mathbf{v}}=\frac{1}{2m_\alpha^2}\int\frac{d\mathbf{k}}{(2\pi)^3}\int\frac{d\mathbf{k}'}{(2\pi)^3}\times N(\mathbf{k})N(\mathbf{k}')W_\alpha(\mathbf{k},\mathbf{k}',0)\left(\mathbf{k}''\frac{\partial}{\partial\mathbf{v}}\right)^2 f_\alpha. \qquad (31)$$

Assuming that the distribution is axially symmetric with respect to the direction of the electric field $\mathbf{E}$ that generates the turbulence, we can write Eq. (31) for the function $f_\alpha(\mathbf{v},t)=f_\alpha(v_\perp,v_z,t)$ in the form

$$\frac{\partial f_\alpha}{\partial t}+\frac{e_\alpha}{m_\alpha}E\frac{\partial f_\alpha}{\partial v_z}=\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left[v_\perp D_{\perp\perp}^{(\alpha)}\frac{\partial f_\alpha}{\partial v_\perp}\right]+\frac{\partial}{\partial v_z}\left[D_{zz}^{(\alpha)}\frac{\partial f_\alpha}{\partial v_z}\right], \qquad (32)$$

where the diffusion tensor components in the velocity space are given by the following formulas:

$$D_{\perp\perp}^{(\alpha)}=\frac{1}{4m_\alpha^2}\int\frac{d\mathbf{k}}{(2\pi)^3}\int\frac{d\mathbf{k}'}{(2\pi)^3}\times N(\mathbf{k})N(\mathbf{k}')W_\alpha(\mathbf{k},\mathbf{k}',0)[\mathbf{n}\mathbf{k}'']^2,$$

$$D_{zz}^{(\alpha)}=\frac{1}{2m_\alpha^2}\int\frac{d\mathbf{k}}{(2\pi)^3}\int\frac{d\mathbf{k}'}{(2\pi)^3}\times N(\mathbf{k})N(\mathbf{k}')W_\alpha(\mathbf{k},\mathbf{k}',0)(\mathbf{n}\mathbf{k}'')^2. \qquad (33)$$

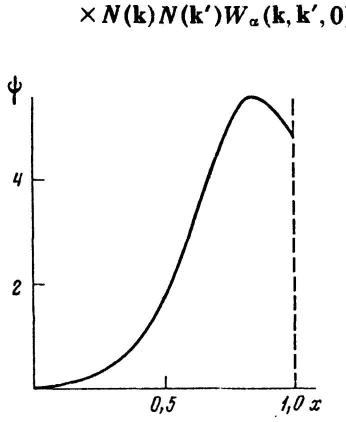

FIG. 1. Distribution of the number of ion-acoustic waves over the wave-vector angles.

Clearly, the solution to Eq. (32) has the form

$$f_\alpha(v_\perp,v_z,t)=\frac{n_\alpha m_\alpha^{3/2}}{(2\pi)^{3/2}\varkappa T_{\perp\alpha}(t)(\varkappa T_{\parallel\alpha}(t))^{1/2}}\times\exp\left[-\frac{m_\alpha v_\perp^2}{2\varkappa T_{\perp\alpha}(t)}-\frac{m_\alpha[v_z-(e_\alpha/m_\alpha)Et]^2}{2\varkappa T_{\parallel\alpha}(t)}\right], \qquad (34)$$

where the time dependence of the temperatures characterizing the anisotropy is specified by the equations

$$\varkappa\frac{d}{dt}T_{\perp\alpha}(t)=2m_\alpha D_{\perp\perp}^{(\alpha)},\qquad \varkappa\frac{d}{dt}T_{\parallel\alpha}(t)=2m_\alpha D_{zz}^{(\alpha)}. \qquad (35)$$

Regarding solution (34) we note an important difference between ion heating in a plasma with two species of ions and ion heating in a one-component plasma. As formula (34) shows, no slowly decreasing power-function distribution is formed in the case under discussion. In this sense the theory of a plasma with two species of ions has in the initial heating stages no problem of stability loss due to the formation of an anomalously high number of resonance ions.[21]

Let us now consider Eqs. (35). To solve these we must find explicit expressions for the diffusion tensor components. Employing Eqs. (9) and (11) and the turbulence spectrum described by Eqs. (21), (22), and (24), we find that for particles with the lower number density, which determine the nonlinear damping rate γ_{NL}, Eqs. (33) yield

$$D_{\perp\perp}^{(2)}=\frac{\varkappa T_2(0)}{2m_2}\frac{A_\perp}{\tau_2},\qquad D_{zz}^{(2)}=\frac{\varkappa T_2(0)}{2m_2}\frac{A_\parallel}{\tau_2}, \qquad (36)$$

where $T_2(0)$ is the initial temperature of ions of the second species (with the lower number density), and $\tau_2=n_2\varkappa T_2(0)/en_e v_s E$ is the characteristic time it takes for the initial temperature to double, and the explicit form of the coefficients $A_\perp$ and $A_\parallel$ is determined by the distribution function of IAT in the wave-vector angles, according to the following relations:

$$A_\perp=\frac{\lambda}{K_N}(M_0^2-3M_0M_2+M_0M_4+4M_2^2-3M_2M_4-2M_1^2+4M_1M_3-2M_3^2),$$

$$A_\parallel=\frac{\lambda}{K_N}(M_0M_2-M_0M_4-M_2^2+3M_2M_4-M_1^2+2M_1M_3-3M_3^2). \qquad (37)$$

Bearing in mind the expressions for the moments M_n in the limits of large [see Eq. (30)] and small Knudsen turbulence numbers, where

$$M_n=\frac{4K_N}{3\pi\lambda(1+\delta)}\left[\varepsilon^{\alpha-1}-\frac{n-1}{\alpha}(1-\varepsilon^\alpha)-\sum_{s=1}^{n+2}(-1)^s\frac{(n-1)(n+2)!}{s!\,s(n+2-s)!}\right], \qquad (38)$$

we obtain from (37)

$$A_\perp=\frac{8}{3\pi},\qquad A_\parallel=\frac{52}{15\pi}\alpha,\qquad K_N\ll\lambda(1+\delta)^2,$$

$$A_\perp\approx1.0,\quad A_\parallel\approx0.1,\quad K_N\gg\lambda(1+\delta)^2. \qquad (39)$$

We see that, first, $A_\perp$ and $A_\parallel$ are practically time independent and, second, $A_\perp \gg A_\parallel$ in both limits. Allowing for these properties of $A_\perp$ and $A_\parallel$, from Eqs. (35) and (36) we get linear-in-time laws of increase of both temperatures, with the transverse temperature increasing faster (cf. Ref. 22):

$$T_{\perp 2}(t)=T_2(0)\left(1+A_\perp \frac{t}{\tau_2}\right), \quad T_{\parallel 2}(t)=T_2(0)\left(1+A_\parallel \frac{t}{\tau_2}\right). \tag{40}$$

This property of preferential increase in ion energy in the direction transverse to that of IAT anisotropy is universal. It is inherent in a plasma with a single species of ions too, and, as noted in Ref. 9, follows from the fact that in the induced scattering of sound on ions the frequency of the waves changes little and the wave vectors proper are aligned primarily along the IAT anisotropy axis.

We will use this property to discuss the heating of ions of the first species. In deriving the equations for $T_{\perp 1}$ and $T_{\parallel 1}$ we will completely ignore the dependence on $T_{\parallel \alpha}$ of the partial contributions to the dielectric constant, $\delta\varepsilon(0,\mathbf{k})$ [Eq. (14)], which determine the scattering probability $W_1(\mathbf{k},\mathbf{k}',0)$. The diffusion tensor components for the ions of the first species can then be uniquely expressed in terms of their values for ions of the second species as follows:

$$D^{(1)}_{\perp\perp(zz)} = \left(\frac{e_2 n_2 m_2}{e_1 n_1 m_1}\frac{T_{\perp 1}}{T_{\perp 2}}\right)^2 D^{(2)}_{\perp\perp(zz)}. \tag{41}$$

Equations (35) in turn assume the following form:

$$\frac{d}{dt} T_{\perp(\parallel)1} = A_{\perp(\parallel)} \frac{T_{\perp 1}^2}{\tau_1 T_1(0)\,(1+A_\perp t/\tau_2)^2}, \tag{42}$$

where the time constant determining the heating of ions of the first species is very large:

$$\tau_1=\tau_2 \frac{T_2(0) m_1}{T_1(0) m_2}\left(\frac{e_1 n_1}{e_2 n_2}\right)^2 \gg \tau_2, \tag{43}$$

with $T_1(0)$ the initial temperature. For the transverse temperature from Eq. (42) we find

$$T_{\perp 1}(t)=T_1(0)\left[1-\frac{A_\perp t/\tau_1}{1+A_\perp t/\tau_2}\right]^{-1} \tag{44}$$

For $A_\perp t \gg \tau_2$ the temperature $T_{\perp 1}(t)$ is already close to its maximum value $T_{\perp 1}(t\to\infty) = T_1(0)(1-\tau_2/\tau_1)^{-1} \approx T_1(0)(1+\tau_2/\tau_1)$, which, incidentally, practically does not differ from the initial value. Obviously, the longitudinal temperature $T_{\parallel 1}$ changes even less. The fact that the ions of the first species heat up insignificantly is not surprising: since wave damping due to induced scattering is determined by ions of the second species, these are the ones that heat up the most.

To sum up, the presence of two species of ions introduces special features into the theory of IAT of a plasma. One new result is the anisotropic nature of the velocity distribution of ions obtained in this investigation. Such a distribution could lead to electromagnetic instability, but its discussion is beyond the scope of the present paper. At the same time, the appearance in the theory of two temperatures characterizing the anisotropy would seem to indicate certain similarities between the theory of turbulent heating of ions and the theory developed earlier for a plasma with a single species of ion.[22] The temperatures in Ref. 22, however, were interpreted as the averaged components of the contributions to the kinetic energy of the ions, while the distribution function for the ions remained indeterminate. It must be noted, though, that the general pattern of the theory corresponds to the one discussed in Refs. 1 and 2. This makes it possible, for one thing, to foresee the consequence of the theory, not discussed in this paper, for the formation and heating of resonance ions and the formation of escaping electrons.

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