Statistical characteristics of phase front dislocations of a wave field

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We study the local and statistical characteristics of the zeroes of the amplitude of a wave field in which strong distortions (dislocations) of the phase front occur. We obtain for the usual case of stable dislocations expressions for the curvature, the torsion, and the velocity of the zero-lines (lines on which the amplitude of the wave field vanishes) of the corresponding scalar complex field satisfying a wave equation. We propose a method for evaluating the average values of various quantities given on the zero-lines of a spatially uniform and stationary random field. For a random field obeying Gaussian statistics we obtain expressions connecting the dislocation number density, the total length of the zero-lines, and their mean square velocity with the spectral and angular dependence of the radiation intensity. We give and discuss the results for these quantities in the particular cases of isotropic and planar distributions of the radiation intensity, for paraxial beams, for monochromatic radiation, and for white noise.

INTRODUCTION

In interference fields there exist practically always singular points in space in which the field intensity vanishes exactly. Near the field zeroes the wavefront (the equiphase surface) undergoes characteristic distortions which have been called dislocations of the phase front.[1] The existence of these singular points in the field which are stable to small perturbations is a characteristic feature of any physical kind of wavefield.[1–4] In many cases the space-time distribution of the dislocations reflects the global structural singularities in the field configuration, being a "skeleton," to use Berry's graphic expression, on which to hang the wave picture of the field.[5] An elucidation of the basic properties of the space-time distribution of the field zeroes and a study of the motion of the dislocations caused by varying the parameters of the interference fields is therefore a very interesting and promising problem which will open up a qualitatively different aspect of the structure of wave fields. The properties of the dislocations are also of practical interest: dislocations are very noticeable "markers" of the field (when one goes around a zero the phase of the field undergoes a rotation[1] of $\pm 2\pi$), and using dislocations one can detect the variation of the wave field and thereby the causes for these changes.

As we have said the problem of finding the statistical characteristics of the singular points of a random field is very important for a clearer understanding of the general picture of the behavior of the phase and the amplitude of the wave field. In an earlier paper Zel'dovich and Baranova[3] calculated the density of the zeroes of a statistically uniform random wave field (speckle structure of a laser beam). It is that paper which stimulated us in our study of calculating averages on zero-carriers (using the terminology of Ref. 3). We were able to improve the approach proposed in Refs. 3 and 6 and to obtain new results. The main one of those is an estimate of the density of dislocations for broad beams (a weak divergence of the beam was assumed in Ref. 3). Another result is a derivation of general expressions for various functionals given on the zero-carrier of a field with dislocations. In particular, in this paper we calculate the total length of the zero-lines (lines on which the amplitude of the wave field vanishes) per unit volume and the mean square velocity of the shift of the zero-lines in random Gaussian fields.

ZERO-CARRIER AND PROPERTIES OF ZERO-LINES

We consider some properties of the zeroes of a scalar complex field

$$\psi(\mathbf{R}, t) = u(\mathbf{R}, t) + iv(\mathbf{R}, t),$$

satisfying the wave equation

$$\Delta\psi(\mathbf{R}, t) - \frac{1}{c_0^2}\frac{\partial^2\psi(\mathbf{R}, t)}{\partial t^2} = 0, \tag{1}$$

where $c_0 = \text{const}$ is the wave velocity in the medium. We "freeze" the picture of the field at some time t. The set of equations

$$\begin{aligned} &\mathrm{Re}\,\psi(\mathbf{R}, t) \equiv u(\mathbf{R}, t) = 0, \\ &\mathrm{Im}\,\psi(\mathbf{R}, t) \equiv v(\mathbf{R}, t) = 0 \end{aligned} \tag{2}$$

determines implicitly the set of points $\mathbf{R} = \{x,y,z\}$ in which the field vanishes exactly (zero-carrier). This set has in three-dimensional space a dimensionality between zero (point) and two (surface) (we exclude here the case of an exponentially small field in some regions in space).

A zero-dimensional zero-carrier is unstable against small changes in the parameters. A stable two-dimensional zero-carrier often occurs as an artefact of the idealization of the conditions (e.g., modes in a waveguide), but in most other problems it also shows instability. For these reasons we exclude points and surfaces from our considerations and restrict ourselves to an analysis of the characteristics of zero-lines.

We shall assume that the zero-lines are formed by the intersection of regular parts of the surfaces (2) for which

$$\mathbf{A} \equiv \mathbf{A}(\mathbf{R}, t) = \nabla u(\mathbf{R}, t) \neq 0, \quad \mathbf{B} \equiv \mathbf{B}(\mathbf{R}, t) = \nabla v(\mathbf{R}, t) \neq 0.$$

If the vector product of the gradients, $\mathbf{C} = [\mathbf{AB}]$ is nonvanishing, the vector

$$\mathbf{l} = \mathbf{C}/C, \quad \mathbf{C} = [\mathbf{AB}] = [\nabla u, \nabla v], \quad C = |\mathbf{C}| \neq 0 \tag{3}$$

is the only vector tangential to the zero-line and defines its direction.

It is well known[1,2] that the existence of directed zero-lines in space entails the existence of a conserved topological

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"charge" of the dislocations in the points where the zero-line intersects with a chosen surface. And what is more, if, for instance, that plane is $z = 0$ with the z axis directed upwards, the topological charge is the same as the quantity $Q = \text{sign}(C_z)$. The sign of the charge $Q = +1$ thus corresponds to the intersection of the zero-line with the surface in the direction from underneath upwards, and $Q = -1$ to the direction from above downwards. In the definition of the number density of the dislocations the sign of the charge is usually neglected.[3]

We denote by s the natural parameter on the zero-line; let $\mathbf{r}(s) = \{r_x(s), r_y(s), r_z(s)\}$ be the parametric solution of the set of Eqs. (2). The first derivative $\mathbf{r}' \equiv d\mathbf{r}(s)/ds$ is equal to the unit vector (3) tangential to the zero-line:

$$\mathbf{r}' = \mathbf{l} = \mathbf{C}/C.$$

The second derivative with respect to s defines the curvature vector:

$$\mathbf{r}'' \equiv d^2\mathbf{r}(s)/ds^2 = K\mathbf{n}_1,$$

where K is the curvature and $\mathbf{n}_1$ is a unit vector along the main normal. Substituting $\mathbf{R} = \mathbf{r}(s)$ into Eqs. (2) reduces them to an identity. Twice differentiating this identity with respect to the parameter s and adding the equation $\mathbf{r}''\mathbf{l} = 0$ we get a set of three equations for the curvature vector $\mathbf{r}''$, the solution of which has the form

$$K = D/C, \quad D = |\mathbf{D}|, \quad \mathbf{D} = \mathbf{A}v_{ss} - \mathbf{B}u_{ss}, \tag{4a}$$

$$\mathbf{n}_1 = [\mathbf{bl}], \quad \mathbf{b} = \mathbf{D}/D. \tag{4b}$$

We have here denoted by u_{ss} (and also by v_{ss}) the value of the second derivative with respect to the direction of $\mathbf{l}$ in a point on the zero-line:

$$u_{ss} = (\mathbf{l}\nabla)^2 u(\mathbf{R}, t)|_{\mathbf{R}=\mathbf{r}(s)}.$$

It is clear from (4b) that the vector $\mathbf{b}$ is the unit binormal vector.

The solution of the set of equations obtained by differentiating the above mentioned identities three times with respect to s and adding the condition $\mathbf{r}'''_\perp \mathbf{l} = 0$ (here $\mathbf{r}'''_\perp$ is the projection of $\mathbf{r}'''$ on the plane which is orthogonal to the vector $\mathbf{l}$) gives an expression for the quantity $\mathbf{r}'''_\perp$ and the torsion of the zero-line, $\varkappa = K^{-2}[\mathbf{r}'\mathbf{r}''\mathbf{r}'''_\perp]$:

$$\varkappa = (KD)^{-1}(\beta u_{ss} - \alpha v_{ss}),$$
$$\alpha = 3Ku_{sn_1} + u_{sss}, \quad \beta = 3Kv_{sn_1} + v_{sss}. \tag{5}$$

Here we have

$$u_{sss} = (\mathbf{l}\nabla)^3 u(\mathbf{R}, t)|_{\mathbf{R}=\mathbf{r}(s)}, \quad u_{sn_1} = (\mathbf{n}_1\nabla)(\mathbf{l}\nabla)u(\mathbf{R}, t)|_{\mathbf{R}=\mathbf{r}(s)},$$

and the operator ∇ is assumed to act only on the field u (and, respectively, on v for the quantities v_{sss} and v_{sn_1}). The values of the higher derivatives on the zero-line must, of course, satisfy equations following from the wave equation (1).

The time-dependence of the field causes the zero-line to move in space. After a time dt the total change of the real and imaginary parts of the field on the zero-line $\mathbf{r}(s)$ is equal to zero:

$$du = \mathbf{A}d\mathbf{R} + u'dt = 0, \quad u' \equiv \partial u(\mathbf{R}, t)/\partial t,$$
$$dv = \mathbf{B}d\mathbf{R} + v'dt = 0, \quad v' \equiv \partial v(\mathbf{R}, t)/\partial t. \tag{6}$$

Not being interested in a glide of the dislocations (a displacement of the zero-lines along themselves), we analyze the shift $d\mathbf{R}$ at right angles to the vector $\mathbf{l}$. Adding the equation $\mathbf{l}d\mathbf{R} = 0$ to (6) and solving the resulting set of equations for $d\mathbf{R}/dt$ we obtain an expression for the rate of displacement of the points of the zero-line in space:

$$\mathbf{w}(\mathbf{R}, t) = \left.\frac{d\mathbf{R}}{dt}\right|_{\mathbf{R}=\mathbf{r}(s)} = \frac{E}{C}\mathbf{n}_w, \quad E = |\mathbf{E}|, \quad \mathbf{E} = \mathbf{A}v' - \mathbf{B}u',$$
$$\mathbf{n}_w = [\mathbf{el}], \quad \mathbf{e} = \mathbf{E}/E, \quad \mathbf{n}_w^2 = 1. \tag{7}$$

We note the equation

$$|\sin\chi| = |[\mathbf{be}]| = \frac{c}{DE}|v'u_{ss} - u'v_{ss}|, \tag{8}$$

where χ is the angle between the velocity vector $\mathbf{w}$ and the curvature vector (4), $K\mathbf{n}_1$. In those cases when the vector product of the gradients, $\mathbf{C} = [\mathbf{AB}]$, tends to zero while the quantities D and E remain finite, the curvature (4) and the absolute magnitude of the velocity (7) of the zero-line (4) and (7) tend to infinity while the angle between them (8) tends to zero. This can occur, for instance, when from a point on a plane contour the zero-line (see the numerical example in Ref. 2) broadens swiftly, remaining approximately planar.

The curvature (4), torsion (5), and velocity (7) of the zero-lines do not explicitly contain the values of the field $\psi = u + iv$ and they are thus invariants of the global transformation $\psi \to \tilde{\psi} = \psi e^{i\gamma}$, $\gamma = \text{const}$. One sees easily that if $\gamma = \gamma(\mathbf{R},t)$ is a single-valued function of its arguments, the space-time distribution of the zeroes of the field and their characteristics (4), (5), and (7) remain unchanged even for the more general transformation $\psi \to \tilde{\psi} = \psi\exp[i\gamma(\mathbf{R},t)]$ since such a change of phase has no effect on the absolute magnitude of the field $|\psi|$. We use this property in what follows for evaluating averages on the zero-carrier.

We note that since $C \neq 0$ [see (3)] holds on the zero-line we can always ensure through a choice of a suitable constant γ and of the orientation of the coordinate system $\{x,y,z\}$ that in a given fixed point $\mathbf{R}_0$ on the zero-line all three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{l}$ are mutually orthogonal. In that point the expressions for the curvature, the torsion, and the velocity of the zero-line then simplify considerably (the values of the derivatives are taken in the point $\mathbf{R} = \mathbf{R}_0$):

$$K^2 = \left(\frac{u_{zz}}{u_x}\right)^2 + \left(\frac{v_{zz}}{v_y}\right)^2, \tag{9}$$

$$w^2 = \left(\frac{u'}{u_x}\right)^2 + \left(\frac{v'}{v_y}\right)^2, \tag{10}$$

and here we have

$$\nabla\tilde{\psi} = u_x + iv_y, \quad \tilde{\psi} \equiv \psi(\mathbf{R}, t)e^{-i\gamma}, \quad \gamma = \text{const}$$

AVERAGES ON THE ZERO-CARRIER

We show how one can evaluate average values of functions given on the zero-carrier. To do this we need to introduce for the complex scalar field $\psi(\mathbf{R}) = u(\mathbf{R}) + iv(\mathbf{R})$ a "natural" coordinate system $\{u,v,s\}$ in which in each point of space $\mathbf{R} = \{x,y,z\}$ the directions of the coordinate axes of the $\{u,v,s\}$ system coincide, respectively, with the directions of the gradients of the real and the imaginary parts of the field, $\mathbf{A}(\mathbf{R}) = \nabla u(\mathbf{R})$, $\mathbf{B}(\mathbf{R}) = \nabla v(\mathbf{R})$, and the direction $\mathbf{C}(\mathbf{R}) = [\mathbf{A}(\mathbf{R})\mathbf{B}(\mathbf{R})]$ which is perpendicular to them (we neglect here an unimportant constant factor which equalizes the dimensionalities of the field and of space). The s coordinate is the natural parameter on a spatial curve which may be called a C-line on which $u(\mathbf{R}) = \text{const}_1$ and $v(\mathbf{R}) = \text{const}_2$ simultaneously. For $u(\mathbf{R}) = v(\mathbf{R}) = 0$ the curvilinear coordinate s goes over into the zero-line. The Jacobian of the transition from the $\{u,v,s\}$ variables to the spatial $\{x,y,z\}$ variables is equal to

$$\frac{\partial(u,v,s)}{\partial(x,y,z)} = C, \qquad du\,dv\,ds = C\,dx\,dy\,dz, \tag{11}$$

where $C = |\mathbf{C}|$ is the absolute magnitude of the vector $\mathbf{C} = [\mathbf{AB}]$ from (3).

The one-to-one correspondence $\{u,v,s\}\leftrightarrow\{x,y,z\}$ is a local property and when going over to the whole space the inverse functions $x(u,v,s)$, $y(u,v,s)$, and $z(u,v,s)$ contain many (in the limit, infinitely many) branches. Moreover, in some points $\mathbf{R}$ of space (of measure zero) the vector product $\mathbf{C} = [\mathbf{AB}]$ may vanish. In statistical problems, when we average, these features of the $\{u,v,s\}$ coordinate system often turn out to be unimportant.

We consider the number N of dislocations on some surface, for instance, the plane $z = 0$. Following Ref. 3 we define the number N_z by the formula

$$N_z = \int \delta(u)\delta(v)du\,dv$$
$$= \int_{S_0} \delta(u(x,y))\delta(v(x,y)) \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy. \tag{12}$$

If the field is given, by expanding the δ function we obtain the obvious relation:

$$N_z = \sum_i \int_{S_0} \delta(x-x_i)\delta(y-y_i)dx\,dy, \tag{13}$$

where the (x_i,y_i) are the coordinates of the zeroes of the field in the given region of the area S_0 over which the integration is carried out.

Being interested in the average number of zeroes of a random field we average (12), using the joint distribution function W of the field and its derivatives. Denoting the set of these derivatives by ξ we find from (12) that

$$\langle N_z\rangle = S_0 \int W(u,v,\xi)\delta(u)\delta(v) \left|\frac{\partial(u,v)}{\partial(x,y)}\right| du\,dv\,d\xi$$
$$\equiv S_0 \left\langle \delta(u)\delta(v) \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \right\rangle. \tag{14}$$

We have assumed in (14) that the characteristics of the random field are spatially uniform. Integrating over the u, v variables and noting that the "contracted" Jacobian $\partial(u,v)/\partial(x,y)$ is equal to the z component of the vector $\mathbf{C}$ [see (11)] we can write the average number (14) of dislocations in the form

$$\langle N_z\rangle = S_0 \langle |C_z(\xi)|\rangle, \qquad C_z(\xi) \equiv C_z = \frac{\partial(u,v)}{\partial(x,y)}, \tag{15}$$

where the presence of ξ in the angular brackets indicates averaging over the derivatives:

$$\langle\ldots\rangle \equiv \int W(0,0,\xi)(\ldots)d\xi. \tag{16}$$

If we are interested in the number of dislocations taking their charge $Q = \operatorname{sign}(C_z)$ [see (3)] into account we must in (15) remove the modulus sign of the Jacobian C_z.

We can generalize Eq. (12), obtaining other averaged characteristics of the field connected with the zero-lines. For instance, the total length of all zero-lines in a volume V_0 is given by the expression

$$L = \int \delta(u)\delta(v)du\,dv\,ds$$
$$= \int_{V_0} \delta(u(x,y,z))\delta(v(x,y,z)) \frac{\partial(u,v,s)}{\partial(x,y,z)} dx\,dy\,dz. \tag{17}$$

For an arbitrary random field we get by analogy with (14) and (15)

$$\langle L\rangle = \int_{V_0} \langle C(\xi)\rangle dx\,dy\,dz, \qquad C(\xi) \equiv C = \frac{\partial(u,v,s)}{\partial(x,y,z)}, \tag{18}$$

and for spatially uniform statistics we have

$$\langle L\rangle = V_0 \langle C(\xi)\rangle. \tag{19}$$

The integral (19) will be evaluated below for Gaussian statistics.

Let $F(u,v,\xi)$ be an arbitrary function of the components of the field and its derivatives. The average value of this function on the zero-carrier, $\langle F\rangle_c$, is given by the expression

$$\langle F(u,v,\xi)\rangle_c = \langle L\rangle^{-1} \int_{V_0} \langle F(0,0,\xi)C(\xi)\rangle dx\,dy\,dz. \tag{20}$$

In the case of spatially uniform statistics (20) gives

$$\langle F(u,\ v,\ \xi)\rangle_c = \langle F(0,\ 0,\ \xi)C(\xi)\rangle / \langle C(\xi)\rangle. \tag{21}$$

The factor $F(u,v,\xi)$ can be the curvature (4), or the torsion (5) of the zero-lines, the absolute square of the field gradient, $|\nabla\psi|^2 = \mathbf{A}^2 + \mathbf{B}^2$, the velocity vector (7) of the shift of a zero-line, or its square, and also other quantities which are significantly connected with the zero-lines.

We note the connection between the number of dislocations $\langle N_z\rangle$ on the plane and the total length of the zero-lines. We multiply and divide the integrand in (14) and (15) by $C(\xi)$ and integrate over a unit section l_0 of the z axis. Using the equation $V_0 = S_0 l_0$ and Eqs. (18) and (21) we then obtain

$$\langle N_z\rangle = l_0^{-1}\langle L\rangle\langle|\cos\theta_z|\rangle_c, \tag{22}$$

where $\cos\theta_z \equiv C_z/C$ is the cosine of the angle θ_z between the direction $\mathbf{l}$ of the tangent to the zero-line and the chosen z axis.

GAUSSIAN NOISE MODEL

We choose the simplest model of acoustic noise, which is a superposition of plane waves arriving from all possible directions $\mathbf{n}$:

$$\psi(\mathbf{R},t) = u(\mathbf{R},t) + iv(\mathbf{R},t)$$
$$= \oint_{4\pi} \int_0^\infty a(\mathbf{n},\omega)\exp[i(k\mathbf{nR} - \omega t)]d^2n\,d\omega, \tag{23}$$

where $a(\mathbf{n},\omega)$ is the random complex amplitude of a plane wave, $k=\omega/c_0$ is the wavenumber, $\mathbf{n}$ is a unit vector ($\mathbf{n}^2=1$), and $d^2n=\sin\theta d\theta d\varphi$ is the element of solid angle. We assume the complex amplitudes $a(\mathbf{n},\omega)$ satisfy Gaussian statistics with a zero average and a phase distributed uniformly over the interval $(0,2\pi)$. We also make the usual assumption that the amplitudes of the plane waves are δ-correlated in direction and frequency:

$$\langle a(\mathbf{n},\omega)a^*(\mathbf{n}',\omega')\rangle=J(\mathbf{n},\omega)\delta(\mathbf{n}-\mathbf{n}')\delta(\omega-\omega'). \quad (24)$$

The angular brackets here indicate averaging over an ensemble and $J(\mathbf{n},\omega)$ is the ray intensity (the average energy flux per unit frequency range and unit solid angle).

The correlator of the field (23) equals

$$\langle\psi(\mathbf{R}_1,t_1)\psi^*(\mathbf{R}_2,t_2)\rangle=\int_0^\infty d\omega\oint_{4\pi}d^2nJ(\mathbf{n},\omega)\exp[ik\mathbf{n}(\mathbf{R}_1-\mathbf{R}_2)-i\omega(t_1-t_2)]. \quad (25)$$

Equation (25) is in accordance with radiative transfer theory.[7] The correlator (25) depends only on the differences in the coordinates of the points $\mathbf{R}_1-\mathbf{R}_2$ and in the times t_1-t_2, and its value at equal times and for $\mathbf{R}_1=\mathbf{R}_2$ equals

$$I_0=\langle|\psi(\mathbf{R},t)|^2\rangle=\int_0^\infty d\omega\oint d^2n\,J(\mathbf{n},\omega) \quad (26)$$

(spatially uniform and stationary field).

Averages of the products of the field and its derivatives and also the products of field derivatives, relating to the same point in space and to the same time, can be expressed in terms of averages over the angular and spectral distribution of the ray intensity, for instance (here $\mathbf{R}=\{x,y,z\}\equiv\{x_1,x_2,x_3\}$)

$$\left\langle\frac{\partial\psi(\mathbf{R},t)}{\partial x_i}\psi^*(\mathbf{R},t)\right\rangle=i\int_0^\infty d\omega\oint_{4\pi}d^2n\,kn_iJ(\mathbf{n},\omega)=iI_0\overline{kn_i},$$

$$\left\langle\frac{\partial\psi(\mathbf{R},t)}{\partial x_i}\frac{\partial\psi^*(\mathbf{R},t)}{\partial t}\right\rangle=\int_0^\infty d\omega\oint_{4\pi}d^2n\,\omega kn_iJ(\mathbf{n},\omega)=I_0\overline{\omega kn_i}, \quad (27)$$

$$i=1,2,3,$$

where the bar indicates an average over angles and frequencies using the ray intensity as a weight function:

$$\overline{f(\mathbf{n},\omega)}=\int_0^\infty d\omega\oint_{4\pi}d^2nf(\mathbf{n},\omega)J(\mathbf{n},\omega)\Big/\int_0^\infty d\omega\oint_{4\pi}d^2n\,J(\mathbf{n},\omega). \quad (28)$$

The averages $\overline{\omega}$, $\overline{\omega^2}$, $\overline{k^2n_in_j}$ $(i,j=1,2,3)$, together with just the expressions (26) and (27) which we have written out, completely determine the parameters of the random Gaussian field (23) and (24).

DISLOCATION DENSITY, LENGTH AND MEAN SQUARE VELOCITY OF THE ZERO-LINES

To calculate these quantities we need the ten-dimensional normal probability density of the field and its first derivatives which can be written in the following vector form:

$$W_{10}(\xi)=\frac{1}{(2\pi)^5(\det\hat{\mathbf{K}})^{1/2}}\exp\left(-\frac{1}{2}\xi^T\hat{\mathbf{K}}^{-1}\xi\right), \quad (29)$$

where $\xi=\{\xi_u,\xi_v\}$ and

$$\xi_u=\left\{u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\frac{\partial u}{\partial x_3}\right\}=\{u,u',\mathbf{A}\},$$

$$\xi_v=\left\{v,\frac{\partial v}{\partial t},\frac{\partial v}{\partial x_1},\frac{\partial v}{\partial x_2},\frac{\partial v}{\partial x_3}\right\}=\{v,v'\mathbf{B}\}$$

are ten-component and five-component column vectors consisting of the components of the field and its space-time derivatives. We assume that the vector $\langle\xi\rangle$ of the values of the random field (23) and (24) and its derivatives vanishes identically. The superscript "T" indicates transposition so that the quantity $\xi^T=\{\xi_u^T,\xi_v^T\}$ is a row vector. Finally, $\hat{\mathbf{K}}^{-1}$ is the matrix which is the inverse of the covariance matrix $\hat{\mathbf{K}}$, which has the form

$$\hat{\mathbf{K}}=\begin{pmatrix}\hat{\mathbf{K}}_1 & \hat{\mathbf{K}}_2^T\\ \hat{\mathbf{K}}_2 & \hat{\mathbf{K}}_1\end{pmatrix}, \quad (30)$$

where the square 5×5 submatrices $\hat{\mathbf{K}}_1$, $\hat{\mathbf{K}}_2$ are equal to

$$\hat{\mathbf{K}}_1=\frac{I_0}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & \overline{\omega^2} & -\overline{k\omega n_j}\\ 0 & -\overline{k\omega n_i} & \overline{k^2n_in_j}\end{pmatrix},$$

$$\hat{\mathbf{K}}_2=\frac{I_0}{2}\begin{pmatrix}0 & \omega & -\overline{kn_j}\\ -\omega & 0 & 0\\ \overline{kn_i} & 0 & 0\end{pmatrix} \quad (31)$$

the $\overline{k^2n_in_j}$ are the elements of a 3×3 matrix $(i,j=1,2,3)$, and $\overline{k\omega n_i}$ and $\overline{kn_i}$ $(i=1,2,3)$ are column or row vectors, depending on their position in the matrices $\hat{\mathbf{K}}_1$ and $\hat{\mathbf{K}}_2$.

It is well known[8] that any symmetric positive definite matrix $\hat{\mathbf{K}}$ can be brought to diagonal form by an orthogonal transformation $\hat{\mathbf{T}}^T\hat{\mathbf{K}}\hat{\mathbf{T}}$ $(\hat{\mathbf{T}}^T\hat{\mathbf{T}}=\hat{\mathbf{1}})$. For the quadratic form $\xi^T\hat{\mathbf{K}}^{-1}\xi$ this means a rotation of the base of the vector space, $\xi\to\tilde{\xi}=\hat{\mathbf{T}}^T\xi$. We split off a partial rotation, leading to a decorrelation of the vectors ξ_u and ξ_v. This is accomplished by multiplying the field $\psi(\mathbf{R},t)$ of (23) by a factor $\Lambda=\exp[-i(\mathbf{k}_c\mathbf{R}-\omega_0t)]$, where $\mathbf{k}_c\equiv\{(k_c)_i\}\equiv\{\overline{kn_i}\}$ and $\omega_0\equiv\overline{\omega}$. The probability density $\tilde{W}(\xi)$ of the combined distribution of the "modified" field $\tilde{\psi}=\Lambda\psi$ and its derivatives is obtained from $W(\xi)$ by the formal substitution $kn_i\to kn_i-(k_c)_i$, $\omega\to\omega-\omega_0$. For the normal distribution (29) this leads to putting the submatrix $\hat{\mathbf{K}}_2$ equal to zero (decorrelation of ξ_u and ξ_v) while the submatrix $\hat{\mathbf{K}}_1$ becomes equal to

$$\hat{\mathbf{K}}_1=\frac{I_0}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & \overline{\omega^2}-\overline{\omega}^2 & -(\overline{k\omega n_j}-\overline{\omega}\,\overline{kn_j})\\ 0 & -(\overline{k\omega n_i}-\overline{\omega kn_i}) & (\overline{k^2n_in_j}-\overline{kn_i}\,\overline{kn_j})\end{pmatrix}. \quad (32)$$

As noted above, multiplying the field ψ by a "good" phase factor Λ leaves the space-time distribution of the zeroes of the field unchanged and in a number of cases also does not affect the shape of the function $F(u,v,\xi)\equiv F(\psi,\psi^*,\eta,\eta^*)$ (curvature, torsion, zero-line velocity); here η is the set of the derivatives of the complex field ψ. In the general case the condition for the invariance of the function $F(\psi,\psi^*,\eta,\eta^*)|_{\psi=0}$ under the transformation $\psi\to\tilde{\psi}=\Lambda\psi$ can be written in the form

$$F(0,0,\eta,\eta^*)=F(0,0,\Lambda\eta,\Lambda^*\eta^*),$$

since the nth derivative of the field,

$$\eta_n\equiv\partial^n\psi/\partial x_1^\alpha\partial x_2^\beta\partial x_3^\gamma\partial t^\delta,\ \alpha+\beta+\gamma+\delta=n,$$

transforms for $\psi=0$ as $\tilde{\eta}_n=\Lambda\eta_n$. When this condition is satisfied the partial rotation of the basic, leading to a decorrelation of the vectors ξ_u and ξ_v, appreciably simplifies the calculation of averages (20) and (21) on the zero-carrier.

If the function $F(u,v,\xi)$ is invariant under an arbitrary orthogonal transformation $\hat{\mathbf{T}}$ the covariance matrix $\hat{\mathbf{K}}$ must be brought to diagonal form:

$$\hat{\mathbf{K}}=\{K_{\alpha\beta}\}=(I_0/2)\lambda_\alpha\delta_{\alpha\beta},\ \alpha,\ \beta=1,\ 2,\ldots,\ 10,$$

where the λ_α are the roots of the characteristic equation

$$\det[(2/I_0)\hat{\mathbf{K}}-\lambda\hat{1}]=0.$$

In this case the probability density (29) splits into a product of ten independent Gaussian distributions:

$$W_{10}(\xi)=\prod_{\alpha=1}^{10}\frac{1}{(\pi I_0\lambda_\alpha)^{1/2}}\exp\left(-\frac{\xi_\alpha^2}{I_0\lambda_\alpha}\right),\qquad(33)$$

and the calculation of the averages on the zero-lines simplifies even more [the Jacobian C of (11) is invariant under an arbitrary orthogonal transformation $\hat{\mathbf{T}}$].

1. Dislocation density. In the $\{x,y,z\}$ coordinate system in which the matrix of the spatial derivatives

$$\hat{\mathbf{p}}=\{p_{ij}\}=\{\overline{k^2n_in_j}\},\ i,\ j=x,\ y,\ z$$

is diagonal we find from (15) for $S_0=1$ and from (33) the number density of the dislocations in the $z=$ const plane:

$$\langle N_z\rangle=(2\pi)^{-1}(\lambda_x\lambda_y)^{1/2},\qquad(34)$$

where the λ_i are the roots of the characteristic equation

$$\det(\hat{\mathbf{p}}-\lambda\hat{1})=\lambda^3-G\lambda^2+H\lambda-D_3=0,$$
$$G=\lambda_x+\lambda_y+\lambda_z,$$
$$H=\lambda_x\lambda_y+\lambda_y\lambda_z+\lambda_z\lambda_x,\ D_3=\lambda_x\lambda_y\lambda_z.\qquad(35)$$

The coefficients G, H, and D_3 are invariant under a rotation of the $\{x,y,z\}$ coordinate system; we shall use this in what follows.

In an arbitrary coordinate system we obtain instead of (34)

$$\langle N_i\rangle=(2\pi)^{-1}(\det A_{ii})^{1/2},\ i=x,\ y,\ z,\qquad(36)$$

where A_{ii} is the minor of the p_{ii} element in the determinant of the matrix $\hat{\mathbf{p}}=\{p_{ij}\}$, $p_{ij}=\overline{k^2n_in_j}-\overline{kn_i}\,\overline{kn_j}$. If the angular distribution of the ray intensity is spherically symmetric [$J(\mathbf{n},\omega)=J(-\mathbf{n},\omega)$] we have $\overline{kn_i}=0$ and Eq. (36) simplifies:

$$\langle N_i\rangle=(2\pi)^{-1}(\overline{k^2n_j^2}\,\overline{k^2n_k^2})^{1/2},\qquad(37)$$

(i,j, and k are all different).

In the particular case of an isotropic noise field when we have $J(\mathbf{n},\omega)=J(\omega)$ the average number of dislocations per unit area is equal to

$$\langle N\rangle=\frac{k_1^2}{6\pi},\quad k_1^2=c_0^{-2}\int_0^\infty J(\omega)\omega^2d\omega\Big/\int_0^\infty J(\omega)d\omega.\qquad(38)$$

For white noise concentrated in a frequency range $\omega_0-\Delta\omega/2\leqslant\omega\leqslant\omega_0+\Delta\omega/2$ the quantity k_1 is given by the equation

$$k_1^2=k_0^2[1+{}^1/_3(\Delta\omega/2\omega_0)^2],$$

where $k_0=\omega_0/c_0$ is the wavenumber. If the ray intensity is uniformly distributed over the $n_z>0$ hemisphere, we have

$$\langle N_z\rangle=k_1^2/6\pi,\ \langle N_x\rangle=\langle N_y\rangle=k_1^2/3\cdot2^{3/2}\pi.\qquad(39)$$

We note that the number density of the dislocations in the $z=$ const plane has not changed [see (38)].

For almost monochromatic radiation ($\Delta\omega\ll\omega_0$) we have $k_1\approx k_0$ and in the case of a narrow beam propagating along the z axis when the radiation is concentrated in a small solid angle $\theta_x\theta_y$ the number of dislocations in the plane $z=$ const is equal to

$$\langle N_z\rangle\approx(2\pi)^{-1}k_0^2[\det(\overline{\theta_i\theta_j}-\bar{\theta}_i\bar{\theta}_j)]^{1/2},\ i,\ j=x,\ y.$$

This is equivalent to the results of Ref. 3.

2. Length of the zero-lines. To calculate the total length $\langle L\rangle$ of the zero-lines per unit volume we use the invariance of the Jacobian C of (11) under a rotation of the $\{x,y,z\}$ coordinate system. We then find from (19) for $V_0=1$ and (33)

$$\langle L\rangle=\pi^{-1}(H/3)^{1/2}\delta_L,\qquad(40)$$

where H is the coefficient of λ in the characteristic equation (35) while the numerical factor ($3^{1/2}/2\leqslant\delta_L\leqslant1$) is equal to

$$\delta_L=\frac{3^{1/2}}{4\pi^3}\oint_{4\pi}\oint_{4\pi}q\,d^2n\,d^2m,$$

$$q^2=(\tau_x^2\lambda_y\lambda_z+\tau_y^2\lambda_z\lambda_x+\tau_z^2\lambda_x\lambda_y)/H,\qquad(41)$$

where $\mathbf{n}$ and $\mathbf{m}$ are unit vectors over which we integrate in (41), while we have $\boldsymbol{\tau}\equiv\{\tau_x,\tau_y,\tau_z\}\equiv[\mathbf{nm}]$.

For an isotropic distribution of the ray intensity $J(\mathbf{n},\omega)$ the numerical factor δ_L takes its maximum value 1, and we have

$$\langle L\rangle=k_1^2/3\pi,\qquad(42)$$

where k_1 is given by (38).

The minimum value $3^{1/2}/2$ of the factor δ_L is reached in the case of a "plane" distribution of the ray intensity, when the field (23) is independent of the z coordinate:

$$\langle L\rangle=(2\pi)^{-1}(\det A_{zz})^{1/2}=\langle N_z\rangle.\qquad(43)$$

The result (43) becomes clear, if we bear in mind that for such a distribution of the ray intensity all zero-lines are straight lines parallel to the z axis and the number of their intersections per unit area with a $z=$ const plane is the dislocation density $\langle N_z\rangle$ of (34) and (36) ($\langle N_x\rangle=\langle N_y\rangle=0$). In that case we find from (22) for $l_0=1$ that

$$\langle|\cos\theta_z|\rangle_c=\langle N_z\rangle/\langle L\rangle=1$$

as should be the case.

We note the following property. In contrast to the dislocation density (36), the total length of the zero-lines is by definition independent of the choice of the directions of the coordinate axes. However, the sum of the squares of the dislocation densities in three mutually orthogonal planes which, according to (36) and (35), is equal to

$$\langle N_x\rangle^2+\langle N_y\rangle^2+\langle N_z\rangle^2=(2\pi)^{-2}\sum_i \det A_{ii}=H/(2\pi)^2, \quad (44)$$

is also independent of the orientation of these planes. Dividing (44) by $\langle L\rangle$ from (40) and using Eq. (22) for $l_0=1$ we find that

$$\sum_i\left(\frac{\langle N_i\rangle}{\langle L\rangle}\right)^2=\sum_i\langle|\cos\theta_i|\rangle^2=\delta_c, \quad {}^3/_4\leqslant\delta_c\leqslant 1, \quad (45)$$

where $\delta_c=3/4\delta_L^2$ and the $\cos\theta_i$ are the direction cosines of the tangent vector of the zero-line.

It follows from (45) that in the case of a strongly anisotropic distribution of the ray intensity, when the values of the $\langle N_i\rangle$ differ greatly, the zero-lines are preferentially stretched along the normal to the surface in which the dislocation density reaches a maximum. We note that the difference $1-\delta_L$ is equal to the sum of the dispersions of the direction cosines:

$$1-\delta_L=\sum_i\sigma_i^2, \quad \sigma_i^2\equiv\langle(|\cos\theta_i|-\langle|\cos\theta_i|\rangle_c)^2\rangle_c, \quad i=x,y,z.$$

3. Mean square velocity of the zero-lines. We calculate the mean square of the displacement velocity of the zero-lines in the case when the distribution of the ray intensity $J(\mathbf{n},\omega)$ of (24) is in the form of a product of independent functions of the frequency and of the angular variables. The elements $\overline{k\omega n_i}$ and $\overline{kn_i}$ of the submatrix $\hat{\mathbf{K}}_1$ of (32) then become equal to zero in the system of coordinates in which the matrix of the spatial derivatives, $\hat{\mathbf{p}}=\{p_{ij}\}$, is diagonal, and we find from (7), (16), (21), and (33) that

$$\langle\mathbf{w}^2\rangle_c=2(\overline{\omega^2}-\overline{\omega}^2)(G/H)\delta_w, \quad (46)$$

where G and H are coefficients in the characteristic equation (35) and

$$\delta_w=\oint\!\!\oint\left(\frac{q_1^2}{q}\right)d^2n\,d^2w/2\oint\!\!\oint q\,d^2n\,d^2m. \quad (47)$$

Here $q_1^2\equiv(\lambda_x n_x^2+\lambda_y n_y^2+\lambda_z n_z^2)/G$ while q is given by (41).

For an isotropic distribution of the ray intensity we have $\delta_w=1$ and the mean square of the zero-line velocity is equal to

$$\langle\mathbf{w}^2\rangle_c=6(\overline{\omega^2}-\overline{\omega}^2)/k_1^2. \quad (48)$$

In the case of white noise in a frequency band $\Delta\omega$ Eq. (48) for the velocity takes the form ($\omega_0\equiv\overline{\omega}$)

$$\langle\mathbf{w}^2\rangle_c=\frac{1}{2}\frac{(\Delta\omega)^2}{k_1^2}=c_0^2\frac{(\Delta\omega)^2}{2\omega_0^2}\Bigg/\left[1+\frac{1}{3}\left(\frac{\Delta\omega}{2\omega_0}\right)^2\right], \quad (49)$$

and the velocity of the zero-lines for $\Delta\omega\ll\omega_0$ is thus much smaller than c_0.

If one narrows the angular distribution to a planar one when, for instance, $\lambda_z\ll\lambda_y\leqslant\lambda_x$ we get for $\lambda_z\to 0$ from (46) and (47) the estimate

$$(2\lambda_x)^{-1}(\overline{\omega^2}-\overline{\omega}^2)\mathcal{J}\left(\frac{\lambda_y}{\lambda_z}\right)\leqslant\langle\mathbf{w}^2\rangle_c\leqslant(\lambda_y)^{-1}(\overline{\omega^2}-\overline{\omega}^2)\mathcal{J}\left(\frac{\lambda_x}{\lambda_z}\right), \quad (50)$$

where

$$\mathcal{J}(x)=16\pi^2\int_0^{\pi/2}\frac{1}{\sin\varphi}\tan^{-1}\left(\frac{\sin\varphi}{x^{1/2}}\right)d\varphi\approx 16\pi^2\ln\left(\frac{4}{x^{1/2}}\right), \quad x\ll 1.$$

For $\lambda_z\to 0$ the zero-lines are almost parallel to the z axis and (50) gives an estimate for the mean square of the velocity of the dislocations in the $z=$ const plane which thus tends to infinity as $\lambda_z\to 0$.

In the limit of monochromatic radiation we have $\overline{\omega}^2-\overline{\omega}^2=0$ and the velocity (46) of the zero-lines is equal to zero.

CONCLUSION

The number density of the dislocations, and the length and the mean square velocity of the zero-lines are some of the many averages on the zero-lines which characterize the behavior of the dislocations and the zeroes of a random wave field and they have an obvious physical meaning. For instance, in the case of monochromatic acoustic radiation propagating in a medium with a constant density ρ the mean square of the field gradient, $\langle|\nabla\psi|^2\rangle_c$ is proportional to the mean square of the vibrational velocity of the particles in the medium on the zero-lines and the mean value of the Jacobian $C(\xi)=C$ of (11) on the zero-lines, $\langle C(\xi)\rangle_c=\langle C^2(\xi)\rangle/\langle C(\xi)\rangle$ [see (16) and (21)] differs only by a constant factor $2\rho c_0$ from the "intensity" of the vortex motion of the energy on the zero-lines:

$$C=2\rho c_0|\mathrm{rot}\,\mathbf{I}|,\quad \mathbf{I}=(2\rho c_0)^{-1}\mathrm{Im}(\psi^*\nabla\psi),$$

where $\mathbf{I}$ is the flux density of the power of the acoustic field. For fields with Gaussian statistics these and many other quantities can be evaluated using the formulae given above.

As an exotic characteristic we give (without derivation) the value of the mean square of the curvature (4) of the zero-lines for a spherically symmetric distribution of the ray intensity $J(\mathbf{n},\omega)$ [to evaluate it we need a 18-dimensional (!) probability density function $W_{18}(\xi)$]:

$$\langle K^2\rangle_c={}^2/_{15}k_1^2[9\overline{\omega^4}-5(\overline{\omega^2})^2/(\overline{\omega^2})^2. \quad (51)$$

[k_1 is given by (38)] which in the case of white noise in a frequency band $\Delta\omega$ takes the form

$$\langle K^2\rangle_c={}^8/_{15}k_0^2(1+11x/3+14x^2/45)/(1+x/3),$$

$$x\equiv(\Delta\omega/2\omega_0)^2\leqslant 1. \quad (52)$$

For $x\to 0$ (almost monochromatic field) we have $\langle K^2\rangle_c=8k_0^2/15$ and the mean square radius of curvature of the zero-lines is a quantity which is $\approx\lambda/4{,}6$, i.e., roughly a quarter of the wavelength λ.

The approach developed in the present paper may turn out to be useful also for studying the behavior of the above mentioned $\mathbf{C}$ lines [$\psi=$ const; see before Eq. (11)] which are connected with the local maxima and other interesting features of the wave field.

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