Coherent interaction of planar defects in a superconductor

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It is shown that the transition temperature T_c of a superconductor oscillates as a function of the distance d between two planar defects present in it. The dependence of the oscillations on the amplitude for electron scattering by a planar defect is investigated. It is shown that the amplitude of the oscillations exhibits critical behavior near the Anderson transition, described in Ref. 8, at which the order parameter becomes localized at atomic distances near planar defects.

1. INTRODUCTION

Quantum oscillations of the superconducting transition temperature T_c have often been observed in coated films,[1,2] sandwiches,[1] and superlattices.[3] One possible experimental geometry is shown in Fig. 1a: an interlayer B of thickness d is introduced into a superconductor A (consisting of a material different from b) of thickness $L \lesssim \xi_0$, where ξ_0 is the coherence length; the transition temperature T_c oscillates as a function of d. This effect was studied theoretically by Kagan and Dubovskii.[4] Their approach was to reduce the problem to calculating T_c for a thin film with a variable boundary condition. The material B was assumed to be nonsuperconducting and the proximity effect was completely ignored; the change in T_c was then related to the change in the density of states of the superconductor A; the density of states depends, to order a/L (a is the interatomic distance), on the boundary condition at the A and B interface; and this boundary condition, in turn, varies periodically as d changes, depending on how commensurate d is with the electron wavelength. These considerations give a reasonable qualitative explanation of the effect, though a systematic analysis does not substantiate them.[5]

In this paper we predict that the transition temperature T_c of the superconductor A, which contains two planar defects, oscillates as a function of the distance d ($d \ll L \ll \xi_0$) between the defects (Fig. 1b). Comparison with Fig. 1a shows that the materials A and B need not be different in order for quantum oscillations to exist, in contrast to Kagan and Dubovskii's picture,[4] where this difference is important. Actually, in both cases (Fig. 1a and b) the oscillations of T_c are a consequence of the coherent interaction of the two planar defects.

2. BASIC RESULTS

Let the superconductor A be a metal with a quadratic spectrum $\varepsilon(k) = k^2/2m$ and Fermi momentum k_F, and let the planar defects be δ-like in the transverse direction, along which we orient the z-axis (the point $z = 0$ lies midway between the defects). Then the boundary condition for the single-particle wave function $\varphi(z)$ of the transverse motion [the complete wave function is $\Psi(\mathbf{r}) = \varphi(z)\exp(ik_\parallel r_\parallel)$, where $k_\parallel = (k_x, k_y)$ and $r_\parallel = (x,y)$] has the following form at the point $z = d/2$:

$$\varphi(d/2+0) = \varphi(d/2-0),\ \varphi'(d/2+0) - \varphi'(d/2-0) = \varkappa\varphi(d/2)$$

and similarly at the point $z = -d/2$. The parameter $\varkappa$ has the dimension of a wave vector and characterizes the scattering properties of the defect: For $\varkappa > 0$ the defect has the form of a potential barrier and for $\varkappa < 0$ the defect has the form of a potential well.

The form of the quantum oscillations depends strongly on the ratio of $\varkappa$ and k_F (Fig. 2): For $\varkappa \gg k_F$ the oscillations have a sawtoothed form (Fig. 2a); for $0 < \varkappa \ll k_F$ the oscillations are nearly sinusoidal with amplitude $\sim(\varkappa/k_F)^2$ and they occur against the background of a constant shift $\sim\varkappa/k_F$ (Fig. 2b); for $\varkappa < 0$, $|\varkappa| \ll k_F$ (Fig. 2c) the form of the oscillations is the same as in the preceding case, but due to the existence of bound states near the defects a smooth component, varying on the scale of the localization radius $|\varkappa|^{-1}$ of the bound states, appears. For $\varkappa < 0$, $|\varkappa| \gg k_F$ the oscillations have a sawtoothed form, but in contrast to the case $\varkappa \gg k_F$ these oscillations are damped (Fig. 2d).

It is not difficult to give a physical interpretation of these results. For $|\varkappa| \gg k_F$ the transmission coefficient of a planar defect is small and there are two weakly bound systems (Fig. 3b): a three-dimensional superconductor with a continuous spectrum for $|z| > d/2$ and a quasi-two-dimen-

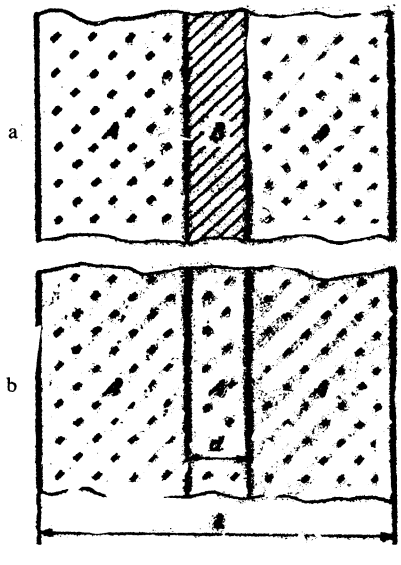

FIG. 1. Geometry of the experiment for observing quantum oscillations of T_c.

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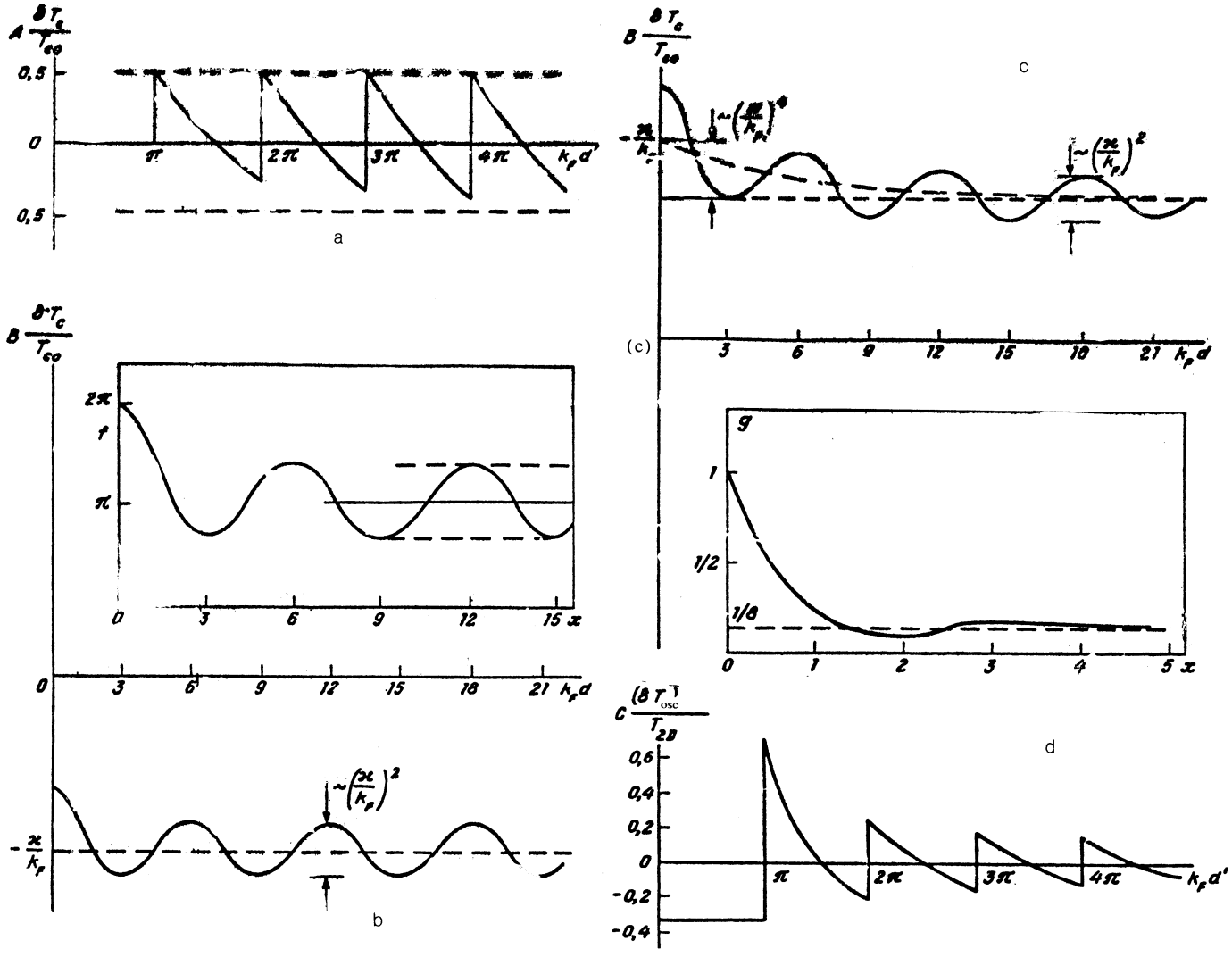

FIG. 2. Form of the quantum oscillations for the geometry in Fig. 1b with different ratios of $\varkappa$ and k_F: a) $\varkappa \gg k_F$, b) $0 < \varkappa \ll k_F$, c) $\varkappa < 0$, $|\varkappa| \ll k_F$, d) $\varkappa < 0$, $|\varkappa| \gg k_F$; here $A = \lambda_0 k_F L/\pi$, $B = \lambda_0 k_F L$, and $C = (\pi/16)^2 (|\varkappa|/k_F)^5 \lambda_0 k_F L$.

sional system for $|z| < d/2$, whose spectrum consists of a collection of two-dimensional bands of size-quantization levels, depending on the longitudinal momentum $k_\parallel$. As d increases the splitting between the levels decreases, and when the bottom of the next two-dimensional band passes through the Fermi level the value of T_c increases abruptly as a result of the abrupt increase in the density of states.[1)] As $|\varkappa|$ decreases the size-quantization levels broaden, and this results in smoothing of the oscillations.

For $|\varkappa| \ll k_F$ it is convenient to consider the electronic wave functions as being the result of repeated reflection of a plane wave from the defects. Interference of the incident wave $A \exp ik_F z$ and the doubly reflected wave $A_1 \exp(2ik_F d + ik_F z)$ plays the main role (two reflections are necessary in order to gather information about the presence of two defects). The result of interference is a periodic function of d, manifested in the form of the oscillations of T_c. Since with each reflection the amplitude of the reflected wave is a fraction $\sim \varkappa/k_F$ of the incident wave, we have $A_1 \sim A(\varkappa/k_F)^2$ and the amplitude of the oscillations is found to be of second order in $\varkappa/k_F$.

For a single planar defect (inset in Fig. 4), as $\varkappa$ varies in the direction of negative values as Anderson quasiphase transition, described in detail in Refs. 8, 5, and 7—a transition into a state in which the order parameter is localized at the atomic distance near a defect—occurs at the point

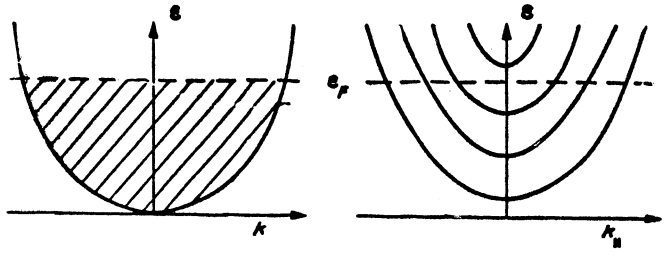

FIG. 3. For $\varkappa \gg k_f$ there are two independent subsystems: the three-dimensional system for $|z| > d/2$ with a continuous spectrum (on the left) and a quasi-two-dimensional system for $|z| < d/2$, whose spectrum consists of a collection of two-dimensional bands (on the right).

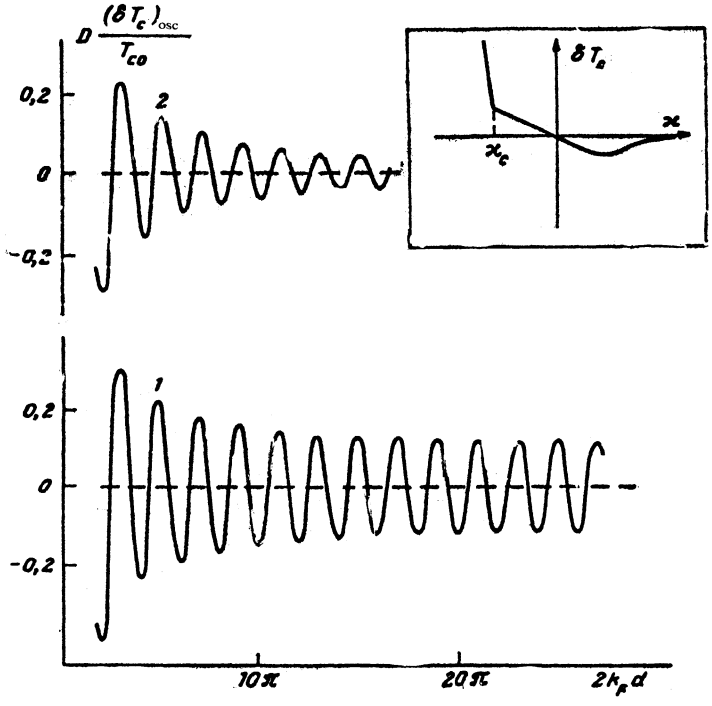

FIG. 4. Form of quantum oscillations near transition $|\varkappa - \varkappa_c| \ll \varkappa_c$ curve *1*—$\varkappa > \varkappa_c$, curve *2*—$\varkappa < \varkappa_c$. The inset shows T_c versus for a single defect; here $D = \lambda_0 k_F L/\pi$.

$\varkappa_c = -4k_F/\pi$. Near the transition the oscillating part of T_c has the following form for $k_F d \gg 1$ (Fig. 4):

$$\frac{(\delta T_c)_{osc}}{T_{c0}} = \frac{1}{\lambda_0 k_F L}\left\{\gamma(2k_F d)\,\theta(\Delta\varkappa) + \text{const}\,\frac{\gamma(2k_F d)}{|\Delta\varkappa| d}\right\}, \quad (1)$$

where $\gamma(x)$ is a periodic function with period 2π and $\Delta\varkappa = \varkappa - \varkappa_c$. As one can see from Eq. (1), the asymptotic form of the oscillations for large values of d changes at the point $\varkappa_c$: The oscillations are periodic for $\varkappa > \varkappa_c$ and decay as $1/d$ for $\varkappa < \varkappa_c$. For small d the amplitude of the oscillations diverges at the transition point [neglecting the broadening of the transition, which is of the order of $\delta\varkappa \sim \varkappa_c (a/L)^{1/2}$; see Ref. 8].

The experimental results for twinning planes[9] and the theoretical considerations in Ref. 8 indicate the possibility of increasing the superconducting transition temperature T_c by introducing planar defects into superconductors. The present analysis shows that the interaction of planar defects can increase T_c even in the case when one defect lowers T_c; the relative increase in T_c is of the order of a/L and can be significant at the limit of applicability of the theory ($L \sim d \sim a$). From this viewpoint the situation under discussion (Fig. 1b) is more advantageous than the case studied in Ref. 4 (Fig. 1a), when oscillations of T_c with amplitude $\sim a/L$ are superposed on a bulk contribution $\sim d/L$ (Ref. 5), which in actual situations (T_c of the material B is lower than T_c of the material A) is negative.

3. INITIAL EQUATIONS

A method for calculating T_c of a superconductor containing planar defects was devised in Refs. 5 and 8. If the transverse size d_0 of a defect is small compared with L ($L \lesssim \xi_0$), where L is the thickness of the superconductor in the case of a single effect or the distance between periodically arranged defects, then the Gor'kov equation can be solved with minimum assumptions about the nature of the defect by using d_0/L as the small parameter. The simplest expression for T_c is obtained when there are no bound states near a defect:[5,8]

$$\frac{\delta T_c}{T_{c0}} = \frac{T_c - T_{c0}}{T_{c0}} = \frac{1}{\lambda_0^3 L}\int dz V_0 N(z)\,[V(z)N(z) - V_0 N_0], \quad (2)$$

where $N(z)$ is the local density of states at the Fermi level, $V(z)$ is the coordinate-dependent four-fermion interaction constant of the BCS theory, N_0 and V_0 are the values of $N(z)$ and $V(z)$ in the absence of a defect, and $\lambda_0 = V_0 N_0$; the integral extends over the neighborhood of the defect. The local density of states is expressed in terms of the eigenfunctions $\Psi_n(\mathbf{r})$ and eigenvalues ε_n of the single-particle Hamiltonian:

$$N(\varepsilon, \mathbf{r}) = \sum_n |\Psi_n(\mathbf{r})|^2 \delta(\varepsilon - \varepsilon_n). \quad (3)$$

If m local levels are present near a planar defect (for fixed $k_\parallel$), the transition temperature T_c of the system is determined by the point at which a nontrivial solution of the linear system of $(m+1)$ equations for $\psi(0)$ and Δ_s ($s = 1, 2, \ldots, m$) appears:

$$\Delta_s = \left[\sum_{s'=1}^{m} \lambda_{ss'}\Delta_{s'} + \lambda_{s0}\psi(0)\right]\ln\frac{1.14\omega_D}{T},$$

$$\psi(0)\left[1 - \lambda_0 \ln\frac{1.14\omega_D}{T}\right] = \frac{d_0}{L}\left[\sum_{s'=1}^{m} \lambda_{0s'}\Delta_{s'} + \lambda_{00}\psi(0)\right]\ln^2\frac{1.14\omega_D}{T}, \quad (4)$$

where λ_{ij} is expressed in terms of $V(z)$, $N(z)$, the eigenfunctions $\varphi_s(z)$, and the eigenvalues $\varepsilon_s(k_\parallel)$ of the bound states:

$$\lambda_{ss'}=N_{2D}^{s'}(0)\int dzV(z)\,|\varphi_s(z)|^2|\varphi_{s'}(z)|^2,$$

$$\lambda_{s0}=\int dz[V(z)N(z)-V_0N_0]\,|\varphi_s(z)|^2,$$

$$\lambda_{0s'}=d_0^{-1}N_{2D}^{s'}(0)\int dzV(z)[N(z)-N_{loc}(z)]V_0|\varphi_{s'}(z)|^2, \tag{5}$$

$$\lambda_{00}=d_0^{-1}\int dzV_0[N(z)-N_{loc}(z)][V(z)N(z)-V_0N_0], \tag{6}$$

$$N_{loc}(z)=\sum_{s=1}^{m}N_{2L}^{s}(0)\,|\varphi_s(z)|^2,$$

where $N_{2D}^{s}(0)$ is the two-dimensional density of states, corresponding to the spectrum $\varepsilon_s(k_\parallel)$, at the Fermi level. The expressions (4)–(6) follow from the formulas (25) of Ref. 5 after $K_{loc}(zz')$ in the form (4) is substituted into it, where only terms with $s=s'$, corresponding to local levels, are retained in the sum. The parameter d_0, having the dimension of length, is introduced in order to make the quantities λ_{ij} dimensionless and actually does not appear in the equations; in the case at hand; it is convenient to take $d_0\sim a$, and then all $\lambda_{ij}\sim 1$.

In what follows the formulas (2)–(6) are applied to a "planar defect," consisting of two elementary planar defects separated by a distance d (Fig. 1c). We considered the quantity $V(z)$ to be constant and equal to V_0, since the difference between $V(z)$ and V_0 will not cause quantum oscillations, and the effect of such a difference can be easily taken into account qualitatively in the final results.

4. FORM OF QUANTUM OSCILLATIONS FOR $\varkappa\gg k_F$

In the case $\varkappa>0$ there are no bound states near planar defects, and T_c is given by the formula (2); the difference between $N(z)$ and N_0 is determined by the states of the continuous spectrum (we shall designate this situation by the index $\langle\langle c\rangle\rangle$):

$$N(z)=N_0+N_c(z). \tag{7}$$

Finding the eigenfunctions $\Psi_n(\mathbf{r})$ and eigenvalues ε_n of the single-particle problem, substituting them into Eq. (3), carrying out the trivial integration over $k_\parallel$, switching from summation over the transverse quantum number to integration, and neglecting terms of order $\sim d/L$, we obtain for $N_c(z)$ for $|z|-d/2<0$

$$N_c(z)=\frac{m}{(2\pi)^2}\int_0^{k_F}dq\left\{2\cos^2 qz\,\frac{\varkappa q\sin qd-\varkappa^2\cos^2(qd/2)}{u_1^+(q)u_1^-(q)}+2\sin^2 qz\,\frac{-\varkappa q\sin qd-\varkappa^2\sin^2(qd/2)}{u_2^+(q)u_2^-(q)}\right\}, \tag{8a}$$

and for $x=|z|-d/2>0$

$$N_c(z)=\frac{m}{(2\pi)^2}\int_0^{k_F}dq\left\{\frac{v_1(q)\cos 2qx+v_2(q)\sin 2qx}{u_1^+(q)u_1^-(q)}+\frac{v_3(q)\cos 2qx+v_4(q)\sin 2qx}{u_2^+(q)u_2^-(q)}\right\}, \tag{8b}$$

where we have introduced the following notation:

$$\begin{aligned}
u_1^{\pm}(q)&=(\varkappa\pm iq)\cos(qd/2)-q\sin(qd/2),\\
u_2^{\pm}(q)&=(\varkappa\pm iq)\sin(qd/2)+q\cos(qd/2),\\
v_1(q)&=q^2\cos^2(qd/2)-[\varkappa\cos(qd/2)-q\sin(qd/2)]^2,\\
v_2(q)&=2q\cos(qd/2)\,[\varkappa\cos(qd/2)-q\sin(qd/2)],\\
v_3(q)&=q^2\sin^2(qd/2)-[\varkappa\sin(qd/2)+q\cos(qd/2)]^2,\\
v_4(q)&=2q\sin(qd/2)\,[\varkappa\sin(qd/2)+q\cos(qd/2)].
\end{aligned} \tag{9}$$

For $\varkappa\gg k_F$ the integrands in Eqs. (8) are localized near the points

$$q_n=\pi n/d',\quad d'=d+2/\varkappa\quad(n=1,2,\ldots) \tag{10}$$

and can be approximated by set of delta functions [in order to simplify the computations it is convenient to add the fractions in Eqs. (8)]. As a result we find

$$N_c(z)=\begin{cases}N_0\dfrac{\pi}{k_Fd'}\displaystyle\sum_{n=0}^{M}[1-(-1)^n\cos 2q_nz]-N_0, & |z|<d/2,\\ -N_0\dfrac{\sin 2k_Fx}{2k_Fx}, & x=|z|-d/2>0,\end{cases} \tag{11}$$

where $M=[k_Fd'/\pi]$ and $[\ldots]$ indicates the integer part of the enclosed number. Substituting the expression (11) into Eq. (2) we obtain

$$\frac{\delta T_c}{T_{c0}}=\frac{\pi}{\lambda_0k_FL}h\left(\frac{k_Fd'}{\pi}\right),\qquad h(x)=[x]\left(\frac{[x]+1/2}{x}-1\right). \tag{12}$$

The oscillations have a sawtoothed form (Fig. 2a).

5. QUANTUM OSCILLATIONS FOR $0<\varkappa<k_F$

We represent $N_c(z)$ in the form

$$N_c(z)=F(z)+G(z), \tag{13}$$

where $F(z)$ is determined by the expressions (8) with the integration extending from 0 to ∞. It can be shown by the method of contour integration (see the Appendix) that for $\varkappa>0$

$$F(z)\equiv 0. \tag{14}$$

The function $G(z)$ is calculated by expanding the integrands in Eqs. (8) in powers of $\varkappa/q$. Substitution into Eq. (2) gives the result

$$\frac{\delta T_c}{T_{c0}}=\frac{1}{\lambda_0k_FL}\left\{-\frac{\varkappa}{k_F}+\left(\frac{\varkappa}{2k_F}\right)^2 f(2k_Fd)\right\}, \tag{15}$$

i.e., the oscillations are of second order in $\varkappa/k_F$. The function $f(x)$, determining the form of the oscillations (Fig. 2b), is given by the expression

$$f(x)=-\sin x+\int_0^{x}dz[\pi-\mathrm{Si}(z)-\mathrm{Si}(x-z)]^2/2+\int_0^{\infty}dz[\pi-\mathrm{Si}(z)-\mathrm{Si}(z+x)]^2 \tag{16}$$

[$\mathrm{Si}(x)$ is the sine integral] and has the following asymptotic form:

$$f(x)=\begin{cases}2\pi-x, & x\ll 1,\\ \pi-\sin x, & x\gg 1.\end{cases} \tag{17}$$

6. OSCILLATIONS OF T_c IN THE PRESENCE OF SHALLOW LEVELS ($\varkappa<0, |\varkappa| \ll k_F$)

For $\varkappa<0$ bound states with

$$\varepsilon_1=-\varkappa^2\beta_1^2/2m, \quad \varepsilon_2=-\varkappa^2\beta_2^3/2m,$$

where β_1 and β_2 are solutions of the equations

$$\tanh\frac{|\varkappa|d}{2}\beta_1=\frac{1-\beta_1}{\beta_1}, \quad \coth\frac{|\varkappa|d}{2}\beta_2=\frac{1-\beta_2}{\beta_2}, \tag{18}$$

appear near the defects. The level ε_2 exist only for $|\varkappa|d>2$, and for $|\varkappa|d<2$ it emerges into the continuous spectrum.

The transition temperature is determined by the expressions (4) and (6) with $m=2$. Neglecting effects of order $\sim a/L$ we obtain

$$T_c=\max(T_{c0}, T_{2D}), \tag{19}$$

where T_{2D} is the transition temperature in a system of Tamm levels neglecting their interaction with the volume:

$$T_{2D}=1.14\omega_D\exp(-1/\lambda_{2D}).$$
$$\lambda_{2D}=\frac{\lambda_{11}+\lambda_{22}}{2}+\left[\left(\frac{\lambda_{11}-\lambda_{22}}{2}\right)^2+\lambda_{12}\lambda_{21}\right]^{1/2}. \tag{20}$$

For $T_{c0}>T_{2D}$ the order parameter propagates through the entire system; for $T_{c0}<T_{2D}$ the order parameter is localized on the atomic scale near defects.[5,7,8] The last condition is realized at $\varkappa<\varkappa_c=-4k_F/\pi$ for large d ($k_Fd\gg1$) and for $\varkappa<\varkappa_c/2$ for small d ($k_Fd\ll1$). Neglecting terms $\sim a/L$ results in smearing of the transition over a scale

$$\delta\varkappa\sim\varkappa_c(a/L)^{1/2}.$$

In the delocalized state ($T_c>T_{2D}$), neglecting terms $\sim a/L$ leads to the following result for T_c in the presence of two local levels ($|\varkappa|d>2$):

$$\frac{\delta T_c}{T_{c0}}=\frac{d_0}{L}\frac{1}{\lambda_0^3}\left[\lambda_{00}+\frac{\lambda_{12}\lambda_{20}\lambda_{01}+\lambda_{21}\lambda_{02}\lambda_{10}-\lambda_{01}\lambda_{10}(\lambda_{22}-\lambda_0)-\lambda_{02}\lambda_{20}(\lambda_{11}-\lambda_0)}{(\lambda_{11}-\lambda_0)(\lambda_{22}-\lambda_0)-\lambda_{12}\lambda_{21}}\right]. \tag{21}$$

In the presence of one level ($|\varkappa|d<2$)

$$\frac{\delta T_c}{T_{c0}}=\frac{d_0}{L}\frac{1}{\lambda_0^3}\left(\lambda_{00}+\frac{\lambda_{01}\lambda_{10}}{\lambda_0-\lambda_{11}}\right). \tag{22}$$

In the absence of levels only the term with λ_{00} remains in Eqs. (21) and (22), and we return to the formula (2).

For $\varkappa<0$ the local density of states can be represented in the form

$$N(z)=N_0+N_c(z)+N_{loc}(z), \tag{23}$$

where $N_c(z)$ and $N_{loc}(z)$ are determined by the expressions (8) and (6). We now separate $N_c(z)$ as in Eq. (13). For $\varkappa<0$ the following result holds for $F(z)$ (see the Appendix):

$$F(z)=-N_{2D}(0)[|\varphi_1(z)|^2+|\varphi_2(z)|^2\theta(|\varkappa|d-2)]=-N_{loc}(z). \tag{24}$$

Substituting Eqs. (23), (13), and (24) into Eq. (21) leads to the result

$$\frac{\delta T_c}{T_{c0}}=\frac{1}{\lambda_0^3L}\left\{V_0^2\int dz[N_0+G(z)]G(z)+V_0N_{2D}(0)\frac{\lambda_{10}^2+\lambda_{20}^2}{\lambda_0}\right\}. \tag{25}$$

The first term in parentheses coincides with the result for $\varkappa>0$, the second term describes the smooth component of the function $T_c(d)$ and varies over the scale $|\varkappa|^{-1}$ (it is written out to lowest order in $\varkappa$). Substituting Eqs. (23), (13), and (24) into Eq. (22) gives a different result from Eq. (25) in that the term λ_{20}^2 does not occur. The calculation of λ_{10} and λ_{20} does not present any difficulties, since $|\varphi_1(z)|^2$ and $|\varphi_2(z)|^2$ vary over the scale $|\varkappa|^{-1}$, while the function

$$G(z)=N(z)-N_0$$

is localized on a scale k_F^{-1} near the planar defects. Taking into account the result (15) for $\varkappa>0$, we arrive at the formula

$$\frac{\delta T_c}{T_{c0}}=\frac{1}{\lambda_0k_FL}\left\{-\frac{\varkappa}{k_F}+\left(\frac{\varkappa}{2k_F}\right)^2f(2k_Fd)+\pi\left(\frac{\varkappa}{k_F}\right)^4g(|\varkappa|d)\right\}, \tag{26}$$

where the function $f(x)$ is the same function as in Eq. (15) and the function $g(x)$ (Fig. 2c) is determined by the expression

$$g(|\varkappa|d)\equiv\left(\frac{\beta_1^2}{|\varkappa|d\beta_1+1-|\varkappa|d/2}\right)^2+\left(\frac{\beta_2^2}{|\varkappa|d\beta_2+1-|\varkappa|d/2}\right)^2\theta(|\varkappa|d-2) \tag{27}$$

[by virtue of Eq. (18) β_1 and β_2 are functions of $|\varkappa|d$] and has the following asymptotic forms:

$$g(x)=\begin{cases}1-3x, & x\ll1,\\ 1/8[1+(5/4)x^2-6x+6)e^{-x}], & x\gg1.\end{cases} \tag{28}$$

7. OSCILLATIONS OF T_c IN THE STRONG LOCALIZATION REGIME ($\varkappa<0, |\varkappa|\gg k_F$)

In the case $|\varkappa|/k_F\gg1$ the scale $|\varkappa|^{-1}$ is small compared with the period of the oscillations. Confining our attention to the region $d\gg|\varkappa|^{-1}$, we neglect the overlapping of the wave functions of the bound states localized on the two defects. Then in Eqs. (4) and (5)

$$\lambda_{11}=\lambda_{12}=\lambda_{21}=\lambda_{22}, \quad \lambda_{01}=\lambda_{02}, \quad \lambda_{10}=\lambda_{20}. \tag{29}$$

For $\varkappa\gg k_F$ the order parameter is localized near the planar defects, and for this reason T_c in the zeroth-order approximation is equal to T_{2D} [see Eq. (20)]. To first order in a/L we have[2)]

$$\frac{T_c-T_{2D}}{T_{2D}}=\frac{d_0}{L}\frac{2\lambda_{10}\lambda_{01}}{\lambda_{2D}^3(\lambda_{2D}-\lambda_0)}, \quad \lambda_{2D}=2\lambda_{11}. \tag{30}$$

The integrals for λ_{10} and λ_{01} in Eq. (5) are easy to calculate because the wave functions of the bound states are strongly localized near $z=\pm d/2$; the oscillations of T_c are determined by the behavior of the quantity $N_c(z)$, for which the expression (11) is valid, near the points $z=\pm d/2$. As a result

$$\frac{(\delta T_c)_{\text{osc}}}{T_{2D}} = \frac{1}{\pi^2 \lambda_0 k_F L} 256 \left(\frac{k_F}{|\varkappa|} \right)^5 h_1\left(\frac{k_F d'}{\pi} \right), \qquad (31)$$

where

$$h_1(x) = \frac{(2[x]+1)([x]+1)[x]}{6x^3} - \frac{1}{3}.$$

The oscillations have a sawtoothed form, but their amplitude decreases as $1/d$ (Fig. 2d).

8. ASYMPTOTIC BEHAVIOR OF THE OSCILLATIONS FOR LARGE d AND THE NEIGHBORHOOD OF THE PHASE TRANSITION

For $|\varkappa|d \gg 1$, $k_F d \gg 1$ the form of the oscillations can be determined for arbitrary ratio of k_F and $\varkappa$. Since the overlapping of the wave functions of the bound states localized on different defects can be neglected, the relations (29) hold. For $0 > \varkappa > \varkappa_c$ the formula (21) implies

$$\frac{\delta T_c}{T_{c0}} = \frac{d_0}{\lambda_0^3 L}\left[\lambda_{00} + \frac{2\lambda_{10}\lambda_{01}}{\lambda_0 - \lambda_{2D}} \right], \quad \lambda_{2D} = 2\lambda_{11}. \qquad (32)$$

For $\varkappa > 0$ only the first term in parentheses remains in Eq. (32), while for $\varkappa < \varkappa_c$ the result (30) holds.

Oscillations which are undamped for large d are related to the quantity λ_{00}, they are related to the integral appearing in it:

$$\int_{-\infty}^{\infty} N_c(z)\, dz = \frac{m}{(2\pi)^2} 2\left\{ -k_F d + \frac{d}{2} \int_0^{2k_F} dq\, H(q, qd) \right\} + O(d^0), \qquad (33)$$

where

$$H(q, x) = \left[\frac{q^2}{q^2 - \varkappa q \sin x + \varkappa^2 \cos^2(x/2)} + \frac{q^2}{q^2 + \varkappa q \sin x + \varkappa^2 \sin^2(x/2)} \right]$$

[for large d the integral of $N_c^2(z)$ becomes constant]. We now expand the function $H(q,x)$ in a Fourier series in the variable x. Then in Eq. (33) there appears a sum of integrals of products of rapidly oscillating exponentials by slowly varying functions, whose asymptotic behavior is determined by the upper limit of integration.[10] The leading term in the asymptotic expansion contains the small parameter $1/d$, which cancels the factor of d in front of the integral (33); in view of the power-law character of the Fourier coefficients, the series arising in Eq. (33) is the expansion of a logarithm and sums to a finite expression, so that the expression in the braces in Eq. (33) can be put into the form

$$\gamma(2k_F d) = \operatorname{arctg} \frac{\sin(2k_F d + 2\varphi)}{1 + 4k_F^2/\varkappa^2 - \cos(2k_F d + 2\varphi)},$$

$$\varphi = \operatorname{arctg}(2k_F/\varkappa). \qquad (34)$$

As a result, the oscillations of T_c for $\varkappa > \varkappa_c$ have the asymptotic form

$$\frac{(\delta T_c)_{\text{osc}}}{T_{c0}} = \frac{1}{\lambda_0 k_F L} \gamma(2k_F d). \qquad (35)$$

For $\varkappa < \varkappa_c$ the parameter λ_{00} is absent in the expression for T_c (30). The oscillating part is contained in the quantities λ_{01} and λ_{10} and is related to the integral

$$V_0 \int_{-\infty}^{\infty} dz\, N_c(z) |\varphi_1(z)|^2 = \frac{\lambda_0}{4k_F}\left\{ -\int_0^{k_F} dq \frac{8q^2 + 4\varkappa^2}{4q^2 + \varkappa^2} + \int_0^{k_F} dq \frac{4q^2}{4q^2 + \varkappa^2} - H(q, qd) \right\}, \qquad (36)$$

where the function $H(q,x)$ is the same function as in Eq. (33). Since Eq. (36) does not contain a compensating factor of d, the oscillations for $\varkappa < \varkappa_c$ decay as $1/d$:

$$\frac{(\delta T_c)_{\text{osc}}}{T_{2D}} = \frac{1}{\lambda_0 L} \frac{256 k_F^3}{\pi^2 \varkappa^2 (\varkappa^2 + 4k_F^2)} \times \frac{\pi|\varkappa| + 4k_F - 4\varkappa \tan^{-1}(2k_F/\varkappa)}{\pi|\varkappa| - 4k_F} \times \frac{1}{\varkappa|d}\, \gamma(2k_F d). \qquad (37)$$

Near the phase transition $\varkappa \approx \varkappa_c$ the results (30) and (32) can be combined into the form

$$\frac{T_c - T_{c0}}{T_{c0}} = \frac{\delta\lambda}{\lambda_0^2}\theta(\delta\lambda) + \frac{d_0}{\lambda_0^3 L}\lambda_{00}\theta(-\delta\lambda) + \frac{d_0}{\lambda_0 L}\frac{2\lambda_{10}\lambda_{01}}{|\delta\lambda|}, \qquad (38)$$

where $\delta\lambda = \lambda_{2D} - \lambda_0$, and

$$(a/L)^{1/2} \ll |\delta\lambda| \ll \lambda_0.$$

Keeping the damped oscillations superposed on the periodic component is justified because of the $1/|\delta\lambda|$ divergence. Substituting the asymptotic forms of the integrals (33) and (36) gives for the oscillations of T_c the result (1) with the constant

$$\text{const} = \frac{8\pi^2}{4+\pi^2}\left(1 - \frac{2}{\pi} \tan^{-1}\frac{\pi}{2} \right). \qquad (39)$$

APPENDIX. CALCULATION OF $F(z)$

The poles of the integrands in Eq. (8) are determined by the roots of the equations

$$u_1^{\pm}(q) = 0, \quad u_2^{\pm}(q) = 0. \qquad (A1)$$

The arrangement of the poles for $\varkappa < 0$, $|\varkappa|d > 2$ is shown in Fig. 5: The poles corresponding to the upper and lower sign in (A1) (light and dark colored circles) lie, respectively, in the upper and lower half-planes; the poles on the imaginary axis, which correspond to bound states [see Eq. (18)] and lie in the "outside" half-plane, are exceptions. Representing the integrals in (8a) and (8b), respectively, as

$$\frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty} dq \left\{ 2\cos^2 qz \frac{-i\varkappa q\, e^{iqd} - \varkappa^2/2 - (\varkappa^2/2) e^{iqd}}{u_1^+(q) u_1^-(q)} + 2\sin^2 qz \frac{i\varkappa q\, e^{iqd} - \varkappa^2/2 + (\varkappa^2/2) e^{iqd}}{u_2^+(q) u_2^-(q)} \right\},$$

$$\frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty} dq \left\{ \frac{v_1(q) e^{2iqx} - iv_2(q) e^{2iqx}}{u_1^+(q) u_1^-(q)} + \frac{v_3(q) e^{2iqx} - iv_4(q) e^{2iqx}}{u_2^+(q) u_2^-(q)} \right\} \qquad (A2)$$

and moving the integration contour into the upper half-plane, it is not difficult to show that the contributions of the "familiar" poles (dark circles in Fig. 5) vanish because of

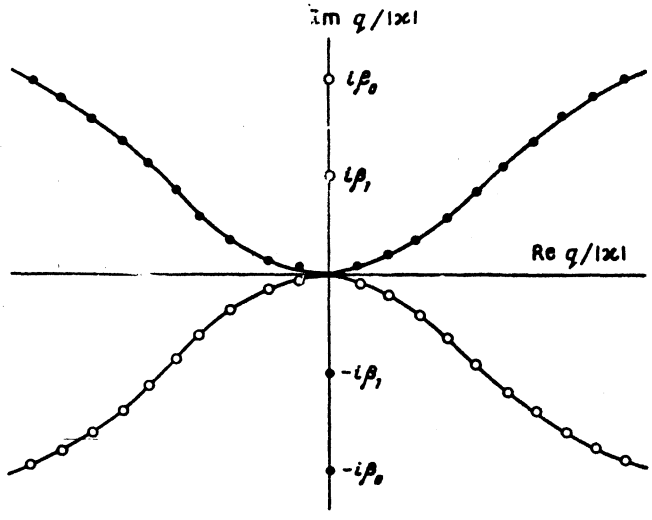

FIG. 5. Arrangement of the poles for the integrals (8) with $\varkappa < 0$, $|\varkappa| d > 2$: the light and dark colored circles designate the poles corresponding to the upper and lower signs, respectively, in Eq. (A1).

Eq. (A1). The contribution of the two "unfamiliar" poles can be expressed in terms of normalized wave functions of the bound states and written in the form (24).

As $|\varkappa|$ decreases the quantity β_2 changes sign at $|\varkappa| d = 2$ and the corresponding dark and light colored circles in Fig. 5 switch places; the contribution of the pole $i|\varkappa|\beta_2$ vanishes, but at the same time the contribution of the upper level to $N_{loc}(z)$ vanishes, so that the result (24) remains valid.

For $\varkappa > 0$ the "unfamiliar" poles in the upper half-plane are absent and the result (14) holds.

[1)] In the limit $\varkappa \to \infty$, T_c is determined by the largest of two quantities: the volume transition temperature T_{c0} and the transition temperature of a size-quantized film:[6] The amplitude of the oscillations is of order a/d and decreases with increasing d. This limit is not described by the theory presented, since for $\varkappa \gtrsim k_F (L/d)^{1/2}$ the starting expressions (see Sec. 3) are no longer applicable because the structure of the superconducting core changes significantly.[5,7] Thus in the situation under study the interaction between the subsystems is significant.

[2)] For $T_{2D} - T_{c0} \gtrsim T_{c0}$ corrections to the starting formulas (4) and (5) appear, but taking them into account is analogous to taking into account the dependence $V(z)$.

[1] V. M. Golyanov, M. N. Mikheeva, and M. B. Tsetlin, Zh. Eksp. Teor. Fiz. **68**, 736 (1975) [Sov. Phys. JETP **41**(2), 365 (1975)]; V. M. Golyanov and M. N. Mikheeva, Zh. Eksp. Teor. Fiz. **70**, 2236 (1976) [Sov. Phys. JETP **43**(6), 1167 (1976)].
[2] A. F. Orlov, A. K. Milai, and V. P. Dmitriev, Fiz. Tverd. Tela **18**, 1470 (1976) [Sov. Phys. Solid State **18**, 854 (1976)]; K. A. Osipov, A. F. Orlov, V. P. Dmitriev, and A. K. Milai, Fiz. Tverd. Tela **19**, 2226 (1977) [Sov. Phys. Solid State **19**, 1304 (1977)].
[3] S. A. Vitkalov, F. A. Pudonin, E. G. Sokol, and I. M. Suslov, Pis'ma Zh. Eksp. Teor. Fiz. **49**, 160 (1989) [JETP Lett. **49**, 188 (1989)].
[4] Yu. M. Kagan and L. B. Dubovskii, Zh. Eksp. Teor. Fiz. **72**, 646 (1977) [Sov. Phys. JETP **45**(2), 339 (1977)].
[5] I. M. Suslov, SFKhT **4**, 2098 (1991).
[6] J. M. Blatt and C. J. Thompson, Phys. Rev. Lett. **10**, 332 (1963).
[7] I. M. Suslov, SFKhT **4**, 1065 (1991).
[8] I. M. Suslov, Zh. Eksp. Teor. Fiz. **95**, 949 (1989) [Sov. Phys. JETP **68**(3), 546 (1989)].
[9] M. S. Khaikin and I. N. Khlyustikov, Pis'ma Zh. Eksp. Teor. Fiz. **138**, 191 (1983) [JETP Lett. **38**, 224 (1983)].
[10] A. B. Migdal and V. Krainov, *Approximation Methods in Quantum Mechanics*, Benjamin, New York, Moscow 1969.

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