

Antiproton annihilation in hydrogen at low temperatures

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The cross section for the annihilation of ultraslow antiprotons (with energies on the order of or less than 10^{-4} eV) with atomic hydrogen is calculated. Effects which are nonadiabatic with respect to the relative motion of the heavy particles are taken into account. The inverse Auger process ($\text{Pr} + e \rightarrow \text{H} + \bar{p}$) is also taken into account. The annihilation probability is found to be smaller by a factor of 10 or more than the values which have been calculated previously in the adiabatic approximation and under the assumption of a one-time production of protonium [B. R. Junkey and J. N. Bardsley, *Phys. Rev. Lett.* **28**, 1227 (1972)]⁶ and [D. L. Morgan, Jr. and V. W. Hughes, *Phys. Rev. D* **2**, 1389 (1970)]⁷ and [W. Kolos, D. L. Morgan, Jr., D. M. Schrader, and L. Wolniewicz, *Phys. Rev. A* **11**, 1792 (1975)].⁸

1. INTRODUCTION

Successful experiments on the synthesis of antihydrogen which have been carried out at the low-energy antiproton storage ring (LEAR) at the European Center for Nuclear Research (CERN) have attracted interest to several theoretical and practical problems in the physics of antimatter. Some ambitious programs of experiments with antihydrogen have been proposed. A first step would be the construction of new installations ("factories") for the production and storage of antimatter.^{1,2}

In order to solve the problem of storing (and thus accumulating) antihydrogen we would obviously like to have some reliable theoretical calculations of the rate at which antiprotons, \bar{p} , and antihydrogen, $\bar{\text{H}}$, annihilate with residual-gas atoms in the interior of the trap at temperatures on the order of a fraction of 1 K, i.e., at energies on the order of and below 10^{-4} eV (Refs. 3 and 4). The reason for the latter condition is that synthesizing antihydrogen requires fairly cold beams of positrons and antiprotons. In addition, if $\bar{\text{H}}$ is to be effectively trapped in an rf or optical trap, the energy spread of the atoms which are synthesized must also be small.

From the theoretical standpoint, solving the problem of \bar{p} or $\bar{\text{H}}$ annihilation with atoms of matter at low temperatures reduces to determining the dynamic properties of systems containing pairs of unlike charged heavy particles (p and \bar{p}). A specific feature of such systems is that in the course of the collision (e.g., $\text{H} + \bar{p}$ or $\text{H} + \bar{\text{H}}$) there is a redistribution of the particles, accompanied by the formation of protonium (Pr), which is a Coulomb atom consisting of a proton and an antiproton. Protonium is produced in highly excited states (with main quantum numbers $n \approx 30$), so the distances between the levels are comparable to the distances between the low-lying levels of hydrogen. As a result, the separation of electron and nuclear motions, which is a technique widely used in atomic physics, cannot be used in this case, and the problem becomes a definitely many-body problem. This circumstance renders this problem one of purely theoretical interest.

In the present paper we examine the annihilation of ultraslow antiprotons with hydrogen. We actually wish to cal-

culate the probability of production of free protonium with a transfer of the kinetic energy of the colliding \bar{p} and p to the atomic electron e :



In free protonium, the proton and the antiproton annihilate essentially instantaneously (at the macroscopic relaxation-time scale). The basic difficulty in the quantum theory of such processes is in dealing with the virtual restructuring, i.e., with the possibility of reactions which are the inverse of (1) and which lead to simply an elastic scattering of the colliding particles:



It is shown below (and this is the basic result of this study) that the occurrence of processes (2) significantly reduces the annihilation probability.⁵ The cross section for the annihilation of an antiproton with hydrogen has been calculated in several papers^{6–8} by the impact-parameter method. That method is based on the following assumptions: a) The relative motion of the nuclei can be treated as classical. b) The effect of the motion of the p and the \bar{p} on the motion of the electron is adiabatic. Correspondingly, one examines the quantum-mechanical states of the electron in the field of two Coulomb centers, which are fixed and unlike.

According to calculations based on these assumptions, if the distance between the Coulomb centers falls below a critical R_c ($R_c = 0.64r_B$, where r_B is the first Bohr radius of the hydrogen atom), the bound states of the electron in the field of the two unlike Coulomb centers disappear. The energy levels of the electron, thought of as a function of the inter-nuclear distance, bunch together exponentially near the critical point.⁹ According to an adiabatic theorem, the probabilities for inelastic processes are negligible as long as the distance between the nuclei is large, so that the distances between the electron levels are large. When the nuclei come to within the critical distance of each other, however, the problem is no longer adiabatic, and the probabilities for inelastic processes increase. The probability for the emission of an electron and the formation of protonium is assumed to be

zero in this model if the internuclear distance is greater than R_c , while this probability is assumed to be equal to one if this distance is less than R_c . According to this assumption, the only trajectories of the antiproton which contribute to the cross section for reaction (1) are those which pass by the proton at a distance smaller than the critical distance.

Objections could be raised to the assumption of the impact-parameter method that the probability for this reaction is one when the nuclei move close to each other. The reason is that in the language of steady-state scattering theory this assumption can be formulated in the following way:

$$S(E, L)=0, \quad L < L_c, \quad (3a)$$

$$|S(E, L)|=1, \quad L > L_c, \quad (3b)$$

where $S(E, L)$ is the S -matrix element which corresponds to the elastic scattering of the partial wave with angular momentum L of the antiproton with respect to the hydrogen and with an energy E , and L_c corresponds to the critical distance R_c . Actually, there is no reason to assume at the outset that condition (3a)—a highly specific condition—holds at even a single value of the energy; to assume that this condition holds over a broad range of energies of the incident antiprotons is much bolder yet. We show below (Appendix A) that, in fact, this condition does not hold in the problem at hand, particularly at small momenta of the antiprotons, such that the condition $kr_B \ll 1$ holds (here k is the momentum of the antiproton).

Calculations carried out in the unitary coupled-channel model show that even if the angular momentum L of the incident antiproton is zero the probability for reaction (1) is smaller by a factor of several units than the estimate found by the impact parameter method.

The impact-parameter method cannot deal with inverse Auger transitions from the state of protonium plus an electron to the state of hydrogen plus an antiproton. As a particular result, that method should lead to results for the probability for protonium production reaction (1) which are too high.

In Secs. 2–4 of this paper we examine the basic aspects of constructing a unitary coupled-channel model: the representation of the wave function of the three-body system, the choice of a convenient coordinate system, and the formulation of a system of equations. In Sec. 5 we discuss the quantum numbers of the channels which are most important in determining the amplitude for reaction (1). In particular, we show that in the energy range of interest here the protonium forms primarily in states with a small angular momentum (0 or 1). In Sec. 6 we present some final remarks and our conclusions. The details of the calculations carried out in the coupled-channel model are in Appendices B and C.

2. WAVE FUNCTION OF THE THREE-BODY SYSTEM

We are interested in reaction (1) at energies of the incident antiproton lower than 10^{-4} eV. The only channels which are open at such energies are the elastic channel and channels in which protonium is formed in states with a main quantum number n less than or equal to 30.

We start from an operator equation for a three-body vector function:

$$|\Phi\rangle = G_c(E)W|\Phi\rangle + |\Phi_0\rangle, \quad (4)$$

where $|\Phi\rangle$ is a three-body vector function, G_c is the Coulomb Green's operator for the ep and $p\bar{p}$ pairs, W is the $e - \bar{p}$ potential, and $|\Phi_0\rangle$ is the "free" wave function of the elastic channel. This free wave function is the product of the wave function of the hydrogen ground state and the Coulomb function of the continuous spectrum of the antiproton.

The primary difficulty which arises in attempts to solve this equation stem from the presence of restructuring processes (1). The possibility that protonium will form gives rise to some additional singularities in the kernel of Eq. (4). These additional singularities correspond to $p\bar{p}$ bound states. In the method suggested below, the wave function of the three-body system is represented as the sum of two components, one describing the elastic channel and the other containing the protonium wave functions explicitly. This choice of components leads to the correct asymptotic behavior of the three-body wave function in the elastic channel and also in the channels in which protonium is formed. In this sense, this choice of components is equivalent to breaking up the wave function into Faddeev components.^{10,11} No singularities associated with the restructuring appear in the kernels of the model equations which are found.

Let us examine this question in more detail. We write the wave function of the three-body system as the sum of two mutually orthogonal components, one describing the elastic channel and the other describing channels in which protonium is formed and channels which are closed. The component of the wave function which describes the elastic channel is

$$\Phi_1(\mathbf{r}, \mathbf{R}) = \varphi_1(\mathbf{r})\chi(\mathbf{R}), \quad (5)$$

where $\varphi_1(\mathbf{r})$ is the wave function of the hydrogen ground state, and the wave function $\chi(\mathbf{R})$ describes the scattering of the antiproton by the hydrogen. It is natural to introduce some projection operators which act in the space of three-body states:

$$\hat{P}_1 = |\varphi_1\rangle\langle\varphi_1|, \quad \hat{P}_2 = \mathbf{1} - \hat{P}_1, \quad \hat{P}_2\hat{P}_1 = \hat{P}_1\hat{P}_2 = \mathbf{0}. \quad (6)$$

These two components of the wave function can be written in the form

$$|\Phi_1\rangle = \hat{P}_1|\Phi\rangle, \quad |\Phi_2\rangle = \hat{P}_2|\Phi\rangle. \quad (7)$$

Equation (4) can be rewritten as a system of coupled equations for the components Φ_1 and Φ_2 :

$$\begin{aligned} |\Phi_1\rangle &= \hat{P}_1 G_c(E)W|\Phi_1\rangle + \hat{P}_1 G_c(E)W|\Phi_2\rangle + |\Phi_0\rangle, \\ |\Phi_2\rangle &= \hat{P}_2 G_c(E)W|\Phi_2\rangle + \hat{P}_2 G_c(E)W|\Phi_1\rangle. \end{aligned} \quad (8)$$

In order to write system of equations (8) in the coordinate representation, we need to choose a coordinate system.

3. THE COORDINATE SYSTEM

The following system of Jacobi coordinates turns out to be convenient (Fig. 1):

$$\mathbf{r} = \mathbf{r}_e - \mathbf{R}_p, \quad (9)$$

$$\mathbf{R} = m\mathbf{r}_e + M\mathbf{R}_p / (m+M) - \mathbf{R}_{\bar{p}}.$$

Here \mathbf{r}_e is the coordinate of the electron, \mathbf{R}_p is the coordinate of the proton, $\mathbf{R}_{\bar{p}}$ is the coordinate of the antiproton, m is the mass of the electron, and M is the mass of the proton. This coordinate system is related in a natural way to the elastic

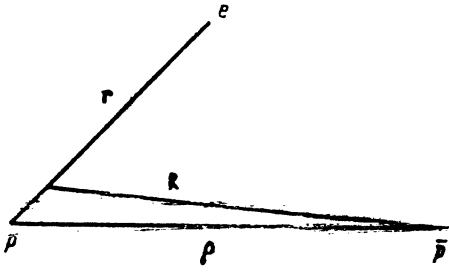


FIG. 1. The coordinate system.

channel ($H + \bar{p} \rightarrow H + \bar{p}$), since it explicitly contains the electron coordinate with respect to the proton, \mathbf{r} . The wave function of the hydrogen atom depends on this coordinate. This coordinate system is convenient because it can also be used to describe the proton-production channel. The reason is that the coordinate $\boldsymbol{\rho}$, of the antiproton with respect to the proton, which appears in the wave function of protonium, agrees within a ratio $m/M \sim 10^{-3}$ with the coordinate \mathbf{R} , of the antiproton with respect to the center of mass of the proton-electron pair. It is this coordinate system that we will be using below.

4. SYSTEM OF EQUATIONS FOR THE COUPLED CHANNELS

Since the component $\Phi_2(\mathbf{r}, \mathbf{R})$ of the three-body wave function describes channels in which protonium is formed, we expand this component in the complete set of protonium wave functions:

$$\Phi_2(\mathbf{r}, \mathbf{R}) = \sum_k g_k(\mathbf{r}) f_k(\mathbf{R}). \quad (10)$$

Here $f_k(\mathbf{R})$ are the protonium eigenfunctions, $g_k(\mathbf{r})$ are unknown expansion coefficients, which represent the electron wave functions in the protonium-production channels, and k is the set of quantum number of protonium (the main quantum number, the angular momentum, and the magnetic quantum number). As a result we find a system of coupled-channel equations for the functions $\chi(\mathbf{R})$ and $g_k(\mathbf{r})$:

$$\begin{aligned} \chi(\mathbf{R}) = & \int G_c(\mathbf{R}, \mathbf{R}') W_{11}(\mathbf{R}') \chi(\mathbf{R}') d\mathbf{R}' \\ & + \int G_c(\mathbf{R}, \mathbf{R}') \varphi_1(\mathbf{r}) \sum_s W(\mathbf{R}' - \mathbf{r}) \\ & \times f_s(\mathbf{R}') g_s(\mathbf{r}) d\mathbf{R}' d\mathbf{r} + \chi_0(\mathbf{R}), \end{aligned} \quad (11)$$

$$\begin{aligned} g_s(\mathbf{r}) = & \int P_2(\mathbf{r}, \mathbf{r}') G_s(\mathbf{r}', \mathbf{r}'') \sum_k W_{sk}(\mathbf{r}'') g_k(\mathbf{r}'') d\mathbf{r}' d\mathbf{r}'' \\ & + \int P_2(\mathbf{r}, \mathbf{r}') G_s(\mathbf{r}', \mathbf{r}'') f_s(\mathbf{R}) W(\mathbf{r}'' - \mathbf{R}) \\ & \times \varphi_1(\mathbf{r}'') \chi(\mathbf{R}) d\mathbf{r}' d\mathbf{r}'' d\mathbf{R}. \end{aligned}$$

Here $G_c(\mathbf{R}, \mathbf{R}')$ is the Coulomb Green's function of the antiproton,

$$W_{11}(\mathbf{R}) = \int \varphi_1(\mathbf{r}) \frac{1}{|\mathbf{r} - \mathbf{R}|} \varphi_1(\mathbf{r}) d\mathbf{r},$$

$\chi_0(\mathbf{R})$ is the free function of the antiproton in the entrance

channel, $G_s(\mathbf{r}, \mathbf{r}')$ is the Coulomb Green's function of the electron in the channel in which protonium is formed in a state with the set of quantum numbers s ,

$$P_2(\mathbf{r}, \mathbf{r}') = 1 - |\varphi_1(\mathbf{r}) \rangle \langle \varphi_1(\mathbf{r}')|,$$

$$W_{nk}(\mathbf{r}) = \int f_n(\mathbf{R}) \frac{1}{|\mathbf{r} - \mathbf{R}|} f_k(\mathbf{R}) d\mathbf{R}.$$

An important property of Eqs. (11) is that they allow a correct description of the asymptotic behavior of the three-body wave function in both the elastic channel and the particle-restructuring channels. A distinction between the coupled-channel model which we are using here and the exact Faddeev equations is that the number of channels considered is limited. In the following section of this paper we show that this distinction is not of fundamental importance, since the form in which we write the equations makes it a simple matter to determine which channels dominate the reaction amplitude and to construct a model which incorporates the effect of specifically those channels. At the same time, the physical content of the equations which we are using turns out to be extremely transparent, so the events which occur can be analyzed at a qualitative level. In Appendix B we present a detailed description of the transformation of system (11) to a form convenient for practical calculations.

5. QUANTUM NUMBERS CHARACTERIZING THE COUPLED-CHANNEL MODEL

A coupled-channel model suitable for actual calculations can be obtained from Eqs. (11) by limiting the number of channels. We will examine the question of just which channels dominate the amplitude for reaction (1) and are the most important.

We note first that we can restrict the discussion to the case in which the total angular momentum of the three-body system is zero. Since the momentum of the incident antiprotons is small, so the relation $kr_B \ll 1$ holds, it will be primarily antiprotons with a zero angular momentum (with respect to the hydrogen) which penetrate to the reaction zone. In the entrance channel, the antiproton in the s state is incident on the hydrogen in its ground state, and the total angular momentum of the three-body system is zero (we are talking about only the angular momentum; we are not considering effects related to the spin of the particles). We would like to know which values of the angular momentum of the protonium which is produced in the reaction are characteristic of our problem. Channels in which protonium is formed open up when the main quantum number is equal to 30. The values of the angular momentum possible for this value of the main quantum number are from 0 to 29. It nevertheless turns out that the probability for reaction (1) in this case is dominated by those channels in which the protonium and the electron are in states with small angular momenta.

To see that this is the case, we go back to Eqs. (11), separating the angular variables:

$$\Phi_1(\mathbf{r}, \mathbf{R}) = \varphi_{1,0}(r) \chi(R) Y_{00}^{00}(\Omega_{\bar{p}}, \Omega_e), \quad (12)$$

$$\Phi_2(\mathbf{r}, \mathbf{R}) = \sum_{k,l} g_{k,l}(r) f_{k,l}(R) Y_{ll}^{00}(\Omega_{\bar{p}}, \Omega_e). \quad (13)$$

Here

$$Y_{l0}^{00} = \frac{1}{(2l+1)^{1/2}} \sum_m Y_{lm}^*(\Omega_{\bar{p}}) Y_{lm}(\Omega_e)$$

is the angular part of the wave function, which corresponds to a zero total angular momentum of the system, constructed from harmonics with an angular momentum of l , and

$$Y_{00}^{00} = Y_{00}^*(\Omega_{\bar{p}}) Y_{00}(\Omega_e)$$

is the angular part of the wave function of the entrance channel, which corresponds to a zero total angular momentum of the system, constructed from harmonics with an angular momentum of 0.

Using (12) and (13), we can put system (11) in the form

$$\begin{aligned} \chi(R) &= \int G_{c,0}(R, R') W_{i1}^{00}(R') \chi(R') dR' \\ &+ \int G_{c,0}(R, R') \varphi_{1,0}(r) \\ &\times \sum_{n,l} W_{0,l}(r, R') f_n(R') g_{n,l}(r) dR' dr + \chi_0(R), \\ g_{n,l}(r) &= \int P_2(r, r') G_{n,l}(r', r'') \\ &\times \sum_{k,l'} W_{nk}^{l,l'}(r'') g_{k,l}(r'') dr'' dr' \\ &+ \int P_2(r, r') G_{n,l}(r', r'') f_{n,l}(R) \\ &\times W_{l,0}(r'', R) \varphi_{1,0}(r'') \chi(R) dr'' dr' dR. \end{aligned} \quad (14)$$

Here

$$\begin{aligned} W_{nk}^{l,l'}(r) &= \int Y_{ll}^{00}(\Omega_e, \Omega_{pr}) f_n(R) \frac{1}{|\mathbf{R}-\mathbf{r}|} f_k(R) Y_{l'l'}^{00} \\ &\times (\Omega_e, \Omega_{pr}) d\Omega_e d\Omega_{pr} dR, \\ W_{0,l}(R, r) &= \int Y_{00}^{00}(\Omega_e, \Omega_{pr}) \frac{1}{|\mathbf{R}-\mathbf{r}|} Y_{ll}^{00}(\Omega_e, \Omega_{pr}) d\Omega_e d\Omega_{pr}, \\ W_{i1}^{00}(R) &= \frac{1}{4\pi} \int \varphi_{1,0}^2 \frac{1}{|\mathbf{R}-\mathbf{r}|} dr d\Omega. \end{aligned} \quad (15)$$

We use the standard expansion of $1/|\mathbf{R}-\mathbf{r}|$ in partial waves:

$$\begin{aligned} \frac{1}{|\mathbf{R}-\mathbf{r}|} &= \sum_{l,m} \frac{4\pi}{2l+1} A_l Y_{lm}^*(\Omega_{\bar{p}}) Y_{lm}(\Omega_e), \\ A_l &= \begin{cases} R^l/r^{l+1}, & R < r, \\ r^l/R^{l+1}, & R > r. \end{cases} \end{aligned}$$

Integrating over the angular variables, and summing over the magnetic quantum number, we find

$$W_{0,l} = A_l / (2l+1)^{1/2}. \quad (16)$$

We see that the potentials $W_{0,l}$, which couple the entrance channel with channels in which the protonium is in a state with an angular momentum l , contain a normalization factor $1/\sqrt{2l+1}$, which reduces the contribution of channels with large angular momenta. We wish to stress that this factor arises because the system is in a state in which the total angular

momentum is zero. However, purely dynamic factors are the primary reason why channels with large angular momenta make only a small contribution. Let us consider a very simple coupled-channel model which is nevertheless sufficient to draw a qualitative picture of how channels with large angular momenta play a role. We assume that the motion of the electrons in the channels in which protonium is formed is free. We also assume that the channels in which protonium is formed are coupled only with the elastic channel—not with each other. The system of equations describing this model is then

$$\begin{aligned} \chi(R) &= \int G_c(R, R') W_{i1}^{00}(R') \chi(R') dR' \\ &+ \int G_c(R, R') \varphi_{1,0}(r) W_{0,l}(r, R') \\ &\sum_{n,l} f_{n,l}(R') g_{n,l}(r) dr dR' + \chi_0(R), \\ g_{n,l}(r) &= \int G_{n,l}^0(r, r') f_{n,l}(R) W_{l,0}(r', R) \varphi_{1,0}(r') dr' dR. \end{aligned} \quad (17)$$

Here $G_{n,l}^0(r, r')$ is the free Green's function of an electron in a channel in which protonium is formed, in a state with a main quantum number n and an angular momentum l . The energy of the electron in this channel is given by $E_n = (M/mn^2 - 1)E_B + E_{\bar{p}}$, where E_B is the Bohr energy of the hydrogen atom, and $E_{\bar{p}}$ is the energy of the incident antiproton. It is a simple matter to reduce this system of equations to a single integral equation for the wave function of the incident antiproton. This equation contains a nonlocal complex potential which describes the effect of the inelastic channels on the scattering of the antiproton in the entrance channel:

$$\begin{aligned} \chi(R) &= \int G_c(R, R') [W_{i1}^{00}(R') \delta(R'-R'') \\ &+ V(R', R'')] \chi(R'') dR'' + \chi_0(R). \end{aligned} \quad (18)$$

The complex optical potential is

$$\begin{aligned} V(R, R') &= \sum_{n,l} \int \varphi_{1,0}(r) W_{0,l}(r, R) f_{n,l}(R) \\ &\times G_{n,l}^0(r, r') f_{n,l}(R') W_{0,l}(R', r') \varphi_{1,0}(r') dr dr'. \end{aligned} \quad (19)$$

For channels with main quantum numbers n from 24 to 30, the momentum of the outgoing electron,

$$k_c = \left(\frac{M}{mn^2} - 1 \right)^{1/2} \frac{1}{r_B}$$

turns out to be small, so the condition $k_c r_B < 1$ holds. The free Green's function of the electron can thus be written in the form¹²

$$G_{n,l}^0(r, r') \approx \frac{r_{<}^{l+1} r_{>}^{-l}}{2l+1} - \frac{(kr)^{l+1} (kr')^{l+1}}{k[(2l+1)!!]^2}.$$

We thus find the following result for the optical potential:

$$\operatorname{Re} V(R) \propto \frac{1}{(2l+1)(2l+1)},$$

$$\operatorname{Im} V(R) \propto \frac{(kr_B)^{2l+2}}{[(2l+1)!!]^2(2l+1)}.$$

The additional factor of $1/(2l+1)$ here arises because of dependence (16). We see that the real and imaginary parts of the optical potential fall off with increasing angular momentum l . The imaginary part of the optical potential, which describes the absorption of antiprotons in the entrance channel, falls off particularly rapidly with increasing l . The obvious reason for this dependence of the optical potential on the angular momentum of channels with a small momentum of the outgoing electrons is the presence of a centrifugal barrier, which sharply reduces the probability that a slow electron with a high angular momentum will be in the reaction zone.

For channels with main quantum numbers n less than 23, the momentum of the outgoing electrons is not small. Such electrons can penetrate through the centrifugal barrier into the reaction zone. Nevertheless, the contribution of such channels is small. The reason is that the expression for the optical potential contains the wave function of a protonium bound state, $f_{n,l}$. This function decays over a distance on the order of $R_{n,l} \sim (n^2 m/M) r_B$. The contribution of such channels, with main quantum numbers $n \leq 23$, is determined by the value of $(R_{n,l}/r_B)^{2l}$. That this is so can be verified by expanding the wave function of the incident antiproton, χ , and also the Green's function $G_c(R, R')$ in a complete set of Coulomb functions,

$$\chi(R) = \sum_k B_k f_{k,0}(R), \quad (20)$$

$$G_c(R, R') = \sum_k \frac{f_{k,0}(R) f_{k,0}(R')}{E - E_k},$$

and by writing an equation for the expansion coefficients B_k ,

$$B_k = \sum_m \frac{B_m}{E - E_k} \int f_{k,0}(R) W_{11}(R) f_{m,0}(R) dR$$

$$+ \sum_m \frac{B_m}{E - E_k} \int f_{k,0}(R) \varphi_{1,0}(r) W_{01}(R, r) \sum_{n,l} f_{n,l}(R)$$

$$\times G_{n,l}^0(r, r') f_{n,l}(R') W_{10}(R') \varphi_{1,0}(r')$$

$$\times f_{m,0}(R') dR dr dR' dr'. \quad (21)$$

To find the estimate written above, we need to replace W_{01} in the second term in (21) by its partial-wave expansion in (16) and carry out an integration over the variables R and r . For $n < 23$, the ratio $R_{n,l}/r_B$ is smaller than 0.6 (for any l), and the quantity $(R_{n,l}/r_B)^{2l}$ is a small quantity on the order of 10^{-1} at an l value as low as 2.

The reason why the channels with small values of the main quantum number (less than 23) make only a small contribution is thus that the protonium wave function corresponding to these channels decays over distances small in

comparison with the Bohr radius of hydrogen, and the overlap of such channel wave functions with the wave function of the entrance channel is only slight. These arguments lead to the conclusion that the protonium forms in states with a small angular momentum (0 or 1).

What are the main quantum numbers of the channels which dominate the reaction amplitude? It was shown above that the contribution of channels with main quantum numbers n below 23 is determined by the expression $(n^2 m/M)^{2l}$. Accordingly, channels with main quantum numbers $n < 10$ are ignored in concrete calculations. All channels with $n > 30$ are closed. Detailed numerical calculations show that it is sufficient to restrict the discussion to channels with $n < 40$. We also note that our model ignores states of the continuous spectrum of the $p\bar{p}$ pair. These states obviously do not describe inelastic channels in which protonium forms; in other words, incorporating such states leads to merely corrections to the real part of the optical potential in (19). An estimate of the contribution of the continuous spectrum to the optical potential on the basis of simple models yields a value on the order of 10%.

Our coupled-channel model is thus a system of equations which describes the entrance channel and channels in which protonium is formed in states with main quantum numbers from 10 to 40 and with an angular momentum of 0 or 1. Calculations show that protonium forms with the highest probability in states with main quantum numbers from 18 to 27. The details of the calculations are presented in Appendix B.

6. RESULTS OF THE CALCULATIONS AND FINAL COMMENTS

The results of the numerical calculations show that at incident-antiproton energies below 10^{-4} eV the probability for reaction (1) is proportional to the momentum of the antiproton, with a proportionality factor of 0.41:

$$|S_r|^2 = 1 - |S|^2 = 0.41kr_B. \quad (22)$$

Following the logic of an adiabatic description of reaction (1), we would assume that the probability for the formation of protonium in an ultraslow head-on collision of hydrogen and an antiproton is one. We see that for energies of the incident antiprotons from 0 to 10^{-4} eV the probability for the inelastic reaction is considerably less than one, according to calculations in the coupled-channel model. For example, at an energy of 10^{-4} eV (10^{-4} eV corresponds to the momentum of an antiproton with $k = 0.3r_B^{-1}$), the probability for this reaction is 0.12. There will evidently be a substantial difference in the results for the cross section for reaction (1) also. The reaction cross section calculated in the coupled-channel model is

$$\sigma = 0.41\pi r_B^2 / kr_B. \quad (23)$$

In the adiabatic approximation,⁶⁻⁸ the reaction cross section has a $1/v^2$ behavior in the limit of a low relative velocity of the colliding particles. In the coupled-channel model, in contrast, the behavior of the cross section is $1/v$ [see (23)]. The reason why the calculations by the impact-parameter method lead to a large probability for protonium production is that those calculations ignore the actual dy-

namics of the production of protonium, in particular, the inverse Auger effect.

We went through the following calculations to determine just how important the inverse Auger effect is. We write the optical potential of the coupled-channel model, found in intermediate calculations, in the form

$$V(R, R') = U(R, R') + i\alpha I(R, R'), \quad (24)$$

where $U(R, R')$ is the real part of the optical potential, $I(R, R')$ is its imaginary part, and the parameter α takes on values from 0 to 1.

By varying the parameter α , we varied the extent to which the inelastic channels affect the scattering of the antiproton in the entrance channel. For example, the value $\alpha = 0$ corresponds to purely elastic scattering, while $\alpha = 1$ corresponds to the actual value of the extent of channel coupling.

Figure 2 shows the probability for reaction (1) as a function of the parameter α for the energy $E = 10^{-4}$. The nonmonotonic behavior of the probability for the inelastic reaction as a function of the depth of the imaginary part of the optical potential is attributed to a competition between the forward and inverse Auger effects. At first, $|S_r|^2$ increases with increasing α , since forward Auger transitions are predominant. As the channel coupling becomes stronger, inverse Auger transitions make a progressively larger contribution to the probability for elastic scattering and thereby reduce the probability for the reaction, $|S_r|^2$. We see from Fig. 2 that the true value of the reaction probability ($\alpha = 1$) lies on the descending branch of the plot, well to the right of the peak. The decrease in the reaction probability in comparison with the peak is therefore due specifically to the inverse Auger effect.

Interestingly, the true value of the imaginary part of the optical potential is larger by a factor of nearly 30 than the value of I_{\max} at which the reaction probability reaches its maximum. This fact can be explained quite simply. Let us examine the requirements which the optical potential must meet for the reaction probability to be at its maximum value. Since the momentum of the antiproton is small ($kr_B \ll 1$), the reaction probability can reach a value close to one only if the S -matrix, thought of as a function of the momentum of the antiproton, has a pole near the origin in the complex plane. (In the opposite case, the element of the S -matrix corresponding to elastic scattering could be written in the form $S = 1 - 2ikr_B a$, where a is complex: $a = a_1 + ia_2$. The

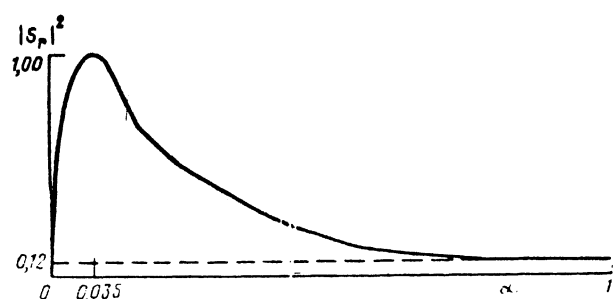


FIG. 2. Reaction probability versus the depth of the imaginary part of the optical potential, for an energy of 10^{-4} eV.

probability for the inelastic reaction would then be $|S_r|^2 = 4kr_B a_2 \ll 1$.) If the S -matrix has a pole near the origin in the complex momentum plane, we can write the following expression for an element S : $S \propto (\gamma - ik)/(\gamma + ik)$. Since inelastic processes are possible in this case, γ is complex: $\gamma = \gamma_1 + i\gamma_2$.

From the condition that the probability for the reaction be at a maximum ($S = 0$, $|S_r|^2 = 1$), we find the following values for γ_1 and γ_2 :

$$\gamma_1 = 0, \quad \gamma_2 = k. \quad (25)$$

Clearly, γ_2 is determined completely by the complex potential: $\gamma_2 = \gamma_2(V)$. Equation (25) thus means that the complex potential must be a fairly "lively" function of the momentum of the incident antiproton. It is not difficult to see that in this case the complex potential is a very weak function of the antiproton momentum. Specifically, the dependence of potential (19) on the momentum (or energy) of the antiproton is determined by the energy dependence of the channel Green's functions $G_{n,l}(E_n)$, where $E_n = (M/mn^2 - 1)E_B + E_p$, E_B is the Bohr energy of the hydrogen atom, and $E_p < 10^{-5}E_B$. For arbitrary values of the main quantum number n corresponding to open channels, we have $(M/mn^2 - 1)E_B \gg E_p$. The dependence of the channel Green's functions and of the complex potential V on the antiproton momentum thus turns out to be weak because the energy of the antiprotons is small. This conclusion means that condition (25) and thus the condition $S = 0$, $|S_r|^2 = 1$ definitely do not hold at low energies of the incident antiproton. The reason is that the imaginary part of the complex potential does not have the small factor kr_B , and at small antiproton momenta the imaginary part of the optical potential, I , turns out to be much larger than I_{\max} (the value at which the reaction probability reaches its maximum). In Appendix A this circumstance is examined analytically in the example of an exactly solvable problem with a separable complex potential.

In summary, the reason why the inverse Auger effect is important is that the motion of the antiproton is slow.

In conclusion I wish to thank I. S. Shapiro for suggesting the problem and for numerous useful discussions. I also thank L. P. Presnyakov for several useful comments.

APPENDIX A

Let us analyze the condition for the maximum probability of inelastic processes, using as an example the exactly solvable problem of the scattering by a complex separable potential whose imaginary part describes an absorption. Such a potential can be thought of as a crude approximation of the complex potential of the actual problem. This separable potential is

$$\begin{aligned} \hat{V} &= (U + iI) |\xi\rangle \langle \xi|, \\ \langle r | \xi \rangle &= b e^{-br/r}, \quad b = 1/r_B. \end{aligned}$$

For the S -matrix element corresponding to elastic scattering we find the expression

$$S = \frac{(k - ib)^2 - (U + iI)}{(k + ib)^2 - (U + iI)}.$$

The condition that S be equal to zero leads to the following

expressions for the real and imaginary parts of the complex potential:

$$U = (k^2 - b^2)/Mb, \quad I = -2k/M.$$

We see that a necessary condition for the value $S = 0$ is that the imaginary part of the complex potential be proportional to the velocity of the scattered antiproton. As was shown above, this condition does not hold in the real problem in the energy range under consideration.

APPENDIX B

Let us examine the details of the calculation in the coupled-channel model. We first transform the system of coupled-channel equations, (14), to a more convenient form. The idea of this transformation is to eliminate the explicit appearance of the diagonal interaction $W_{11}(R)$ and $W_{nn}(r)$ from the equations, through the introduction of some new Green's functions. Equations (14) then become

$$\begin{aligned} \chi(R) &= \int \mathcal{G}_{c,0}(R, R') \varphi_1(r) \\ &\times \sum_{n,l} W_{0,l}(r, R') f_{n,l}(R') g_{n,l}(r) dR' dr + \bar{\chi}_0(R), \\ g_{n,l}(r) &= \int \mathcal{G}_{n,l}(r, r') P_2(r', r'') \\ &\times \sum_{\substack{k \neq n, \\ l' = 0, 1}} W_{nk}^{l, l'}(r'') g_{k, l'}(r'') dr' d r'' \\ &+ \int \mathcal{G}_{n,l}(r, r') P_2(r', r'') f_{n,l}(R) \\ &\times W_{l,0}(r'', R) \varphi_{1,0}(r'') \chi(R) dr' d r'' dR. \end{aligned} \quad (B1)$$

The Green's functions of the antiproton $\hat{\mathcal{G}}_{c,0}(R, R')$ and that of the electron $\hat{\mathcal{G}}_{n,l}(r, r')$, are defined by

$$\begin{aligned} \hat{\mathcal{G}}_{c,0} &= (\hat{H}_{\bar{p}} - E_{\bar{p}})^{-1}, \\ \hat{H}_{\bar{p}} - E_{\bar{p}} &= -\frac{1}{2M} \frac{d^2}{dR^2} - \frac{1}{R} + W_{11}(R) - E_{\bar{p}}, \\ \hat{\mathcal{G}}_{n,l} &= (\hat{H}_e - E_n)^{-1}, \\ \hat{H}_e - E_n &= -\frac{1}{2m} \frac{d^2}{dr^2} - \frac{1}{r} + \frac{l(l+1)}{2mr^2} + \hat{P}_2 W_{nn}^{l,l}(r) - E_n, \\ \hat{P}_2 &= 1 - |\varphi_1\rangle \langle \varphi_1|. \end{aligned}$$

To simplify the calculations, we have approximated the potentials $W_{nk}^{l,l}(r)$, which describe the coupling of the various channels in which protonium is formed, by separable potentials. In Appendix C we analyze the validity of this approximation.¹¹ We find that replacing the local potential by a separable one in the scattering problem leads to satisfactory results if the condition $ka_0 < 1$ holds, where a_0 is the effective range of the potential, and k is the momentum of the scattered particle. The model separable potential is

$$\hat{W}_{nk}^{l,l} = v_{nk}^{l,l} |\xi_l\rangle \langle \xi_l|,$$

$$\langle r | \xi_l \rangle = j_l(k_0 r) W_{3020}^{l,l}(r),$$

$$v_{nk}^{l,l} = \frac{W_{nk}^{l,l}(r_B)}{W_{3020}^{l,l}(r_B)} \left[\int j_l^2(k_0 r) W_{nk}^{l,l}(r) dr \right]^{-1/2}.$$

Here $l = 0, 1, j_0(z) = \sin(z), j_1(z) = \sin(z)/z - \cos(z)$, and $k_0 = 0.3/r_B$. In this approximation, k_0 is a parameter. Numerical calculations show that the result depends weakly on k_0 if $k_0 r_B < 1$. Once the potentials $W_{nk}^{l,l}(r)$ have been replaced by separable potentials, that part of the system of coupled-channel equations which describes channels in which protonium is formed reduces to algebraic form. Equations (B1) then become

$$\begin{aligned} \chi(R) &= \int \mathcal{G}_{c,0}(R, R') \varphi_1(r) \\ &\times \sum_{n,l} W_{0,l}(r, R') f_{n,l}(R') g_{n,l}(r) dR' dr + \bar{\chi}_0(R), \\ g_{n,l}(r) &= \hat{\mathcal{G}}_{n,l} \hat{P}_2 |\xi_l\rangle \sum_{\substack{k, l', \\ m, l''}} v_{nk}^{l,l'} (1-A)^{-1}_{l', l'', km} \langle \xi_{l''} | \hat{\mathcal{G}}_{m, l''} Q_{m, l''} \\ &+ \hat{\mathcal{G}}_{n,l} \hat{P}_2 Q_{n, l}, \\ A_{l', l'', nk} &= v_{nk}^{l,l'} \langle \xi_l | \hat{\mathcal{G}}_{n, l} \hat{P}_2 | \xi_{l''} \rangle, \\ Q_{n, l}(r) &= \varphi_1(r) \int f_{n, l}(R) W_{l,0}(R, r) \chi(R) dR. \end{aligned}$$

Substituting the expressions for the electron functions $g_{n,l}(r)$ into the equation for the antiproton function $\chi(R)$, we find a single integral equation for $\chi(R)$, which can be solved numerically.^{11,13} We will not reproduce that lengthy equation here.

APPENDIX C

Let us examine the validity of replacing a local short-range potential by a separable potential in a scattering problem. For simplicity we restrict the discussion to the one-channel, one-dimensional case. We consider the Lippmann-Schwinger equation describing scattering by a potential W :

$$|g\rangle = \hat{G}_0 W |g\rangle + |p_0\rangle.$$

Here \hat{G}_0 is a free Green's function, p_0 is a plane wave, and W is the short-range potential. By "short-range" here we mean that the effective range of the potential is small in comparison with the wavelength of the scattered particle. We expand the wave function in plane waves:

$$|g\rangle = \sum_{p, l} g_{p, l} |p, l\rangle,$$

where $g_{p, l}$ satisfies the equation

$$g_{p, l} = 2m \int \frac{\langle p l | W p' l \rangle g_{p', l}}{k^2 - p'^2} dp' + \delta(p - k).$$

Here m and k are the mass and momentum, respectively, of the scattered particle.

Since the condition $r_w k \ll 1$ holds (r_w is the range of the potential), in the region with $p, p' \sim k$ the matrix element

$\langle pWp' \rangle$ turns out to be proportional to the product $p'^{l+1}(p')^{l+1}$ (as is easily verified):

$$\langle plWp'l \rangle \propto (pp')^{l+1}. \quad (C1)$$

The integral equation is dominated by momenta $p' \sim k$. The matrix element $\langle plWp'l \rangle$ can thus be approximated by

$$\langle plWp'l \rangle \approx \langle plWkl \rangle \langle klWp'l \rangle / \langle klWk'l \rangle.$$

This replacement is equivalent to introducing a separable potential. In the coordinate representation, this potential is

$$U(r, r') = W(r) j_l(kr) j_l(kr') W(r') / \int j_l^2(kr) W(r) dr.$$

Here $j_l(kr)$ is a Bessel function describing free motion with an angular momentum l .

It is not difficult to see that the arguments above also apply to the multichannel case. The only necessary condition is that the products of the channel momenta and the ranges of the corresponding potentials be small.

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