Equidistant spectra of anharmonic oscillators

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We set up an example of a potential of the anharmonic oscillator type for which the time-independent Schrödinger problem leads to a strictly equidistant spectrum for all the excited states. The general solution of this problem is strictly periodic in time (i.e., there is no spreading of arbitrary wave packets). We discuss a direct method for building anharmonic potentials that lead to equidistant spectra and of operators similar to the creation and annihilation operators in the Fock representation.

1. As is known,[1,2] one of the main objects of quantum mechanics, the harmonic oscillator, has an equidistant spectrum of energy eigenvalues, and the general solution of the respective evolution problem is periodic in time (to within an advance of the constant phase of the wave function). A characteristic feature of the harmonic oscillator as an object of classical mechanics is its isochronism, that is, the independence of the oscillation period of the amplitude.

It is also known[1,2] that in quantum mechanics the transition to anharmonic oscillators generally results in nonequidistant spectra and in the spreading of arbitrary wave packets, while in classical mechanics this leads to nonisochronous oscillations. But it should not be assumed that there is a one-to-one correspondence between the isochronism of classical oscillators and the equidistant spectra of quantum oscillators.

Indeed, according to the solution in Ref. 3 of the problem of reconstructing the potential from a given energy dependence of the oscillation period, there are many isochronous asymmetric potentials but only one symmetric isochronous potential, the potential of a harmonic oscillator. Below, however, we construct an example of a symmetric nonisochronous potential of an anharmonic oscillator for which the eigenvalue spectrum is in fact equidistant and the general solution of the evolution problem is strictly periodic in time, that is, the spreading of arbitrary wave packets is excluded.

Among the many isochronous asymmetric potentials, which generally lead to nonequidistant spectra, there are those that meet the condition of equidistance.

We believe that singling out the class of anharmonic oscillators having equidistant spectra and excluding the spreading of arbitrary wave packets is of definite interest in view of the problem of "quantum chaos"[4] and, possibly, the problem of quantizing inherently nonlinear fields. The reason is that with anharmonic oscillators having an equidistant spectrum it is possible to go over to the Fock representation[2,5] by introducing into the spectrum a shift operator, a natural generalization of the well-known creation and annihilation operators.

Below we discuss an approach to the problem of constructing one-dimensional potentials that lead to equidistant spectra of the Hamiltonian operator and the related creation and annihilator operators. We also analyze in detail the energy eigenvalue problem for the potential of an anharmonic oscillator that excludes the spreading of arbitrary wave functions.

2. For the Schrödinger equation

$$i\Psi_t = H\Psi \tag{2.1}$$

with a time-independent operator H we define the conditions in which the following class of solutions exists:

$$\Psi(x,\ t+2\pi/\omega) = e^{i\alpha}\Psi(x,\ t), \tag{2.2}$$

where ω and α are the parameters of the selected class of solutions. The fundamental period of a solution, $T = 2\pi/\omega$, gives the time of return of the system to the initial state (to within a constant-phase increment α).

Suppose that in the associated time-independent problem,

$$\Psi_E(x,\ t) = e^{-iEt}\psi_E(x),\ H\psi_E = E\psi_E \tag{2.3}$$

corresponds to a strictly point spectrum of H and that

$$(E,\ \psi_E) \equiv \{\ldots(E_n,\ \psi_n)\ldots\} \tag{2.4}$$

is the set of eigenelements of problem (2.3) ordered in integer values of n. Integrating Eq. (2.1) over the fundamental period of the selected class of solutions (2.2) yields

$$i(e^{i\alpha}-1)\Psi(x,t) = H\int_0^{2\pi/\omega} d\tau\Psi(x,t+\tau),$$

which for the case of steady states $\Psi_E(x,t)$ leads to the equation

$$\mu(E,\ \omega,\ \alpha)E\psi_E \equiv \frac{\exp(i\alpha)-1}{\exp(-2\pi iE/\omega)-1}\,E\psi_E = H\psi_E. \tag{2.5}$$

This coincides with the initial eigenvalue problem (2.3) for H if $\mu = 1$, a requirement that is met if we deal with Hamiltonian operators H leading to the following point spectrum:

$$E_n = \omega Z(n) + \omega_0,\ \omega_0 = -\omega\alpha/2\pi, \tag{2.6}$$

where $Z(n)$ is an arbitrary integer function of the integer argument n that numbers the eigenelements (2.4) of problem (2.3).

When there is only one degree of freedom and the Hamiltonian operator is of the form

$$H = {}^1/_2 p^2 + U(x),\ p = -i\partial_x \tag{2.7}$$

the class of solutions (2.2) exists if there is a solution to the problem of determining potentials $U(x)$ that lead to a point "integer valued" spectrum (2.6) of the Hamiltonian operator (2.7). If, in addition, the set of eigenelements (2.4) and (2.6) of problem (2.3) is complete, the solution of the Cauchy problem with arbitrary initial data belongs to the

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selected class of solutions and arbitrary wave packets do not spread. Obviously, the set of such potentials includes those that lead to equidistant spectra $[Z(n) = n]$.

In what follows we basically restrict our discussion to equidistant spectra. To analyze the ensuing inverse problem of constructing potentials $U(x)$ that rapidly grow as $x \to \pm\infty$ and lead to a strictly equidistant point spectrum of the energy eigenvalues we employ the following approach.

Suppose that there are operators L that transform the solutions of the Schrödinger equation (2.3) with an arbitrary value of parameter E into solutions of the Schrödinger equation with a parameter equal to $E+\omega$:

$$H\psi_{E+\omega}=(E+\omega)\psi_{E+\omega}. \tag{2.8}$$

Thus,

$$L: \psi_E \to \psi_{E+\omega}=L\psi_E. \tag{2.9}$$

Substituting (2.9) into (2.8) and comparing the obtained expressions with Eq. (2.3) multiplied by L, we find that the operators L must satisfy the following operator equations:

$$[H, L]=\omega L, \ [H, L]=HL-LH. \tag{2.10}$$

Note that the solutions of the operator equations (2.10) do not generally transfer the eigenelements (E_n, ψ_n) of problem (2.3) into other eigenelements of the same problem. Indeed, the action of L on the eigenfunctions ψ_n may be due to a change in the asymptotic behavior of $L\psi_n$ as $x \to \pm\infty$ or $x \to x_s$, where the x_s are singular points of the L. But under certain restrictions the solutions of the operator equations (2.10) determine a class of potentials $U(x,\omega)$ leading to equidistant spectra. Here the action of operator $L_+ = L$ and that of the Hermitian conjugate $L_- = \tilde{L}^*$ are linked to the following mapping:

$$L_\pm: \psi_n \to C_{n\pm1}\psi_{\pm1}=L_\pm\psi_n, \ E_n \to E_{n\pm1}=E_n\pm\omega, \tag{2.11}$$

where the $C_{n\pm1}$ are constants determined by the normalization of the eigenfunctions, and $E_n = \omega n + \omega_0$. Note that the operators $L_\pm$ satisfy the following relations:

$$[H, L_+L_-]=[H, L_-L_+]=0. \ [(L_+L_-), (L_-L_+)]=0. \tag{2.12}$$

To simplify presentation of the results, the explicit formulation of the restrictions ensuring that the solutions of the operator equations (2.10) act according to (2.11) is replaced by a direct verification of the solutions. Obviously, operators $L_\pm$ that act according to (2.10) and (2.11) must be considered a natural generalization of the creation and annihilation operators acting in a Fock space, and generated by the eigenvectors of harmonic oscillators, to the case of the eigenvectors of anharmonic oscillators with an equidistant spectrum.

3. Let us now analyze Eqs. (2.10), which for a Hamiltonian of type (2.7) allow for formal solutions in terms of polynomials of finite degree in the momentum operator $p \equiv -i\partial_x$:

$$L(x, p)=\sum_{m=0}^{M} L_m(x, \omega)p^m. \tag{3.1}$$

Substituting (3.1) into Eq. (2.10) leads for the functions $L_m(x,\omega)$ to an overdetermined linear system of ordinary differential equations whose coefficients depend on potential $U(x)$ and its derivatives. The condition needed for this system to have a solution leads to a nonlinear (in the general case) equation for the sought potential $U(x,\omega)$.

According to the above remarks, we must prove that the spectrum of the Hamiltonian operator (2.7) corresponding to the solutions of this equation is equidistant and that the operators $L_\pm$ act on the eigenfunctions according to (2.11).

In the simplest case where $M=1$, substitution of (3.1) into Eq. (2.10) leads to the system of equations

$$\frac{1}{2}\frac{d^2L_0}{dx^2}-iL_1\frac{dU}{dx}=\omega L_0, \quad i\frac{dL_0}{dx}+\frac{1}{2}\frac{d^2L_1}{dx^2}=\omega L_1, \quad i\frac{dL_1}{dx}=0. \tag{3.2}$$

Solving this system, we arrive at the well-known result

$$L=l_0+\omega x+ip, \ U-U_0=l_0(\omega x)+{}^1/_2(\omega x)^2, \tag{3.3}$$

where l_0 and U_0 are constants of integration, whose values can be set at zero. Thus, at $M=1$ the solutions of the operator equations (2.10) lead to a potential of a harmonic oscillator with an equidistant spectrum and with operators L_+ and L_- satisfying (2.11).

In the case where $M=2$, substitution of (3.1) into Eq. (2.10) yields the following system of equations:

$$\frac{1}{2}\frac{d^2L_0}{dx^2}-iL_1\frac{dU}{dx}-L_2\frac{d^2U}{dx^2}=\omega L_0, \quad i\frac{dL_0}{dx}+\frac{1}{2}\frac{d^2L_1}{dx^2}-2iL_1\frac{dU}{dx}=\omega L_1, \quad i\frac{dL_1}{dx}+\frac{1}{2}\frac{d^2L_2}{dx^2}=\omega L_2, \quad i\frac{dL_2}{dx}=0. \tag{3.4}$$

The condition needed for this system to have a solution leads to an equation for potential $U(x)$:

$$(x+x_0)\frac{dU}{dx}+2U=\frac{1}{2}\omega^2(x_0+x)^2+\text{const}. \tag{3.5}$$

Solving this equation yields the singular potential

$$U-U_0=\frac{1}{8}\omega^2(x+x_0)^2-\frac{1}{8}(\omega x_0)^2+\frac{V_0}{(x+x_0)^2}-\frac{V_0}{x_0^2}, \tag{3.6}$$

in which $V_0 = x_0(\frac{1}{8}\omega^2x_0^2 - \frac{1}{4}\omega + \frac{1}{2}\delta_0)$, and U_0, δ_0, and x_0 are arbitrary constants, with x_0 determining the position of the singular point of the potential. Finally, here is a formula for L:

$$L=p^2-i\omega(x+x_0)p+\frac{1}{4}\omega^2(x+x_0)^2-\frac{1}{4}(\omega x_0)^2+\frac{2V_0}{(x+x_0)^2}-\frac{2V_0}{x_0^2}+\delta_0. \tag{3.7}$$

The singular potential (3.6) is symmetric with respect to point $x = x_0$ and consists of two potential wells separated at the singularity point by an infinitely high barrier. It can easily be verified that in the classical case the potential wells are isochronous. In the quantum case, (3.6) leads to a Schrödinger equation of the form

$$\frac{d^2\psi}{d\xi^2}+\left(-\frac{1}{4}\omega^2\xi^2+2\mathscr{E}-\frac{V_0}{\xi^2}\right)\psi=0 \tag{3.8}$$

[see Eqs. (2.3) and (2.7)], where we have used the notation

$$\xi=x+x_0,\ \mathscr{E}=E+{}^1/_8(\omega x_0)^2+V_0/x_0^2-U_0. \tag{3.9}$$

In connection with Eq. (3.8) we consider the eigenvalue problem for parameter $\mathscr{E}$ on the ray $0<\xi<+\infty$:

$$\lim_{\xi\to+\infty}\psi=\lim_{\xi\to+0}\psi=0. \tag{3.10}$$

After the independent variable ξ is replaced with $Z=\omega^{1/2}\xi$ in

$$\psi=\xi^\alpha\exp(-\omega\xi^2/4)F(\xi),\ 2\alpha=1+(1+8V_0)^{1/2} \tag{3.11}$$

we arrive at the equation

$$Z\frac{d^2F}{dZ^2}-(Z^2-2\alpha)\frac{dF}{dZ}+\varepsilon Z=0,\qquad \varepsilon=\frac{2\mathscr{E}}{\omega}-\alpha, \tag{3.12}$$

which allows for polynomial solutions $F_n(Z)$ at

$$\varepsilon_n=2n\ \text{ or }\ \mathscr{E}_n=(n+{}^1/_2\alpha)\omega,\ n=0,1,2\dots .$$

Hence, each of the potential wells forming the singular potential (3.6) leads to an equidistant spectrum of the Hamiltonian operator. Here are some polynomials $F_n(Z)$, which together with (3.11) determine the respective eigenfunctions:

$$\begin{aligned}&F_0=1,\ \varepsilon_0=0;\ F_1=Z^2-(1+2\alpha),\ \varepsilon_1=2;\\ &F_2=Z^4-2(3+2\alpha)Z^2+(1+2\alpha)(3+2\alpha),\ \varepsilon_2=4;\\ &F_3=Z^6-3(5+2\alpha)Z^4+3(5+2\alpha)Z^2\\ &\quad-(1+2\alpha)(3+2\alpha)(5+2\alpha),\ \varepsilon_3=6.\end{aligned} \tag{3.13}$$

4. Now let us analyze a more meaningful situation that emerges if we represent the operator L in the form of a polynomial of the third degree in the momentum operator:

$$L=L_0(x)+L_1(x)p+L_2(x)p^2+L_3(x)p^3. \tag{4.1}$$

The operator relation (2.10) generates the following system of equations:

$$\begin{aligned}&\frac{1}{2}\frac{d^2L_0}{dx^2}-iL_1\frac{dU}{dx}-L_2\frac{d^2U}{dx^2}+iL_3\frac{d^3U}{dx^3}=-\omega L_0,\\ &i\frac{dL_0}{dx}+\frac{1}{2}\frac{d^2L_1}{dx^2}-2iL_2\frac{dU}{dx}-3L_3\frac{d^2U}{dx^2}=-\omega L_1,\\ &i\frac{dL_1}{dx}+\frac{1}{2}\frac{d^2L_2}{dx^2}-3iL_3\frac{dU}{dx}=-\omega L_2,\\ &i\frac{dL_2}{dx}+\frac{1}{2}\frac{d^2L_3}{dx^2}=-\omega L_3,\\ &i\frac{dL_3}{dx}=0.\end{aligned} \tag{4.2}$$

The solutions of this system have the form

$$L_3=-i,\quad L_2=\xi(x),\quad iL_1=3U-{}^1/_2\xi^2(x)+C,$$

$$-\omega L_0=\frac{1}{4}\frac{d^3U}{dx^3}-3U\frac{dU}{dx}+\frac{1}{2}[\xi^2(x)+2C+5\omega]\frac{dU}{dx}-\frac{1}{2}\omega^2\xi^2(x), \tag{4.3}$$

where $\xi(x)=\omega x-C_2$, $C=C_1-\frac{1}{2}C_2^2$, with C_1 and C_2 constants of integration, and $U(x)$ must satisfy the equation

$$\frac{1}{4}\frac{d^4U}{dx^4}-3\frac{d}{dx}\left(U\frac{dU}{dx}\right)+\frac{1}{2}[\xi^2(x)+\xi_0^2]\frac{d^2U}{dx^2}+3\omega\xi(x)\frac{dU}{dx}+3\omega^2U=\frac{1}{2}\omega^2[\xi^2(x)+\xi_0^2], \tag{4.4}$$

in which $\xi_0^2\equiv2(-C+\omega)$.

Thus, when L is represented by a polynomial of the third degree in the momenta, the condition needed for the operator relations (2.10) to have a solution leads to a nonlinear equation of the fourth order that determines the class of permissible potentials. Equation (4.4) has, for instance, the following simple solutions:

$$\begin{aligned}&U=\frac{1}{2}\xi^2(x)+\text{const},\qquad U=\frac{1}{18}\xi^2(x)+\frac{1}{6}\xi_0^2,\\ &U=\frac{1}{2}\xi^2(x)+\frac{\omega^2}{\xi^2}+\frac{1}{6}\xi_0^2.\end{aligned} \tag{4.5}$$

Note that these solutions are representatives of the solutions considered earlier for the cases where the L are operators linear and quadratic in the momentum operator.

Direct substitution verifies that at $\omega=\frac{1}{2}$ and $\xi_0=0$ the symmetric potential

$$U(x)=\frac{1}{8}x^2+\frac{2}{1+x^2}-\frac{4}{(1+x^2)^2}+\frac{1}{3}, \tag{4.6}$$

is one of the regular solutions of Eq. (4.4). Let us verify that with this potential the spectrum of the Hamiltonian operator (2.7) is equidistant. If in the Schrödinger equation (2.3) we allow for (2.7) and (4.6) and assume that

$$\psi_E(x)=\frac{F(x)}{(1+x^2)}\exp\left(-\frac{x^2}{4}\right) \tag{4.7}$$

we have the following equation for $F(x)$:

$$(1+x^2)\left(\frac{d^2F}{dx^2}-x\frac{dF}{dx}+\varepsilon F\right)=4x\frac{dF}{dx},\qquad \varepsilon=2(E+{}^5/_{12}), \tag{4.8}$$

which allows for polynomial solutions at the following values of parameter ε:

$$\varepsilon=0,3,4,5,\dots,n+2,\dots . \tag{4.9}$$

Here are a few first polynomials of this kind:

$$\begin{aligned}&\varepsilon_0=0,\ F_0=1,\\ &\varepsilon_1=3,\ F_1=x^3+3x,\\ &\varepsilon_2=4,\ F_2=x^4+2x-1,\\ &\varepsilon_3=5,\ F_3=x^5-5x,\\ &\varepsilon_4=6,\ F_4=x^6-3x^4-9x^2+3,\\ &\varepsilon_5=7,\ F_5=x^7-7x^5-7x^3+21x,\\ &\varepsilon_6=8,\ F_6=x^8-12x^7+10x^4+60x-15,\\ &\varepsilon_7=9,\ F_7=x^9-18x^7+54x^5+90x^3-135x.\end{aligned} \tag{4.10}$$

Obviously, the eigenvalues $\varepsilon_2=n+2$ correspond to the strictly equidistant part of the spectrum of the Hamiltonian operator for $n\geqslant1$. But a gap separates the ground-state energy $E_0 = -5/12$ from the equidistant part. Note that for $n \geqslant 1$ a polynomial of degree $n+2$ corresponds to an eigenfunction with n zeros. It is easy to verify that the polynomials $F_n(x)$ emerging in this problem can be represented in the form

$$F_n(x) = (-1)^n (1+x^2)^2 \exp\left(\frac{x^2}{2}\right)\frac{d}{dx} \times\left[(1+x^2)^{-1}\frac{d^{n-1}}{dx^{n-1}}\exp\left(-\frac{x^2}{2}\right)\right], \quad n \geqslant 1, \tag{4.11}$$

with $n \geqslant 1$, which clearly indicates the simple but nontrivial link with the Hermite polynomials. Indeed, Hermite polynomials can be defined via the relation[6]

$$He_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2}\right). \tag{4.12}$$

This enables writing the polynomials $F_n(x)$ in the form

$$F_n(x) = -(1+x^2)^2 \exp\left(\frac{x^2}{2}\right)\frac{d}{dx}\left[\frac{He_{n-1}(x)}{(1+x^2)}\exp\left(-\frac{x^2}{2}\right)\right]. \tag{4.13}$$

Combining this with the well-known recurrence formulas for Hermite polynomials, we arrive, for instance, at the following alternative representations of $F_n(x)$:

$$F_n(x) = (x^3+3x)He_{n-1}(x) - (n-1)(1+x^2)He_{n-2}(x) = He_{n+2}(x) + 2(n+2)He_n(x) + (n+2)(n-1)He_{n-2}(x). \tag{4.14}$$

Allowing for the fact that the solutions of the Schrödinger equation with potential (4.6) lead to nonclassical polynomials, we give below the expressions for the generating functions of $F_n(x)$ and the corresponding eigenfunctions $\psi_n(x)$:

$$\begin{aligned} F(x,\tau) &= \sum_{n\geqslant 0}\frac{\tau^n}{n!}F_{n+1}(x) = [(x^3+3x)-(1+x^2)\tau]\exp\left(x\tau-\frac{\tau^2}{2}\right), \\ S(x,\tau) &= \sum_{n\geqslant 0}\frac{\tau^n}{n!}\psi_n(x) = \left[\frac{x^3+3x}{1+x^2}-\tau\right]\exp\left(-\frac{x^2}{4}+x\tau-\frac{\tau^2}{2}\right). \end{aligned} \tag{4.15}$$

Equations (4.15) make it possible to establish all the necessary recurrence formulas for the $F_n(x)$, for example,

$$\begin{aligned} &(x^3+3x)F_{n+2}(x) - (x^4+2x-1)F_{n+1}(x) \\ &+2nxF_n(x) - n(1+x^2)[F_{n+1}(x) \\ &-2xF_n(x) + (n+1)F_{n-1}(x)] = 0, \end{aligned} \tag{4.16}$$

and the formulas for the eigenfunctions with $n \geqslant 1$,

$$\begin{aligned} &\psi_{n+2} - x\psi_{n+1} + n\psi_n = -\exp(-x^2/4)He_n(x), \\ &\frac{d\psi_{n+1}}{dx} + \frac{1}{2}x\psi_{n+1} - n\psi_n = \left[\exp\left(-\frac{x^2}{4}\right)He_n(x)\right]\frac{d}{dx}\left[\frac{x^3+3x}{1+x^2}\right]. \end{aligned} \tag{4.17}$$

Finally, let us consider the normalization integral

$$I_{nn'} = \int_{-\infty}^{+\infty} dx\,\psi_n(x)\psi_{n'}(x) = \int_{-\infty}^{+\infty} dx \exp\left(-\frac{x^2}{2}\right)\frac{F_n(x)F_{n'}(x)}{(1+x^2)^2}. \tag{4.18}$$

Replacing in (4.18) one of the polynomials by (4.11) and employing the formula

$$(1+x^2)\frac{dF_n}{dx} = (n-1)xHe_{n-2}(x), \tag{4.19}$$

which follows from (4.14), we get after integrating by parts $n-1$ times,

$$I_{nn'} = (n-1)(n-1)!(2\pi)^{1/2}\delta_{nn'}. \tag{4.20}$$

where $\delta_{nn'}$ is the Kronecker delta. Thus, the anharmonic-oscillator eigenfunctions normalized to unity have the form

$$\psi_n(x) = \frac{\exp(-x^2/4)}{[(n-1)(n-1)!(2\pi)^{1/2}]^{1/2}}\frac{F_n(x)}{(1+x^2)}, \quad n \geqslant 1. \tag{4.21}$$

Direct calculations of the appropriate normalization integral in the case of the ground state yield

$$\psi_0 = \left(\frac{2}{\pi}\right)^{1/4}\frac{\exp(-x^2/4)}{1+x^2}. \tag{4.22}$$

Note that all the above expressions for the recurrence formulas and generating functions are rigorously defined for states corresponding to the equidistant part of the spectrum of eigenvalues of the Hamiltonian operator.

The ground state of the system,

$$\psi_0 = \frac{\exp(-x^2/4)}{1+x^2}, \quad E_0 = -\frac{5}{12}$$

is separated from the first excited state

$$\psi_1 = \frac{(x^3+3x)}{1+x^2}\exp\left(-\frac{x^2}{4}\right), \quad E_1 = \frac{13}{12}$$

by a gap $E_1 - E_0 = 3/2$, which is three times the energy difference between neighboring levels in the strictly equidistant part of the spectrum, $E_{n+1} - E_n = 1/2$, $n \geqslant 1$. This, however, does not violate the strict periodicity in time of the general solution of the time-dependent Schrödinger equation,

$$\psi(x,t) = \exp\left(-\frac{5it}{12}\right) \times\left\{C_0\psi_0 + \sum_{n\geqslant 1} c_n \exp\left[\frac{i(n+2)t}{2}\right]\psi_n(x)\right\} \tag{4.23}$$

which with arbitrary constants $C_0, C_1, \ldots, C_n, \ldots$ is a strictly periodic function of time with a fundamental period of 24π. Since $\psi(x,t+12\pi) = -\psi(x,t)$, the probability density $|\psi(x,t)|^2$ is a strictly periodic function of time with a period of 12π. Let us verify that in the case of potential (4.6) the operator L defined in (4.1) and (4.3) has the properties specified in (2.12). Calculations show that at $\omega = 1/2$, $\xi_0^2 = 0$, and $C = -\omega = -1/2$,

$$\begin{aligned} &L_3 = -i, \quad L_2(x) = \frac{1}{2}x, \\ &iL_1 = \frac{3}{2} + \frac{1}{4}x^2 + \frac{6}{1+x^2} - \frac{12}{(1+x^2)^2}, \end{aligned} \tag{4.24}$$

$$L_0=x\left[\frac{1}{4}+\frac{1}{8}x^2+\frac{1}{1+x^2}+\frac{2}{(1+x^2)^2}-\frac{24}{(1+x^2)^3}\right].$$

The action of operator L on the eigenfunction ψ_n can be written in the form

$$L\psi_n=\mathfrak{L}_0\psi_n+\mathfrak{L}_1 p\psi_n, \tag{4.25}$$

where

$$\mathfrak{L}_0=x\left[-\frac{1}{1+x^2}-\frac{2}{(1+x^2)^2}+\frac{8}{(1+x^2)^3}\right]+\frac{1}{2}\varepsilon_n x,$$
$$i\mathfrak{L}_1=-\frac{2}{1+x^2}+\frac{4}{(1+x^2)^2}-\varepsilon_n,\quad \varepsilon_n=n+2,\quad n\geqslant 1. \tag{4.26}$$

Allowing for (4.7), we find that Eq. (4.25) assumes the form

$$L\psi_n\frac{\exp(-x^2/4)}{1+x^2}\left[\varepsilon_n\left(xF_n-\frac{dF_n}{dx}+\frac{2xF_n}{1+x^2}\right)+2\frac{1-x^2}{(1+x^2)^2}\frac{dF_n}{dx}\right]. \tag{4.27}$$

Clearly, $L\psi_0=0$ because $\varepsilon_0=0$ and $F_0=1$. Simple calculations yield further:

$$L\psi_1=3\psi_2,\ L\psi_2=4\psi_3,\ L\psi_3=5\psi_4. \tag{4.28}$$

It can be shown by induction that

$$L\psi_n=(2+n)\psi_{n+1}.$$

Thus, L possesses the required properties (2.12); namely, it shifts the eigenstates with $n\geqslant 1$ upward in the spectrum. In what follows we denote this operator by L_+. To build the operator L_- that shifts the eigenstates downward we employ the fact that L_- is defined by Eqs. (4.1) and (4.3) if we replace ω with $-\omega$. At $\xi_0^2=0$ and $C=-\omega$ we obtain

$$L_-=L_++3U_x-\omega\xi(x)+2[(L_1)_++i\omega]p+2L_3p^3,\quad \omega>0. \tag{4.29}$$

In the case of potential (4.6) this expression leads to

$$L_-\psi_n=L_+\psi_n+\frac{\exp(-x^2/4)}{1+x^2}\left[-\varepsilon_n xF_n+(2\varepsilon_n-1)\frac{dF_n}{dx}+4\frac{dF_n/dx-\varepsilon_n xF_n}{1+x^2}-\frac{8dF_n/dx}{(1+x^2)^2}\right], \tag{4.30}$$

where $L_+\psi_n$ is defined in (4.27). Clearly, $L_-\psi_0=0$. Moreover, $L_-\psi_1=0$ since at $n=1$ the expression in brackets on the right-hand side of (4.30) is equal to $-3F_2(x)$ and the corresponding term is $-3\psi_2(x)$. And since according to (4.28) $L_+\psi_1=3\psi_2$, the right-hand side of (4.30) vanishes. Also,

$$L_-\psi_2=4\psi_1,\ L_-\psi_3=2(3+2)\psi_2,\ldots,\ L_-\psi_n=(n-1)(n+2)\psi_{n-1}. \tag{4.31}$$

Thus, for the potential specified by (4.6) the problem of building operators $L_\pm$ that satisfy (2.11) has a solution. In view of (2.12), the operators L_+L_- and L_-L_+ are functions of the Hamiltonian operator H. Performing simple calculations, we find that

$$[L_+L_-]=-2\mathscr{H}(6\mathscr{H}-5),$$
$$N=L_+L_-+L_-L_+=2\mathscr{H}(8\mathscr{H}^2-10\mathscr{H}+1), \tag{4.32}$$

where $\mathscr{H}\equiv H+5/12$. Note that the second operator relation in (4.32) implicitly defines the Hamiltonian operator $\mathscr{H}$ (or H) as a function of operator N.

5. In conclusion we note the possibility of generalizing the above approach on the basis of the following reasoning.

If we apply the operator relations (2.10) to the eigenstates ψ_n in the case where the operators $L_\pm$ satisfy conditions (2.11), we arrive at the following relations:

$$(E_{n\pm1}-E_n\mp\omega)\psi_{n\pm1}=0. \tag{5.1}$$

These relations have solutions if the first-order difference of energy eigenvalues, $E_{n\pm1}-E_n$, is independent of the integral-valued variable n and is equal to $\mp\omega$. Let us examine the formal operator relations

$$[[H,L_\pm],L_\pm]=\pm2\omega L_\pm^2. \tag{5.2}$$

If we apply these to the eigenstates ψ_n in the case where the $L_\pm$ satisfy conditions (2.12), we get

$$(E_{n\pm2}+E_n-2E_{n\pm1}\mp2\omega)\psi_n=0. \tag{5.3}$$

These relations can be solved if the second-order difference of the energy eigenvalues, $E_{n\pm2}+E_n-2E_{n\pm1}$, is independent of the integer-valued variable n and equal to $\pm2\omega$. In other words, in the case of the Hamiltonian operator (2.7) the solutions of the operator equations (5.2) must result in potentials $U(x)$ to which the following point spectrum corresponds:

$$E_n=\omega(n^2+\gamma n)+\omega_0, \tag{5.4}$$

where ω, ω_0, and γ are the spectrum parameters. If γ is an integer, the spectrum (5.4) belongs to the class of "integral" spectra (2.6) specified earlier, the spectra linked with the strict periodicity in time of the general solution of the Schrödinger equation (2.1). Here is an example illustrating this hypothesis. At $M=1$ substituting (3.1) into the operator relation (5.2) yields the following system of equations overdetermined with respect to $L_0(x)$ and $L_1(x)$:

$$\begin{aligned}&{}^1/_2 iL_1''+iL_2'L_0''-(L_0')^2-{}^1/_2 iL_0'''L_1-L_1^2U''-L_1L_1'U'\\&\quad=2\omega(L_0^2-iL_1L_0'),\\&-2L_1'L_0'+{}^1/_2 iL_1''L_1'+iL_1'L_1''-L_0'L_1'+L_1L_0''-{}^1/_2 iL_1L_1'''\\&\quad=2\omega(L_0L_1-iL_1L_1'),\\&L_1L_1''-2(L_1')^2=2\omega L_1^2,\quad (\ldots)'\equiv d/dx.\end{aligned} \tag{5.5}$$

Solving the last two equations in (5.5) yields

$$L_1=\frac{C_1}{\cos\varphi},\quad L_0=\frac{C_0}{1-\sin\varphi},\quad \varphi=(2\omega)^{1/2}x, \tag{5.6}$$

where C_0 and C_1 are arbitrary constants. The solutions of the first equation in (5.6),

$$L_2\frac{d^2U}{d\varphi^2}+L_1\frac{dL}{d\varphi}\frac{dU}{d\varphi}+L_0^2+\left(\frac{dL_0}{d\varphi}\right)^2=-i(2\omega)^{1/2}\left[\frac{d}{d\varphi}(L_0L_1)+2L_1^{-1}\frac{dL_0}{d\varphi}\left(\frac{dL_1}{d\varphi}\right)^2\right], \tag{5.7}$$

determine the family of potentials $U(\varphi)$ that guarantee that the operator equation (5.2) has a solution. The solutions of Eq. (5.7) satisfying the condition that there exists a strictly point spectrum are

$$U(\varphi)=\frac{U_0}{1-\sin\varphi},\quad U_0=-i\left(\frac{\omega}{2}\right)^{1/2}\frac{C_0}{C_1}-\left(\frac{C_0}{C_1}\right)^2. \quad (5.8)$$

Here the admissible values of parameters C_0 and C_1 are limited by the condition that U_0 must be real. The eigenvalue problem for the singular potential (5.8) leads, as is known,[7] to a point spectrum of the type (5.4).

The above examples attest to the validity of the hypothesis that for the Hamiltonian operator (2.7) the solutions of the operator equations of the form

$$\underbrace{[\ldots[H,\ L],\ \ldots,\ L]}_{N}=N!\omega L^N,\quad N\geqslant 1 \quad (5.9)$$

specify, in certain conditions, a class of potentials $U(x)$ leading to a point spectrum of the eigenvalues E_n represented by a polynomial of degree N in the integral-valued variable n.

We have thus demonstrated the possibility of a "direct" approach to the problem of constructing, on the entire straight line or on segments of that line, potentials that lead to polynomial point spectra of the Hamiltonian operator (at least in the case of one degree of freedom). Representing the solutions of the operator equations (5.9) in the form of polynomials in the momentum operator (3.1) results in constructing operators L that are analogs of creation and annihilation operators. But whether an appropriate Fock space can be introduced depends on representing the Hamiltonian operator as a function of the operator products L_+L_- and L_-L_+ and supplementing the condition in which the $L_\pm$ act via (2.11), and this requires additional analysis.

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