Interaction of magnetostatic waves (MSW) with phonons in a transversely magnetized plate

M. I. Kaganov and T. I. Shalaeva

Kapitza Institute for Physical Problems, Russian Academy of Sciences, Moscow

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The interaction of long-wave magnetostatic waves with phonons is analyzed for the case of a transversely magnetized plate in the low temperature limit. Relaxation times showing a nonexponential temperature dependence are calculated.

1. The objective of this paper is to calculate the lifetime of a magnetostatic wave (MSW) in a ferromagnetic plate magnetized normal to the surface. We restrict our discussion to the case of very low temperatures[1)] such that

$$T \ll \hbar\omega_0,\ \hbar[(\omega_0(\omega_0+\omega_M))^{1/2}-\omega_0] \tag{1}$$

and we also assume that

$$\omega_0,\ \omega_M \ll u^2/(2\omega_{ex}a^2). \tag{2}$$

With these assumptions, it has been shown by us earlier[1] that the inhomogeneous exchange contribution to magnon energy may sometimes be neglected when treating the magnon-phonon interaction. We shall have occasion, later on in our study, to cast light on the role of the exchange interaction in situations where it cannot be neglected; for the moment, however, only MSW dispersion without allowance for exchange interaction will be considered. For an MSW of frequency ω and wave vector $\mathbf{k}$ (two-dimensional in the plane of the plate) we have[2]

$$\omega = \left(\omega_0^2 + \omega_0\omega_M \frac{k^2}{k^2+q_n^2}\right)^{1/2} \equiv \omega_n(k), \tag{3}$$

and depending on the symmetry of the (alternating) magnetic field in the plate, one of two transcendental equations determines the quantity q_n ($n = 0,1,\ldots$) here. If the magnetic field potential φ is symmetric with respect to the plane $z = 0$, then q_n is the $(n + 1)$st ($n = 0,1,\ldots$) root of the equation

$$k = q\,\mathrm{tg}(qd), \tag{4}$$

whereas for $\varphi(-z) = -\varphi(z)$, q_n is given by the $(n + 1)$st ($n = 0,1,\ldots$) root of the equation

$$-k = q\,\mathrm{ctg}(qd). \tag{5}$$

The plate is a layer of thickness $2d$ ($|z| < d$), and the z-axis is parallel to $\mathbf{H}\|\mathbf{M}$.

Typical $\omega = \omega_n(k)$ curves are shown in Fig. 1. In the following, the superscripts s and a label solutions of equations (4) and (5), respectively. For $kd \ll 1$, the zeroth-mode frequency $\omega_0^{(s)}$ varies linearly with k,

$$\omega_0^{(s)} \approx \omega_0 + {}^1/_2\,\omega_M kd, \tag{6}$$

and for $n \neq 0$ we have

$$\omega_n^{(s)} \approx \omega_0 + \frac{\omega_M}{2(\pi n)^2}(kd)^2, \quad n>0. \tag{6'}$$

For $kd \gg \pi(n + 1/2)$,

$$\omega_n^{(s)} \approx [\omega_0(\omega_0+\omega_M)]^{1/2}\left\{1 - \frac{\pi^2}{2}\frac{\omega_M}{\omega_0+\omega_M}\frac{(n+{}^1/_2)^2}{(kd)^2}\right\}. \tag{7}$$

It should be noted that the larger the mode number n the more restrictive the condition of applicability of (7). The case of large n is of no interest here because, as will be made plain below, the inclusion of the exchange interaction is obligatory when the lifetime of an $n \gg 1$ mode is to be calculated.

The mode number n is equal to the number of zeros of $\varphi(z)$ over the plate thickness, and the MSW frequency—and hence the energy $\hbar\omega_n(k)$ of the corresponding elementary excitation—decreases with n, so that the zeroth mode has the highest frequency. These features are of course easily interpreted as being due to the anomalous character (noted in particular in Ref. 1) of the propagation of MSWs along the $\mathbf{H}$ direction.

From (5) it follows that, for $kd \ll 1$,

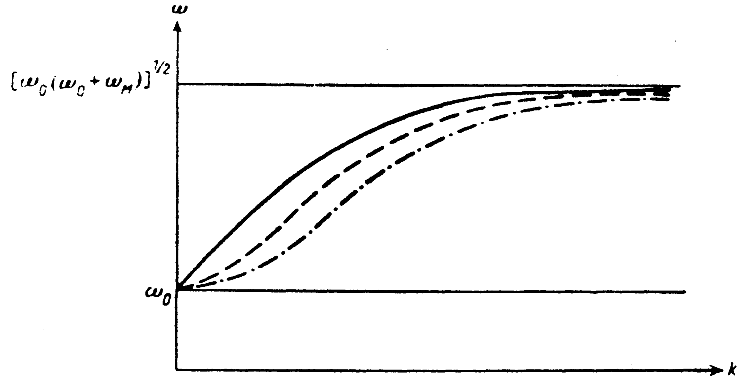

FIG. 1. Plot of $\omega = \omega_n(k)$. The solid line represents the dependence $\omega = \omega_0^{(s)}(k)$, linear for $kd \ll 1$; the dashed line: $\omega = \omega_n^{(s)}(k)$ for $n > 0$; the dash-dot line: $\omega = \omega_n^{(a)}(k)$ for the same n as that taken for $\omega = \omega_0^{(s)}(k)$.

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$$\omega_n^{(a)} \approx \omega_0 + \frac{\omega_M}{2} \frac{(kd)^2}{[\pi(n+1/2)]^2}, \quad n \geqslant 0, \tag{8}$$

and for $kd \gg \pi(n+1)$

$$\omega_n^{(a)} \approx [\omega_0(\omega_0+\omega_M)]^{1/2} \left\{1 - \frac{\pi^2}{2} \frac{\omega_M}{\omega_0+\omega_M} \frac{(n+1)^2}{(kd)^2}\right\}, \quad n \geqslant 0. \tag{8'}$$

The quantization of MSWs can be performed in a standard way by using the approximate Holstein–Primakov representation[3]

$$m^{\pm} \equiv m_x \pm i m_y = (2g\hbar M)^{1/2} \begin{Bmatrix} a^+ \\ a \end{Bmatrix}, \tag{9}$$

$$M_z = M - g\hbar a^+ a,$$

and expanding in terms of the orthonormal eigenfunctions so as to express the operators a^+ and a through the Bose operators a_{kn}^+ and a_{kn}.[2)] For a symmetric (s) MSW, for example, we have

$$a = \frac{1}{V^{1/2}} \sum_{\mathbf{k},n} \{u_{\mathbf{k}n} a_{\mathbf{k}n} e^{i\mathbf{k}\rho} + v_{\mathbf{k}n}{}^{\cdot} a_{\mathbf{k}n}{}^{+} e^{-i\mathbf{k}\rho}\} \frac{\cos(q_n z)}{[1+\sin^2(q_n d)/kd]^{1/2}}, \tag{10}$$

where V is the volume of the plate and $\rho \equiv (x,y)$.

By substituting the Eq. (9) in the Hamiltonian of the magnet,

$$\mathscr{H} = -\frac{1}{2} \int dv \{m_x h_x + m_y h_y + 2m_z H\}, \tag{11}$$

after first expressing the MSW magnetic field $\mathbf{h}$ in terms of $\mathbf{m}$ by use of appropriate magnetostatic equations. The resulting expression is then equated to the standard form

$$\mathscr{H} = \sum_{\mathbf{k},n} \hbar\omega_n(k) a_{\mathbf{k}n}{}^{+} a_{\mathbf{k}n}. \tag{12}$$

We will determine $u_{\mathbf{k}n}$ and $v_{\mathbf{k}n}$ coefficients in u,v-transformation (10):

$$u_{\mathbf{k}n} = \frac{\omega_0 + \omega_n(k)}{2[\omega_0\omega_n(k)]^{1/2}}, \quad v_{\mathbf{k}n} = \left(\frac{k_y - ik_x}{k}\right)^2 \frac{\omega_0 - \omega_n(k)}{2[\omega_0\omega_n(k)]^{1/2}}. \tag{13}$$

Although there are four sound vibration types known to exist in plates (see Ref. 4), we will only be interested in those phonons whose emission or absorption does not alter the symmetry of an MSW. On the other hand, we shall see below that, in limiting cases, the damping of an MSW is dominated by a sound wave whose displacement vector $\mathbf{U}$ is homogeneous with respect to the coordinate z. It is therefore permissible to consider only one vibration type,

$$\mathbf{U} = \frac{1}{V^{1/2}} \sum_{\mathbf{f},m} \left[\frac{\hbar}{2\rho\Omega_m(\mathbf{f})}\right]^{1/2} \mathbf{e}_\mathbf{f} (b_{\mathbf{f}m} e^{i\rho\mathbf{f}} + b_{\mathbf{f}m}{}^{+} e^{-i\rho\mathbf{f}}) \cos(\beta_m z). \tag{14}$$

Here ρ denotes the mass density of the magnetic; $\mathbf{f}$ is the (two-dimensional) wave vector of the phonon; $b_{\mathbf{f}m}$ and $b_{\mathbf{f}m}^+$ are Bose operators; β_m is a root of the equation

$$\sin(\beta_m d) = 0, \tag{15}$$

and the frequency $\Omega_m(\mathbf{f})$ and polarization $\mathbf{e}_\mathbf{f}$ are given by the respective equations

$$\Omega_m(\mathbf{f}) = (f^2 + \beta_m^2)^{1/2} u, \quad \mathbf{e}_\mathbf{f} = -\left(\frac{f_y}{f}, \frac{f_x}{f}, 0\right). \tag{16}$$

We note that the phonons (14) are transverse with respect to $\mathbf{f}$ and that one possible solution of (15) is that for which $\beta_0 = 0$.

The Hamiltonian for one-phonon processes of interest here is quadratic in $\mathbf{m}$ and linear in the strain tensor components

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

(see Refs. 3 and 5). For an elastically isotropic magnet, neglecting inhomogeneous exchange effects, we can confine ourselves for the interaction Hamiltonian to the simplest invariant with one magnetoelastic constant γ.

$$\mathscr{H}_{int} = \gamma \int M_i M_k u_{ik} dv. \tag{17}$$

From this, separating the anharmonic terms quadratic in a and a^+ and linear in b and b^+ and remembering that both a operators belong to a spectrum branch of the same symmetry, we have

$$\mathscr{H}_{int} = \sum_{\mathbf{k},n,\mathbf{k}',n',\mathbf{f},m} \{\Psi_1 a_{\mathbf{k}n} a_{\mathbf{k}'n'}^{+} b_{\mathbf{f}m}^{+} \Delta(\mathbf{k}-\mathbf{k}'-\mathbf{f})$$
$$+ \Psi_2 a_{\mathbf{k}n} a_{\mathbf{k}'n'} b_{\mathbf{f}m}^{+} \Delta(\mathbf{k}+\mathbf{k}'-\mathbf{f}) + \Psi_3 a_{\mathbf{k}n} a_{\mathbf{k}'n'}^{+} b_{\mathbf{f}m} \Delta(\mathbf{k}-\mathbf{k}'+f) + \text{H.a.}\}, \tag{18}$$

where

$$\Psi_1 = (v_{\mathbf{k}n} u_{\mathbf{k}'n'} t_{\mathbf{f}m}^{\cdot} - u_{\mathbf{k}n} v_{\mathbf{k}'n'}^{\cdot} t_{\mathbf{f}m}) \Psi,$$
$$\Psi_2 = (v_{\mathbf{k}n} v_{\mathbf{k}'n'} t_{\mathbf{f}m}^{\cdot} - u_{\mathbf{k}n} u_{\mathbf{k}'n'} t_{\mathbf{f}m}) \Psi.$$
$$\Psi_3 = (v_{\mathbf{k}n} u_{\mathbf{k}'n'}^{\cdot} t_{\mathbf{f}m} - u_{\mathbf{k}n} v_{\mathbf{k}'n'}^{\cdot} t_{\mathbf{f}m}^{\cdot}) \Psi,$$
$$\Psi = \frac{\gamma g\hbar M}{8}\left(\frac{\hbar}{2\rho V}\right)^{1/2}\left(1+\frac{\sin^2(q_n d)}{kd}\right)^{-1/2} \tag{19}$$
$$\times\left(1+\frac{\sin^2(q_{n'} d)}{k'd}\right)^{-1/2}\left\{\frac{\sin(q_n - q_{n'} - \beta_m)d}{(q_n - q_{n'} - \beta_m)d}\right.$$
$$\left. + \frac{\sin(q_n + q_{n'} - \beta_m)d}{(q_n + q_{n'} - \beta_m)d} + \frac{\sin(q_n + q_{n'} + \beta_m)d}{(q_n + q_{n'} + \beta_m)d}\right\},$$
$$t_{\mathbf{f}m} = \frac{(f_x + if_y)^2}{f(\Omega_{\mathbf{f}m})^{1/2}}.$$

Equations (19) justify our neglect of the antisymmetric phonon modes: they drop out of the matrix-element calculation anyway integration over z.

We are now in a position to calculate $\tau_n^{-1}(\mathbf{k})$, the inverse lifetime of an MSW. Taking account of both the direct and inverse processes but retaining only the loss terms in the collision integral, it is found that[3]

$$\frac{1}{\tau_n(\mathbf{k})}$$
$$\frac{2\pi}{\hbar^2} \sum_{\mathbf{k}',n',\mathbf{f},m} \{|\Psi_1|^2 (n_{\mathbf{k}'n'} + N_{\mathbf{f}m} + 1) \Delta(\mathbf{k}-\mathbf{k}'-\mathbf{f}) \delta(\omega_n(k)$$
$$-\omega_{n'}(k') - \Omega_m(f))$$
$$+ |\Psi_2|^2 (N_{\mathbf{f}m} - n_{\mathbf{k}'n'}) \Delta(\mathbf{k}-\mathbf{k}'+\mathbf{f}) \delta(\omega_n(k) - \omega_{n'}(k')$$
$$+\Omega_m(f)) + |\Psi_3|^2 (n_{\mathbf{k}'n'} - N_{\mathbf{f}m}) \Delta(\mathbf{k}+\mathbf{k}'-\mathbf{f}) \delta(\omega_n(k)$$
$$+\omega_{n'}(k') - \Omega_m(f))\}. \tag{20}$$

Here

$$N_{fm}=\frac{1}{\exp[\hbar\Omega_m(f)/T]-1},\quad n_{\mathbf{k}n}=\frac{1}{\exp[\hbar\omega_n(k)/T]-1}\approx\exp\left[-\frac{\hbar\omega_n(k)}{T}\right] \tag{21}$$

are the equilibrium distribution functions for phonons and MSWs, respectively, and we have made use of (1) in proceeding from the Bose function to the exponential function in the last equation.

As is customary, the summation over $\mathbf{k}'$ can be replaced by an integration

$$\sum_{\mathbf{k}'}\to\frac{S}{(2\pi)^2}\int\ldots d^2k'=\frac{S}{(2\pi)^2}\int\ldots k'dk'd\varphi,$$

where $S=V/2d$ is the plate surface area and φ the polar angle. Unfortunately, both the dispersion laws (3) and (15) and the expressions for the amplitudes Ψ_i ($i=1,2,3$) are too complicated to allow a straightforward evaluation of $\tau_n(k)$ even in the limiting cases we have chosen.

In Eqs. (18) through (20), the following three one-phonon processes are covered:

1) an MSW emits a phonon;
2) an MSW absorbs a phonon; and
3) two MSWs merge to emit a phonon.

A discussion of the third process will not be given here because for such an event to occur one MSWs must meet another, and the number of such potential partners is exponentially small—hence a small factor $\exp(-\hbar\omega_0/T)$ in the corresponding probability expression.

2. We begin with phonon emission, with particular emphasis on the $kd\ll1$ case. The relevant energy and momentum conservation laws

$$\mathbf{k}=\mathbf{k}'+\mathbf{f},\quad \omega_n(k)=\omega_{n'}(k')+\Omega_m(f) \tag{22}$$

impose considerable restrictions on the calculations that follow. It is therefore necessary to analyze (22) in detail for various situations that may arise.

a) Both the initial and finite n values are zero ($n=n'=0$) and $kd\ll1$. Combining (6), (16), and (22) gives

$$\frac{1}{2}\omega_M(k-k')d=u\left[(k-k')^2+\left(\frac{\pi m}{d}\right)^2\right]^{1/2} \tag{22'}$$

showing that $k'<k$, with a consequence that this result only holds for $m=0$ in the limit as $k\to0$. Looking back, this justifies our choice of the $\beta_0=0$ mode in (14). For $m=0$ (22') yields then

$$(k^2+k'^2-2kk'\cos\varphi)^{1/2}=\alpha(k-k'),\quad \alpha=\frac{\omega_Md}{2u}, \tag{23}$$

where φ is the angle between the vectors $\mathbf{k}$ and $\mathbf{k}'$. Hence

$$-1\leqslant\cos\varphi=\frac{(k^2+k'^2)(1-\alpha^2)+2\alpha^2kk'}{2kk'}\leqslant1,\quad k'<k. \tag{24}$$

It is seen that for $\alpha<1$ there is no solution, whereas for $\alpha>1$ the emission process gives rise to MSWs with wave vectors k' in the range

$$\left(\frac{\alpha-1}{\alpha+1}k,k\right),$$

and it will be noticed that to every such k' there correspond two angles determined by (23), $\varphi(k')$ and $2\pi-\varphi(k')$—a very natural result if one recalls that the vector $\mathbf{k}$ specifies the only preferred direction in the plane of the plate. The possible values of $k'=k'(k,\varphi)$ are shown schematically in Fig. 2, where the ends of the $\mathbf{k}'$ vectors lie on the curve $\mathscr{K}$ located between the circles of radii k and $[(\alpha-1)/(\alpha+1)]k$. As α approaches unity ($\alpha\geqslant1$), the curve $\mathscr{K}$ contracts to a straight segment $(0,k)$ on the $\varphi=0$ axis, and the equation (24) for the curve $\mathscr{K}$ degenerates into $\cos\varphi=1$ (i.e., $\varphi=0$ and 2π).

The inverse lifetime[3)] $1/\tau^{00}(k)$ due to phonon emission is evaluated by performing the necessary integration in the first term in (20). In doing so, the integration over k' helps dispose of the δ-function that occurs. The result is

$$\frac{1}{\tau^{00}(k)}=\frac{(\gamma gM)^2}{2^{11}\pi\rho\omega_0^2d}\begin{cases}f_1(\alpha)k^3T/(\omega_Md), & kd\ll\dfrac{T}{\hbar\omega_M},\\ f_2(\alpha)\hbar k^4, & \dfrac{T}{\hbar\omega_M}\ll kd\ll1,\end{cases} \tag{25}$$

where

$$f_1(\alpha)=\frac{\alpha^4}{\alpha^2-1}\left\{\left(\frac{\pi}{2}-\arcsin\alpha^{-1}\right)(\alpha^2+1)-2(\alpha^2-1)^{1/2}\right\},$$

$$f_2(\alpha)=\frac{\alpha^4}{(\alpha^2-1)^2}\left\{(\alpha^2+2)(\alpha^2-1)^{1/2}-(2\alpha+1)\left(\frac{\pi}{2}-\arcsin\alpha^{-1}\right)\right\}.$$

Inspection will show that as $\alpha\to1$ expression (25) tends to infinity,

$$f_1(\alpha)\propto(\alpha-1)^{-1/2},\ f_2(\alpha)\propto(\alpha-1)^{-3/2}$$

and one can enquire, why? The inequality $\alpha>1$ is in fact the condition for the Cherenkov sound radiation: for the MSW velocity at $\alpha>1$ we have

$$v_0=\frac{\partial\omega_0}{\partial k}=\frac{\omega_Md}{2}>u,\quad kd\ll1.$$

At $\alpha=1$, a resonance-type situation takes place ($v_0=u$), leading to a divergence in the probability for an MSW to emit a phonon. The dispersion law (6) is only an approximation, however, and we can eliminate the divergence by refining Eq. (6),

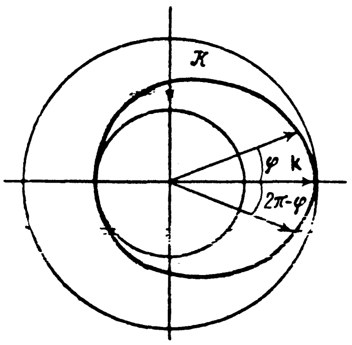

FIG. 2. Curve $\mathscr{K}$: variation of the wave vector k' with the angle φ for phonon emission processes. The vector $\mathbf{k}$ is fixed; $n=n'=0$; $k'(\varphi=0)=k$; $k'(\varphi=\pi)=[(\alpha-1)/(\alpha+1)]k$; $kd\ll1$.

$$\omega_0(k)\approx\omega_0\left\{1+\frac{\omega_M}{2\omega_0}kd-\frac{\omega_M}{8\omega_0}\left(2+\frac{\omega_M}{\omega_0}\right)(kd)^2+\ldots\right\}. \tag{26}$$

Equation (23) then becomes

$$(k^2+k'^2-2kk'\cos\varphi)^{1/2}-(k-k')$$
$$=(\alpha-1)(k-k')-\left(2+\frac{\omega_M}{\omega_0}\right)\frac{d}{4}(k^2-k'^2). \tag{27}$$

This is solvable only if

$$\frac{1}{4}\left(2+\frac{\omega_0}{\omega_M}\right)kd<\alpha-1,\quad \alpha-1\ll 1, \tag{28}$$

which means that, for α values close to unity, the processes under study have a threshold as a function of the wave vector k on going from the short wavelength side. Its value is[4)]

$$k_{\rm th}=\frac{4(\alpha-1)}{d(2+\omega_0/\omega_M)},\quad \alpha-1\ll 1, \tag{29}$$

implying that $k_{\rm th}d\ll 1$ and thereby confirming the validity of the expansions above.

We now recalculate $[\tau^{00}(k)]^{-1}$ by applying expression (27) to the first term in (20). As before, the result will be strongly dependent on the relationship between the temperature and the dispersive part of the magnon energy $\hbar[\omega_0(k)-\omega_0]$. Also, different analytical expressions will be obtained, depending on whether

$$k<\frac{2(\alpha-1)}{d(2+\omega_M/\omega_0)}=\frac{1}{2}k_{\rm th}$$

or

$$k>\frac{k_{\rm th}}{2}$$

(these expressions are of course identical for $k=k_{\rm th}/2$).

Thus, for $T\gg\hbar\omega_M kd$,

$$\frac{1}{\tau^{00}(k)}=\frac{(\gamma gM)^2(dk)^2T}{2^{11}\pi\rho d^3\omega_0^2u}\left[\frac{k_{\rm th}}{2k(\alpha-1)}\right]^{1/2}\times\begin{cases}k\left[\arcsin\left(\dfrac{k_{\rm th}-3k}{k_{\rm th}-k}\right)-\dfrac{\pi}{2}\right], & k\leqslant\dfrac{k_{\rm th}}{2},\\ \pi(k_{\rm th}-k), & \dfrac{k_{\rm th}}{2}\leqslant k<k_{\rm th},\end{cases} \tag{30}$$

and for $T\ll\hbar\omega_M kd$

$$\frac{1}{\tau^{00}(k)}=\frac{(\gamma gM)^2(kd)^2\hbar}{2^{12}\pi\rho d^3\omega_0^2}\left(\frac{k_{\rm th}}{2k(\alpha-1)}\right)^{1/2}\times\begin{cases}k^2\left[\arcsin\left(\dfrac{k_{\rm th}-3k}{k_{\rm th}-k}\right)-\dfrac{\pi}{2}\right], & k\leqslant\dfrac{k_{\rm th}}{2},\\ \pi(k_{\rm th}-k)(3k-k_{\rm th}), & \dfrac{k_{\rm th}}{2}\leqslant k<k_{\rm th}.\end{cases}$$

Comparing these results with equations (25) reveals their identity in the case $kd\ll\alpha-1$: the first and third of equations (30) duplicate, respectively, the first and second of equations (25).

It is seen that, in a more elaborate analysis, the resonance between the magnetostatic and sound waves produces a peak in the wave-vector dependence of the inverse lifetime (Fig. 3). In the limit as $k\to 0$, the inverse relaxation time tends to zero as k^4, and at $k=k_{\rm th}$ it vanishes linearly. Its maximum value is

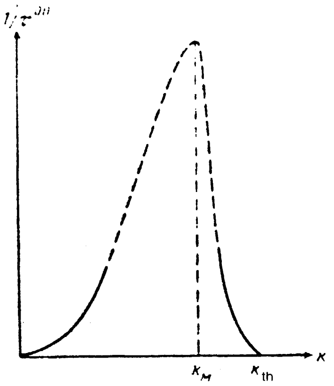

FIG. 3. The dependence of $[\tau^{00}(k)]^{-1}$ on k for phonon emission processes ($\alpha\gtrsim 1$, $kd\ll 1$).

$$k_{\rm th}=\frac{4(\alpha-1)}{d(2+\omega_M/\omega_0)}.$$

$$\frac{1}{\tau^{00}_{max}}\sim\frac{\gamma^2k_{\rm th}^3}{2^{15}\rho(\alpha-1)^{1/2}d}\begin{cases}\dfrac{T}{u}, & T\gg\hbar\omega_M k_{\rm th}d,\\ \dfrac{\hbar k_{\rm th}}{u}, & T\ll\hbar\omega_M k_{\rm th}d,\end{cases} \tag{31}$$

or

$$\frac{1}{\tau^{00}_{max}}\sim\frac{\gamma^2(\alpha-1)^{5/2}}{2^9\rho ud^4(2+\omega_M/\omega_0)^3}\begin{cases}T, & T\gg\hbar\omega_M k_{\rm th}d,\\ \dfrac{4\hbar(\alpha-1)}{d(2+\omega_M/\omega_0)}, & T\ll\hbar\omega_M k_{\rm th}d.\end{cases}$$

Now if the process above exhibits a threshold behavior for $kd\ll 1$, the relevant question is whether a threshold k value exists for α substantially different from unity. A nonexpansion analysis of the conservation laws (22) for $m=0$ will give the answer [cf. Eqs. (22) and (22')]. From (22) it is readily found that

$$\cos\varphi=\frac{k^2+k'^2-(\omega_0(k)-\omega_0(k'))^2/u^2}{2kk'}, \tag{32}$$

so that the condition for a solution is

$$u(k-k')<\omega_0(k)-\omega_0(k') \tag{33}$$

which, applying (3) with $n=0$, is easily shown (33) not to hold for $k>k_{\rm th}$ where $k_{\rm th}$ is a root of the equation

$$uk=\omega_0(k)-\omega_0. \tag{34}$$

Based on the definition (4) for q_0, this last result may be considered as an equation for q_0d and is conveniently rewritten as

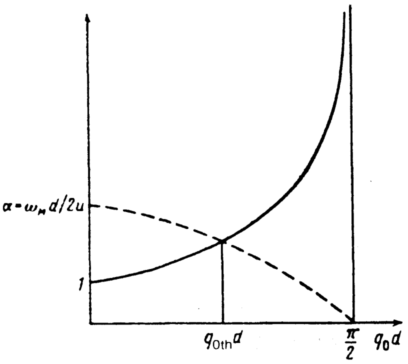

FIG. 4. Graphic solution of Eq. (35). The solid and dashed lines represent the left and right sides of the equation, respectively. It is seen that the lines do not intersect if $\alpha = \omega_M d/2u < 1$.

$$\frac{\omega_M}{\omega_0}\frac{x\,\mathrm{tg}\,x}{\alpha}+1=\alpha\frac{\sin x\cos x}{x},\quad x=q_0 d. \tag{35}$$

Referring to Fig. 4, a graphic solution of (35) clearly shows the existence of a threshold value k_{th} (recall that $\omega_M \sim \omega_0$). If $\alpha \gtrsim 1$, it is only natural that the previous value is retrieved; for arbitrary α values, the root of (35) cannot possibly be written down, and for $\alpha \gg 1$ the dashed curve in Fig. 4 climbs up while the solid one goes somewhat down. It is seen that as $\alpha \to \infty$, the root x tends to $\pi/2$, giving for $\alpha \gg 1$

$$k_{\text{th}} \approx \frac{\omega_0\omega_M}{u[(\omega_0(\omega_0+\omega_M))^{1/2}+\omega_0]}. \tag{36}$$

Assuming that $\omega_0 \sim \omega_M$ we have

$$k_{\text{th}} \sim \frac{\omega_M}{u} \sim \frac{\alpha}{d},$$

indicating that $k_{\text{th}}d \gg 1$ for $\alpha \gg 1$. Figure 5 shows schematically the dependence of $k_{\text{th}}d$ on α for $\omega_0 = 2\omega_M$. It is seen that this dependence differs only slightly from a linear one.

The existence of a threshold MSW wave vector for phonon emission processes must be attributed to the neglect of inhomogeneous exchange interaction. If it is this latter which dominates the spin wave dispersion, then

$$\omega=\omega_{ex}(ak)^2,$$

and the condition for a long wavelength phonon with $v(k) > u$ to be emitted is given by

$$k>u/(2\omega_{ex}a^2)=k_{ex}.$$

The threshold k dependence we predict here can only be observed if k_{th} is smaller than k_{ex}, that is,

$$2\omega_M\omega_{ex}<(u/a)^2, \tag{37}$$

which is actually the same as condition (2). We note also that the condition $\alpha \gg 1$ is consistent with inequalities (2) [or (37)] if

$$d \gg \frac{\Theta_C}{\Theta_D}a,$$

where

$$\Theta_D = \frac{\hbar u}{a}$$

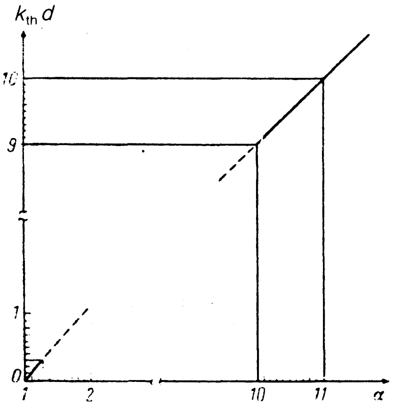

FIG. 5. The threshold wave vector as a function of the parameter α. For $\alpha \to 1$,

$$k_{\text{th}}d=\frac{4(\alpha-1)}{2+\omega_M/\omega_0},$$

and for $\alpha \to \infty$

$$k_{\text{th}}d=\frac{2\alpha}{[(1+\omega_M/\omega_0)^{1/2}+1]}.$$

is of the order of the Debye temperature.[6]

We conclude this section by estimating the phonon emission probability for k values close to k_{th} and α values not too close to unity. Also, the maximum value of $1/\tau^{00}(k)$ will be calculated.

Analysis of the conservation laws shows that, for $k \to k_{\text{th}}$, the domain of integration over k' contracts to a point near $k' = 0$, with

$$k'_{min}=(k_{\text{th}}-k)\frac{u-(\partial\omega_0(k)/\partial k)_{\text{th}}}{u(\alpha+1)}$$
$$k'_{max}=(k_{\text{th}}-k)\frac{u-(\partial\omega_0(k)/\partial k)_{\text{th}}}{u(\alpha-1)}. \tag{38}$$

Making use of this, it is a simple matter to separate the dominant term in the integrand when employing (20). Utilizing integration over φ to dispose of the δ-function we arrive at an expression (an integral over k') with a factor $|\sin\varphi|^{-1}$ playing the dominant role; the angle φ is given by

$$\cos\varphi=\alpha-(k_{\text{th}}-k)\frac{u-(\partial\omega_0(k)/\partial k)_{\text{th}}}{uk'}, \tag{39}$$

where, using (33) and (34),

$$\left(\frac{\partial\omega_0(k)}{\partial k}\right)_{k_{\text{th}}} = \frac{2\alpha\omega_0 u}{uk_{\text{th}}+\omega_0}\,\frac{1}{1+(k/q_0)^2_{\text{th}}}\,\frac{1}{1+dk_{\text{th}}(1+(q_0/k)^2_{\text{th}})}, \tag{40}$$

so that

$$\left(\frac{\partial\omega_0(k)}{\partial k}\right)_{k_{\text{th}}} < u.$$

From (39) it follows that $\varphi = \pi, 0$ on the boundaries of the interval with the implication that $|\sin\varphi|^{-1} \to \infty$ as $k \to k_{\text{th}}$. Except for the factor $|\sin\varphi|^{-1}$, it is permissible to put $k = k_{\text{th}}$ and $k' = 0$ everywhere in the integrand of (20), and

$$\int_{k_{min}'}^{k_{max}'} \frac{dk'}{|\sin\varphi|} = \frac{\alpha\pi}{(\alpha^2-1)^{3/2}}(k_{th}-k)\frac{u-(\partial\omega_0(k))/\partial k)_{th}}{u}. \quad (41)$$

Finally,

$$\frac{1}{\tau_{th}^{00}} = \frac{(\gamma gM)^2\hbar}{2^6\rho d^3}\frac{\alpha k_{th}^3}{(\alpha^2-1)^{3/2}}\frac{1-(\partial\omega_0(k)/\partial k)_{th}u^{-1}}{\omega_0(\omega_0+uk_{th})}$$

$$\times\left[\frac{\sin(q_0d)}{q_0}\right]_{th}^2(k_{th}-k)\begin{cases}1, & T\ll\hbar uk_{th},\\ \dfrac{T}{\hbar uk_{th}}, & T\gg\hbar uk_{th}.\end{cases} \quad (42)$$

Turning back to the case $kd\ll1$ ($\alpha\to1$) we note that the limits of integration can be written as

$$k'_{min} = \frac{1}{2}\frac{\alpha-1}{\alpha+1}k_{th}\left\{1-\left[1-4\left(\frac{\alpha-1}{\alpha+1}\right)^2\frac{k(k_{th}-k)}{k_{th}^2}\right]^{1/2}\right\}$$

$$\approx\frac{(\alpha-1)^3}{8}(k_{th}-k), \quad k'_{max}=k_{th}-k,$$

so that, as in the general case, here again the domain of integration over k' domain contracts to zero as $k\to k_{th}$.

Finally, we can easily estimate the quantity $[\tau^{00}(k)]_{max}^{-1}$ by noting that for $kd\ll1$ the inverse lifetime vanishes as A_0k^4 whereas for $k\to k_{th}$ it behaves like $A_{th}(k_{th}-k)$. Assuming these dependences to be valid over the entire range ($0<k<k_{th}$) (which they are not), the maximum value of k_M may be estimated from

$$A_0k^4=A_{th}(k_{th}-k)$$

or, for $T\ll\hbar uk_{th}$, from

$$\varepsilon x^4=1-x, \quad (43)$$

where

$$x=k/k_{th},$$

$$\varepsilon=\frac{\alpha^3[(\alpha^2+2)(\alpha^2-1)^{1/2}+(2\alpha+1)(\pi/2-\arcsin\alpha^{-1})](\omega_0+uk_{th})}{2^5\pi\omega_0(\alpha^2-1)^{1/2}[1-u^{-1}(\partial\omega_0(k)/\partial k_{th}]\,(\sin(q_0d)/q_0d)_{th}^2}.$$

Clearly Eq. (43) is only tractable in the extreme cases $\varepsilon\ll1$ and $\varepsilon\gg1$, of which the former is easily shown to be not realizable for $\alpha\gtrsim1$. In the latter case (corresponding to $\alpha\gg1$), we have

$$\varepsilon\approx\alpha^5\left(\frac{\omega_0+\omega_M}{\omega_0}\right)^{1/2}\frac{\pi}{2^7}, \quad k_M\approx k_{th}\varepsilon^{-1/4}$$

giving

$$k_M\approx\frac{1}{d}\left[\frac{2^7}{\pi\alpha}\left(\frac{\omega_0}{\omega_0+\omega_M}\right)^{1/2}\right]^{1/4},$$

$$\left[\frac{1}{\tau^{00}(k)}\right]_{max}\approx\frac{(\gamma gM)^2\hbar}{2^5\pi^2\rho\omega_0^2d^5}\left(\frac{\omega_0}{\omega_0+\omega_M}\right)^{1/2}\alpha^2. \quad (44)$$

Recall that in the (opposite) case $\alpha\to1$ the value of $[1/\tau^{00}(k)]_{max}$ is given by (31).

The $1/\tau^{00}(k)$ curve for $\alpha\gtrsim1$ was shown schematically back in Fig. 3. In the case $\alpha\gg1$, the curve retains its shape while shifting very markedly towards large k values.

3. One possibility that arises from the anomalous MSW dispersion law (3) is the creation of an $n'>0$ MSW through the emission of a phonon by an $n=0$ MSW. In this section, a discussion of this process for MSWs of long wavelengths such that $kd\ll1$ will be given.

The same argument as above shows that it is only the sound vibrations homogeneous in z (i.e., the phonons with $m=0$) which can participate in the interaction processes. The conservation laws (22) with $m=0$, $n=0$, $n'\neq0$ yield for k' an equation

$$(k^2+k'^2-2kk'\cos\varphi)^{1/2}=\alpha k, \quad (45)$$

which is analogous to (23) and does not contain n' because the n' dependent term $(k'/n')^2$ has been dropped as negligible in comparison with the others. It will be observed that, in contrast to (23), equation (45) is solvable both for $\alpha>1$ and $\alpha<1$. In the latter case the angle φ is bounded by the condition

$$\sin^2\varphi<\alpha^2.$$

Figure 6 shows the allowable values of k' for $\alpha<1$. For $\alpha>1$,

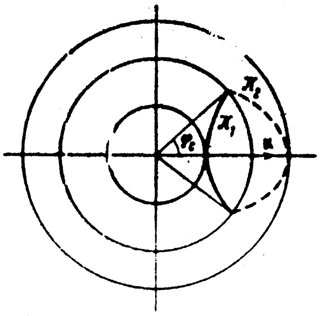

FIG. 6. The curves $\mathscr{K}_1$ and $\mathscr{K}_2$ represent the dependence of k' on the angle φ for phonon emission processes; $n=0$, $n'>0$, $kd\ll1$. There exist two solutions [or two functions $k'(\varphi)$], for one of which

$$\min k'(\varphi=0)=k(1-\alpha),$$

while for the other

$$\max k'(\varphi=0)=k(1+\alpha).$$

At the intersection point

$$k'(\varphi=\arcsin\alpha)=k(1-\alpha^2)^{1/2}, \quad \alpha<1.$$

the emission of a phonon by an MSW with $kd \ll 1$ occurs either with or without a change in the mode number (so that either $n = 0$, $n' \neq 0$ or $n = n' = 0$, respectively), whereas for $\alpha < 1$ only processes changing the wave mode number are possible. Since the probability of a mode number conserving process is much higher, only the result for $\alpha < 1$ is of interest here:

$$\frac{1}{\tau^{0n'}(k)} = \frac{1}{2n'^4}\left(\frac{\gamma}{8\pi}\right)^2 \frac{\omega_M^2}{\omega_0^3} \frac{\alpha^4(\alpha^2+1)}{\rho d^5} \cdot$$

$$\times \begin{cases} (kd)^5 T, & kd \ll \dfrac{T}{\hbar\omega_0} \ll 1, \\ (kd)^6 \hbar\omega_0, & \dfrac{T}{\hbar\omega_0} \ll kd \ll 1. \end{cases} \tag{46}$$

Using the fact that

$$\sum_{n'=1,\dots} (n')^{-4} \approx 1{,}06,$$

it is easily shown that the total probability for emitting MSWs with all $n' \neq 0$ is much less than that for emitting an MSW with $n' = 0$. The latter probability determines completely the value of $1/\tau^0(k)$ for $\alpha < 1$ (when $1/\tau^{00} \equiv 0$) and is given by

$$\frac{1}{\tau^0(k)} \approx \frac{1}{2}\left(\frac{\gamma}{8\pi}\right)^2 \frac{\omega_M^2}{\omega_0^3} \frac{\alpha^4(\alpha^2+1)}{\rho d^5}$$

$$\times \begin{cases} (kd)^5 T, & kd \ll \dfrac{T}{\hbar\omega_0} \ll 1, \\ (kd)^6 \hbar\omega_0, & \dfrac{T}{\hbar\omega_0} \ll kd \ll 1. \end{cases} \tag{46'}$$

Although we have found it unnecessary to cite the value of $1/\tau^{0n'}(k)$ for $\alpha > 1$, it should be recognized that $1/\tau^{0n'} \neq 0$ for $k > k_{\text{th}}$ (see the discussion in the preceding section). This means that the total probability for the generation of a phonon with $k > k_{\text{th}}$ is different from zero. The existence of a threshold should manifest itself as a kink on the wave vector dependence of the inverse lifetime (see the concluding section of this paper).

4. We now consider the emission of a phonon by an $n \neq 0$ MSW and we assume again that $kd \ll 1$. It is readily shown that this process is not allowable unless $m = 0$ and $q'_n > q_n$. To the first nonvanishing order in kd, the conservation laws (22) yield

$$(k^2+k'^2-2kk'\cos\varphi)^{1/2} = \frac{\alpha d}{\pi^2}\left(\frac{k^2}{n^2} - \frac{k'^2}{n'^2}\right), \tag{47}$$

which is only solvable for

$$\sin^2\varphi < \left[\frac{\alpha}{\pi^2}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right)\right]^2 (kd)^2 \ll 1,$$

showing that the range of allowable angles is very narrow so that the MSW propagation direction remains virtually the same after a phonon has been emitted. The allowable values of $\mathbf{k}'$ are shown in Fig. 7. The existence of a solution of (47) has nothing to do with the value of α. We also note (in anticipation of a future result) that, for

$$kd \ll \frac{T}{\hbar\omega_M} \ll 1$$

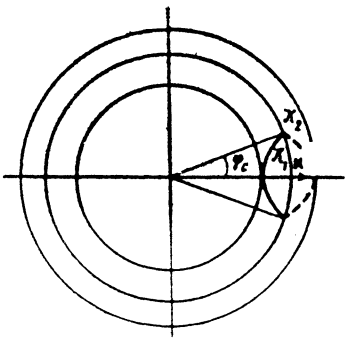

FIG. 7. k' values possible for phonon emission processes for a wave with $n > 0$ ($kd \ll 1$, $n' > n$). There exist two solutions $k'(k)$ represented by the solid line $\mathscr{K}_1$ and dashed line $\mathscr{K}_2$. For one of the solutions,

$$\max k'(\varphi=0) = k\left[1 + \frac{\alpha}{\pi^2}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right)kd\right],$$

for the other,

$$\min k'(\varphi=0) = k\left[1 - \frac{\alpha}{\pi^2}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right)kd\right],$$

and for both

$$k'(\varphi_c) = k\left\{1 - \left[\frac{\alpha}{\pi^2}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right)kd\right]^2\right\}^{1/2}.$$

The value of φ_c is very small:

$$\sin\varphi_c = \frac{\alpha}{\pi^2}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right)kd.$$

the situation differs from all the examples above in that the major contribution into the damping of MSWs with $n > 0$ comes from phonon absorption (rather than phonon emission) processes. Accordingly, we omit the (rather clumsy) formula for $1/\tau^{nn'}(k)$ for the case $kd \ll T/\hbar\omega_M$ and limit ourselves to the condition

$$\frac{T}{\hbar\omega_M} \ll kd \ll 1$$

to obtain

$$\frac{1}{\tau^{nn'}(k)} = \frac{1}{2n^6}\left(\frac{\gamma}{8\pi}\right)^2 \frac{\hbar\alpha^4}{\rho d^5}(kd)^{10}\left(1+\frac{n}{n'}\right)^2\left[1+\left(\frac{n}{n'}\right)^2\right]^2. \tag{48}$$

Since this last expression clearly diverges when summed over n', we are faced, for the first time in our analysis, with a situation where the exchange interaction may no longer be neglected when calculating the total probability of phonon emission by an MSW with $n > 0$ ('total' meaning the summation over all n'). Allowance for the exchange, on the other hand, complicates the theory to such an extent that an analytical calculation of $1/\tau^n(k)$ becomes hardly possible. We note, finally, that, for the wave number n' fixed, equation (48) is of course valid if the exchange-neglect conditions (2) and (37) are fulfilled.

5. There are two condensation points recognizable in the MSW spectrum, $kd \to 0$ and $kd \to \infty$. Our discussion so far has been focused on MSWs of long wavelength. In this section, phonon emission processes with $kd \gg 1$ will be considered. If number-conserving emission ($n' = n$) is allowed,

that the $n = 0$ wave has nothing special about it as compared with the others. Interaction is only possible for $m = 0$ and $q_{n'} > q_n$ and, simultaneously with the condition $kd \gg 1$, the inequality $k'd \gg 1$ is necessarily obeyed. From Eq. (22) it follows that

$$(k^2+k'^2-2kk'\cos\varphi)^{1/2}$$

$$= \frac{\omega_0\omega_M\pi^2(n'^2+2n'-n^2-2n)}{2u[\omega_0(\omega_0+\omega_M)]^{1/2}} \frac{1}{(kd)^2}, \quad n' \geqslant n, \tag{49}$$

reducing the allowable angles to a narrow range of the order of $(kd)^{-3}$. Figure 8 shows the allowable k' values as functions of the angle φ for a fixed vector **k**. The probability for an $n \to n'$ phonon emission event can now be obtained by appropriate integration to give

$$\frac{1}{\tau^{nn'}(k)} = \frac{(\gamma gM)^2\hbar}{2^7\pi d\rho u^2} \times \begin{cases} A^2(kd)^{-6}, & (kd)^2 \ll \dfrac{\hbar u A}{T}, \\ \dfrac{TA}{\hbar u}(kd)^{-4}, & (kd)^2 \gg \dfrac{\hbar u A}{T} \gg 1, \end{cases} \tag{50}$$

where

$$A = \frac{\omega_0\omega_M\pi^2(n'^2+2n'-n^2-2n)}{2[\omega_0(\omega_0+\omega_M)]^{1/2}u}.$$

Note that $1/\tau^{nn} \equiv 0$ because A vanishes when $n = n'$. Since the quantity

$$\frac{1}{\tau^n} = \sum_{n'} \frac{1}{\tau^{nn'}},$$

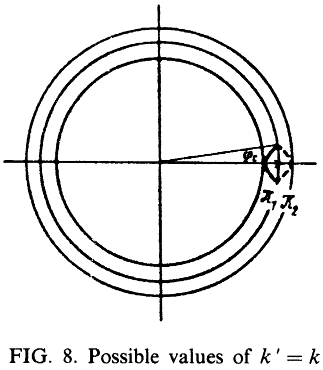

FIG. 8. Possible values of $k' = k'(\varphi)$ for photon emission processes ($kd \gg 1$). There exist two solutions $k'(\varphi)$ (solid line $\mathscr{K}_1$ and dashed line $\mathscr{K}_2$). For $\mathscr{K}_2$

$$\max k'(\varphi=0) = k\left(1 + \frac{B}{k^3}\right).$$

and for $\mathscr{K}_1$

$$\min k'(\varphi=0) = k\left(1 - \frac{B}{k^3}\right).$$

At the intersection point,

$$k'(\varphi_c) = k\left[1 - \left(\frac{B}{k^3}\right)^2\right]^{1/2}, \quad \sin\varphi_c = \frac{B}{k^3},$$

$$B = \frac{\omega_0\omega_M}{2[\omega_0(\omega_0+\omega_M)]^{1/2}} \frac{\pi^2(n'^2+2n'-n^2-2n)}{ud^2}.$$

is divergent, our result is again of no use in the calculation of the total phonon emission time. It is natural, however, that equation (50) is quite adequate for fixed-n' no-exchange calculations—a point of special importance for the situation where the condition $kd \gg 1$ is fulfilled (see the discussion above).

A limiting-case analysis will give an order-of-magnitude estimate of the inverse MSW lifetime due to phonon emission. By assuming $\alpha \gtrsim 1$ and $\omega_0 \sim \omega_M$, it is a simple matter to show that

$$\frac{1}{\tau_I(k)} \sim \frac{\gamma^2\hbar}{\rho d^5} f(kd), \tag{51}$$

with a function $f(kd)$ whose behavior for $kd \ll 1$ or $kd \gg 1$ is easily obtainable from the formulas above and which turns out to have a maximum $f_{\max} \sim 1$ at $kd \sim 1$. Thus

$$\left(\frac{1}{\tau_I}\right)_{max} \sim \frac{\gamma^2\hbar}{\rho d^5}, \tag{52}$$

admittedly a very crude estimate in view of our full disregard for dimensionless factors.

6. Turning now to phonon absorption processes, the conservation laws take the form

$$\mathbf{k} = \mathbf{k}' - \mathbf{f}, \quad \omega_n(\mathbf{k}) = \omega_{n'}(\mathbf{k}') - \Omega_m(\mathbf{f}), \tag{53}$$

and it can be shown that, analogous to the phonon emission case, see section 1, in the limiting cases $kd \ll 1$ or $kd \gg 1$ only z-homogeneous phonons are allowed to participate. For the zeroth mode, the absorption processes with $kd \ll 1$ contribute much less into the damping than do the emission processes. As for the nonzeroth ($n \neq 0$) mode, however, it is precisely the absorption processes with $n' \neq 0$ which control its damping at $kd \ll T/\hbar\omega_M$, and it is therefore worthwhile to discuss them in more detail. For $n > 0$, $n' = 0$ we have

$$(k^2+k'^2-2kk'\cos\varphi)^{1/2} = \alpha k', \quad k' > k. \tag{54}$$

If $\alpha > 1$, there are no restrictions on the angle φ, and the values of k' range from $k/(\alpha+1)$ to $k/(\alpha-1)$. For $\alpha < 1$, the value of φ is restricted by

$$\sin^2\varphi < \alpha^2,$$

and for each pair $\{k,\varphi\}$ there exist two values of k' (see Fig. 9), to be found between $k/(1+\alpha)$ and $k/(1-\alpha)$. The inverse damping time is calculated to be

$$\frac{1}{\tau^n(k)} = \frac{(\gamma gM)^2\alpha^4}{2^6\pi^3\omega_0^2\omega_M\rho} \frac{k^5T}{n^2}, \quad kd \ll \frac{T}{\hbar\omega_M}. \tag{55}$$

One further equation (which we omit here) indicates that for $kd \gg 1$ the phonon absorption contribution to the inverse lifetime is much less than that of phonon emission.

Referring to Fig. 10, the (schematic) dependence $1/\tau^{n(s)}(k)$ for $d \gg d_{\rm ex}$ summarizes the results we have obtained for the two damping processes discussed (it will be recalled that the contribution from the third process is exponentially small). For $\alpha \sim 1$, the maximum of the first curve is in the wave vector region for which $kd \sim 1$ [cf. (51)]. Interestingly, the MSW lifetime is longer the greater the mode number n, that is, the larger the corresponding value of q_n. The lifetime of the $n = 0$ mode is shortest, as one might expect for the mode with the maximum frequency.

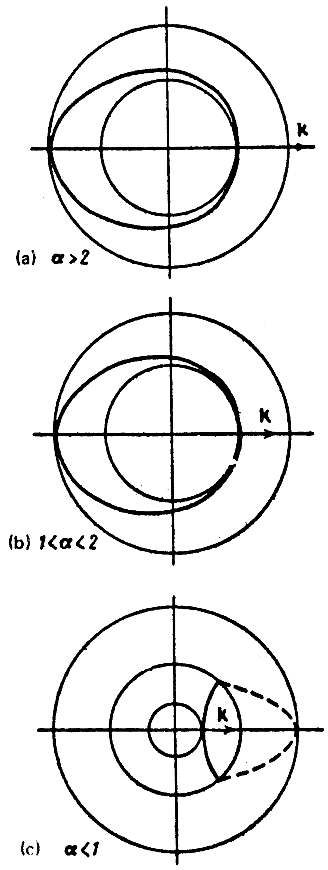

FIG. 9. Solid lines represent the possible values of $k'(\varphi)$ for phonon absorption with $n>0$, $n'=0$, $kd \ll 1$. For $\alpha > 1$

$$\max k'(\varphi=\pi) = \frac{k}{\alpha-1},$$

$$\min k'(\varphi=0) = \frac{k}{\alpha+1}.$$

For $\alpha < 1$ there exist two solutions $k'(\varphi)$. For one of them,

$$\max k'(\varphi=0) = \frac{k}{1-\alpha},$$

for the other

$$\min k'(\varphi=0) = \frac{k}{1+\alpha},$$

and for both

$$k'(\varphi=\arcsin\alpha) = \frac{k}{(1-\alpha^2)^{1/2}}.$$

7. We dwell briefly on the damping of the antisymmetric MSW. The expressions for $u_{\mathbf{k}n}$, $v_{\mathbf{k}n}$, and $\omega_n(\mathbf{k})$ are the same as those for the symmetric MSW, see Eq. (13), and the only difference is that the quantities q_n are solutions to (5) rather than to (4). Also, instead of (10) we have

$$a = \frac{1}{V^{1/2}} \sum_{\mathbf{k}n} [u_{\mathbf{k}n} a_{\mathbf{k}n} \exp(i\mathbf{k}\boldsymbol{\rho}) + v_{\mathbf{k}n}^* a_{\mathbf{k}n}^+ \exp(-i\mathbf{k}\boldsymbol{\rho})] \times \frac{\sin(q_n z)}{[1+\cos^2(q_n d)/(kd)]^{1/2}}. \qquad (10')$$

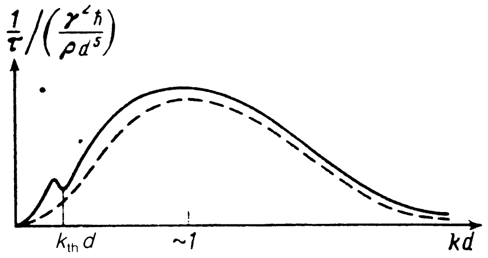

FIG. 10. Damping time as a function of the wavevector for an MSW interacting with phonons (schematic). Solid line represents the behavior of $1/\tau^{0(s)}(k)$ (note the kink at $k = k_{th}$. Dashed line represents $1/\tau^{n(s)}(k)$ for $n > 0$. The maximum value of the inverse lifetime is estimated to within an order of magnitude; the value of $1/\tau/(\gamma^2\hbar/\rho d^5)$ depends strongly on the system parameters.

As is the case for the symmetric wave, the energy $\omega_n(\mathbf{k})$ decreases, instead of increasing, as the mode number n is increased.

Turning to the inverse lifetime calculation, we note that for $kd \ll 1$ the main contribution to the damping comes from phonon emission. The derivation parallels quite closely that given for a symmetric wave with $n > 0$, with πn formally replaced by $\pi n + \pi/2$. The process is only possible for $n' \geqslant n$. It is found that

$$\frac{1}{\tau^n(k)} = \frac{(\gamma g M)^2}{2^{12}\pi^4\omega_0^2} \frac{\hbar a^4 d^5 k^{10}}{\rho(n+1/2)^8} \times \begin{cases} 1, & \dfrac{2\pi^2 n^2 T}{\hbar\omega_M} \ll (kd)^2 \ll 1, \\ \dfrac{2\pi^2 n^2 T}{(kd)^2\hbar\omega_M}, & (kd)^2 \ll \dfrac{2\pi^2 n^2 T}{\hbar\omega_M}. \end{cases} \qquad (56)$$

It is not necessary to treat the $n=0$ MSW separately because in this case too the frequency $\omega_0(k)$ depends quadratically on k in the limit as $k \to 0$ [see Eq. (8)].

To obtain $1/\tau^{nn'}(k)$ for $kd \gg 1$ it suffices to replace $(n + 1/2)$ and $(n' + 1/2)$ by, respectively, $(n+1)$ and $(n'+1)$ in (49). As with the symmetric MSW, the lifetime increases with increasing mode number. Comparison shows that at extremely small (large) kd the lifetime of the symmetric MSW is shorter than (of the same order that) the antisymmetric MSW lifetime.

8. Using equation (17) for $\mathscr{H}_{\text{int}}$, we can estimate the influence of an interaction of two magnons with one phonon on MSW as well as phonon damping can be estimated. If the phonon energy $\hbar\Omega(f)$ is less than the doubled magnon energy, the decay of the phonon into two MSWs is forbidden and the two-magnon processes result in an exponentially large phonon lifetime ('exponentially' referring to temperature).

Turning back to Eqs. (14) and (15) we consider a phonon with a very small wave vector and homogeneous in z $(m=0)$, and take into account the interaction of such a phonon with a symmetric MSW. We have already mentioned that for $f \to 0$ coalescence of two MSWs with phonon formation is impossible because it requires that $uf > 2\omega_0$. For $f \to 0$, the integrals for phonon absorption and phonon emission processes may be calculated by replacing the bracket

$$\left[\exp\left(-\frac{\varepsilon_{k'n'}}{T}\right)-\exp\left(-\frac{\varepsilon_{kn}}{T}\right)\right]$$

by the first term in the expansion in $\hbar\Omega_0(f)/T$, i.e.,

$$\frac{1}{\tau^{ph}(f)}=\frac{(\gamma gM)^2\hbar^2f^2}{2^7\pi d^3\rho T}\sum_{nn'}\int|v_{kn}|^2\left\{\frac{\sin(q_n-q_{n'})d}{q_n+q_{n'}}-\frac{\sin(q_n+q_{n'})d}{q_n+q_{n'}}\right\}^2$$
$$\times\exp\left[-\frac{\hbar\omega_n(k)}{T}\right]\delta(\omega_0(\mathbf{f})+\omega_{n'}(\mathbf{k}-\mathbf{f})-\omega_n(\mathbf{k}))\,d^2k. \qquad (57)$$

Because of the factor $\exp[-\hbar\omega_n(k)/T]$, the integral is dominated by small k's, which enables an expansion in powers of kd to be made in the integrand. Consider the case $n=n'=0$ first. The conservation laws are fulfilled only if $\alpha>1$. Assuming that $fd\ll\alpha-1$, the conservation laws imply that

$$\cos\varphi=1/\alpha. \qquad (58)$$

The angle φ here specifies the direction of the vector $\mathbf{k}$ [integrated over in (57)] relative to the phonon wave vector $\mathbf{f}$. Taking advantage of the δ-function when integrating over φ, we obtain the following expression for the inverse phonon lifetime due to the scattering by an MSW with $n=n'=0$:

$$\frac{1}{\tau^{00}_{ph}(f)}=\frac{3!\,\alpha}{2^8\pi^3}\frac{\gamma^2 f}{d^4\rho\hbar^2}\frac{T^3\exp\left(-\frac{\hbar\omega_0}{T}\right)}{\omega_0{}^2\omega_M(\alpha^2-1)^{1/2}}. \qquad (59)$$

As $\alpha\to1$, this expression exhibits a divergence similar to that encountered when calculating the inverse MSW lifetime. As in the derivation of (30), the condition $\alpha-1\ll1$ calls for a more detailed analysis than for (59). Let $1\gg fd\gg\alpha-1$. The refined conservation laws take the form

$$\cos\varphi=\frac{1}{\alpha}\left[1+\left(1+\frac{\omega_M}{2\omega_0}\right)kd\right], \qquad (60)$$

and the expression for $1/\tau^{00}_{\rm ph}(f)$ becomes

$$\frac{1}{\tau^{00}_{ph}(f)}=\frac{\gamma^2\omega_M{}^3\hbar^2 f(\alpha-1)^{7/2}}{2^{12}\pi^3 2^{1/2}\omega_0\rho d^4(1+\omega_M/2\omega_0)}\frac{\exp(-\hbar\omega_0/T)}{T},\quad \alpha\approx1. \qquad (59')$$

Note that the small factor $(\alpha-1)^{7/2}$ in this formula is offset by the large factor $(\hbar\omega_M/T)^4$ (recall that $T\ll\hbar\omega_M,\hbar\omega_0$).

In the case $n=0$, $n'>0$, the conservation laws are obeyed for any α and the argument of the δ-function vanishes when

$$k=f/\alpha. \qquad (61)$$

The corresponding contribution into the inverse phonon damping time is obtained by integrating and summing over $n'>0$, to give

$$\frac{1}{\tau^{0,n'>0}_{ph}(f)}=\frac{\gamma^2\hbar^2u^4f^6\exp(-\hbar\omega_0/T)}{2^7\pi^4d^3\rho\omega_M\omega_0{}^2T}. \qquad (62)$$

Compared to the terms above, those with $n>0$, $n'>0$ contribute much less to the sum (57) over n, n' and may therefore be neglected, as may the terms arising from the interaction with antisymmetric MSWs.

To summarize, in plates with $\alpha<1$ the inverse damping time is given by Eq. (62), whereas in plates with $(\alpha-1)\ll fd$ it is determined either by (59) if $T\ll\hbar\Omega(f)(fd)^{1/4}$ or by (62) if $T\gg\hbar\Omega(f)(fd)^{1/4}$.

9. Two circumstances are noteworthy.

(a) For all the examples investigated, the lifetime of an MSW is strongly dependent on its wavelength (or its wave vector k). MSW experiments are usually conducted at a fixed frequency ω using the magnetic field as a variable parameter. The magnetic field dependence of the MSW lifetime is obtained by simply expressing k in terms of H by inverting the dispersion law (3) and remembering that $\omega_0=gH$ and $H\leqslant\omega/g$.

(b) Of all the results, greatest interest attaches to those connected with (1) the threshold value for the wavevector, $k=k_{\rm th}$ (and hence for the magnetic field, $H=H_{\rm th}<\omega/g$); (2) quasiresonant peaks in the lifetime dependences on k and H, and (3) kinks at $H=H_{\rm th}$, when one of the scattering mechanisms is switched off.

It should be emphasized once more, finally, that our primary interest has been in those aspects of the MSW—phonon interaction dominated by the peculiarity of the MSW dispersion law. The unusual features of this dispersion law are: the decrease of the frequency with increasing mode number n; the linear dependence of $\omega=\omega_0$ on the wave vector in a symmetric $n=0$ mode; and the existence of two condensation points ($k\to0$ and $k\to\infty$).

When comparing the above results with experiment (something not attempted in this study) it is necessary to take into account the dissipation mechanisms we have ignored here, such as impurity scattering, surface roughness scattering, etc.

[1)]The notation we adopt here is as follows: $\omega_0=g(H+4\pi\beta M)$, g is the magnetomechanical ratio, $\mathbf{H}$ the steady magnetic field inside the plate, $\mathbf{M}$ the spontaneous magnetization at $T=0$, β the anisotropy ratio for the easy axis normal to the plate, $\omega_M=4\pi gM$, u the speed of sound, a the interatomic separation, $\omega_{\rm ex}=J/\hbar$, J is the exchange integral, $J\sim\Theta_C$, Θ_C the Curie temperature.

[2)]$\mathbf{m}=\mathbf{m}(\mathbf{r},t)$ is the varying term in the expression for the magnetic-moment density of the ferromagnetic plate, $\mathbf{M}(\mathbf{r},t)=\mathbf{M}+\mathbf{m}(\mathbf{r},t)$.

[3)]The second superscript indicates the state occupied by the MSW after having emitted a phonon.

[4)]Here and afterwards the frequencies ω_0 and ω_M are assumed to be of the same order of magnitude.

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